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Rapid left expansivity, a commonality between Wolfram's Rule 30 and powers of p/q ☆

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ABSTRACT

We define the class of rapidly left expansive cellular automata, which contains Wolfram's Rule 30, fractional multiplication automata, and many others. Previous results on aperiodicity of columns in space-time diagrams of certain cellular automata generalize to this new class. We also present conditions that imply periodic behavior in cellular automata and use these to prove new results on rapidly left expansive cellular automata that originate from the theory of distribution modulo 1.

1. Introduction

Cellular automata (CA) are self-maps on the set $\Sigma^{\mathbb{Z}}$ of bi-infinite sequences (configurations) over a finite symbol set Σ defined by using a local rule. They are seemingly simple symbolic dynamical systems that can however exhibit complicated behavior (for a survey on CA, see [8]). The purpose of this paper is to point out a certain common feature between two previously studied CA, generalize these examples to a new class of CA in which this feature still appears, and prove new results on this class.

One cellular automaton we consider is Wolfram's Rule 30, a CA over the symbol set $\{0, 1\}$ defined in Example 2.3, which is notoriously resistant to proofs of nontrivial results (for earlier results, see [7,15]) and has recently inspired Stephen Wolfram to offer prizes for the solution of certain problems concerning it [18]. The first problem of these is: If Rule 30 is initialized on the configuration containing a single 1 in the center, then is the center column of the space-time diagram (also called a trace of the configuration) eventually periodic? (For a small part of this space-time diagram, see the left-hand side of Fig. 1.) The answer "no" is expected, and if one instead considers traces of width 2, the answer is indeed "no" by a result of Jen [7] (see Corollary 3.7 in the present paper). The proof in fact works for all initial configurations from the larger set $\mathcal{L}_0(\Sigma)$ of configurations having a leftmost non-zero symbol.

Other examples of CA with complicated behavior are given by fractional multiplication automata $\Pi_{p/q,pq}$ which perform multiplication by p/q in base pq for coprime $p > q > 1$. The complexity of these CA is understandable, because even ignoring the CA point of view it is well known that the distribution of fractional parts (i.e. distribution modulo 1) of sequences of

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the form $(\xi(p/q)^i)_{i\in\mathbb{N}}$ for $\xi > 0$ is a mysterious topic as demonstrated e.g. in Chapter 3 of the book [3]. For example, in the case $p/q = 3/2$, it is not known whether $\xi > 0$ can be chosen so that the fractional parts in the whole sequence remain less than $1/2$ [12].

The dynamics of the automata $\Pi_{p/q,pq}$ are not as intractable as those of Rule 30, and results on them have been proved e.g. in [9–11]. However, a suggestive connection between $\Pi_{p/q,pq}$ and Rule 30 is that the traces of width 1 of fractional multiplication automata for initial configurations from $\mathcal{L}_0(\Sigma)$ were shown not to be eventually periodic in [11] (see Corollary 3.8 in the present paper). In Section 3 we show that both Rule 30 and fractional multiplication automata belong to the larger class of rapidly left expansive cellular automata, and prove a generalization of the previously mentioned results of [7] and [11] concerning traces for this larger class in Theorem 3.5.

The above-mentioned connection between Rule 30 and multiplication by p/q is interesting, because it hints at the possibility of using results on the distribution of fractional parts as an inspiration for new results on the dynamics of Rule 30 and other rapidly left expansive CA. More concretely, we choose as a starting point the results saying that any fractional part repeats in the sequence $(\xi(p/q)^i)_{i\in\mathbb{N}}$ only finitely many times [5] and that the fractional parts of this sequence have infinitely many limit points [14]. In Section 4 we interpret these as symbolic dynamical statements concerning the automata $\Pi_{p/q,pq}$ and not only give alternative, purely symbolic dynamical proofs of these results, but also generalize them to the class of rapidly expansive cellular automata in Corollaries 4.6 and 4.9.

The results of the previous paragraph are corollaries, because they follow by an application of Theorem 3.5 from results that hold for even larger classes of CA and that may be of independent interest. These results are of the form where certain sufficient conditions imply periodic behavior. One earlier example of this type is the classical Morse-Hedlund theorem [13] (see Theorem 4.1 in the present paper), according to which having a small number of subwords in a sequence implies eventual periodicity of the sequence. Another one is the theorem of Boyle and Kitchens [1], according to which every closing CA, and conjecturally every surjective CA, has a dense set of configurations that are periodic under the action of that CA. We show (Theorem 4.5) that if the right tail of a configuration repeats infinitely often under the action of a left expansive CA, the right tail even repeats periodically, and if the CA is also left expansive "with height 0", the whole configuration repeats periodically. For instance, composing any left closing CA with a suitable shift yields a CA which is left expansive with height 0. We also show (Theorem 4.7) that if the sequence of right tails under the action of a CA has finitely many limit points, then arbitrarily wide traces are eventually periodic, and in Remark 4.8 we interpret this result as a generalization of the Morse-Hedlund theorem.

2. Preliminaries

We denote the set of positive integers by $\mathbb{Z}_+$ and define the set of natural numbers by $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. Whenever A and B are sets, B^A denotes the collection of functions from A to B. We often denote the value of a function $f \in B^A$ at $a \in A$ by $f[a]$ instead of $f(a)$. For $b \in B$, we denote by $b^A \in B^A$ the special function satisfying $b^A[a] = b$ for all $a \in A$.

A function $f \in B^{\mathbb{Z}}$ (respectively, $f \in B^{\mathbb{N}}$) is periodic if there is a $p \in \mathbb{Z}_+$ such that $f[i+p] = f[i]$ for all $i \in \mathbb{Z}$ (respectively, $i \in \mathbb{N}$). Then we may also say that f is p-periodic. We say that $f \in B^{\mathbb{Z}}$ (respectively, $f \in B^{\mathbb{N}}$) is eventually periodic if there are $p \in \mathbb{Z}_+$ and $c \in \mathbb{Z}$ (respectively, $c \in \mathbb{N}$) such that $f[i+p] = f[i]$ holds for all $i \geq c$. When $f \in B^{\mathbb{N}}$, such a $c \in \mathbb{N}$ is called a preperiod of f.

We call a nonempty finite set Σ of symbols an alphabet. We will assume without loss of generality that Σ is equal to $\Sigma_n = \{0, 1, \ldots, n-1\}$ for some $n \in \mathbb{Z}_+$, so in particular Σ always contains 0. Bi-infinite sequences over an alphabet Σ are called configurations. A configuration x is formally an element of $\Sigma^{\mathbb{Z}}$ and therefore its value at a coordinate $i \in \mathbb{Z}$ is denoted by $x[i]$. Given an interval $I = [i, j]$ with $i \leq j \in \mathbb{Z}$, x has a finite subsequence denoted by $x[I] = x[i, j] = x[i]x[i+1]\cdots x[j]$. Similarly, for the infinite interval $I = [i, \infty]$, x has a right infinite subsequence denoted by $x[I] = x[i, \infty] = x[i]x[i+1]x[i+2]\cdots$.

Any finite sequence $v = v[0]v[1]\cdots v[n-1]$, where $n \in \mathbb{N}$ and $v[i] \in \Sigma$, is a word over Σ. We say that the word v occurs in a configuration $x \in \Sigma$ at position i if $x[i]\cdots x[i+n-1] = v[0]\cdots v[n-1]$. The set of words of length n over Σ is denoted by Σ^n.

A sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in \Sigma^{\mathbb{N}}$ converges to $x \in \Sigma^{\mathbb{N}}$ if for every $n \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that $x_i[0,n] = x[0,n]$ for all $i \geq N$. We say that x is a limit point of $(x_i)_{i\in\mathbb{N}}$ if some subsequence of $(x_i)_{i\in\mathbb{N}}$ converges to x. We briefly mention that these agree with the usual definitions of convergence and limit points when $\Sigma^{\mathbb{N}}$ is equipped with the prodiscrete topology.

Definition 2.1. Let Σ be an alphabet. We say that a map $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is a *cellular automaton* (with memory m and anticipation n for $m, n \in \mathbb{N}$) if there exists a map $f : \Sigma^{m+n+1} \to \Sigma$ such that $F(x)[i] = f(x[i-m], \ldots, x[i], \ldots, x[i+n])$ for $x \in \Sigma^{\mathbb{Z}}$, $i \in \mathbb{Z}$. Such a map f is an (m,n) *local rule* of F. If we can choose $r \in \mathbb{N}$ so that $r = m = n$, we say that F is a radius-r CA.

Note also that if F has an (m,n) local rule $f : \Sigma^{m+n+1} \to \Sigma$, then F is a radius-r CA for $r = \max\{m, n\}$, with possibly a different local rule $f' : \Sigma^{2r+1} \to \Sigma$.

One of the simplest cellular automata is the shift map $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ defined by $\sigma(x)[i] = x[i+1]$ for $x \in \Sigma^{\mathbb{Z}}$, $i \in \mathbb{Z}$: this clearly has a $(0,1)$ local rule. In the following we present more interesting examples of cellular automata, which will be used to motivate the definition of rapidly left expansive cellular automata in the next section.

Example 2.2 *(Left permutive CA).* A CA $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is (m,n) *left permutive* with $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ if F has an (m,n) local rule $f : \Sigma^{m+n+1} \to \Sigma$ such that for every word $v \in \Sigma^{m+n}$ the function $f_v : \Sigma \to \Sigma$ defined by $f_v(a) = f(av)$ for $a \in \Sigma$ is a bijection.

Example 2.3 *(Elementary CA).* Elementary cellular automata (ECA) are CA $F : \Sigma_2^{\mathbb{Z}} \to \Sigma_2^{\mathbb{Z}}$ with a binary alphabet and with $(1,1)$ local rules. Their study was initiated by Wolfram in the paper [19], which also popularized a systematic naming scheme for them. Using that naming scheme, one notable example is the Rule 30 automaton W_{30} with the $(1,1)$ local rule $f_{30} : \Sigma_2^3 \to \Sigma_2$ defined by

$$
\begin{aligned}
&f_{30}(000) = 0 \quad f_{30}(001) = 1 \quad f_{30}(010) = 1 \quad f_{30}(011) = 1 \\
&f_{30}(100) = 1 \quad f_{30}(101) = 0 \quad f_{30}(110) = 0 \quad f_{30}(111) = 0.
\end{aligned}
$$

Rule 30 is $(1,1)$ left permutive, because both symbols of Σ_2 appear as the result in each of the four columns above.

Before the following example, fractional multiplication CA, it is appropriate to define the notion of configurations left asymptotic to $0^{\mathbb{Z}}$.

Definition 2.4. We say that a configuration $x \in \Sigma^{\mathbb{Z}}$, $x \neq 0^{\mathbb{Z}}$, is *left asymptotic to* $0^{\mathbb{Z}}$ if there exists an $N \in \mathbb{Z}$ such that $x[i] = 0$ for $i < N$ (the exclusion of $x = 0^{\mathbb{Z}}$ is just a matter of convenience). Then the minimal $N \in \mathbb{Z}$ such that $x[N] \neq 0$ is the *left edge* of x, denoted by $\ell(x)$. The set of all configurations over Σ left asymptotic to $0^{\mathbb{Z}}$ is denoted by $\mathcal{L}_0(\Sigma)$.

Configurations left asymptotic to $0^{\mathbb{Z}}$ are analogous to usual representations of positive numbers, where there may be infinitely many digits to the right of the decimal point but always a finite number of digits to the left of the decimal point (and then the representation can be extended to a bi-infinite sequence by adding an infinite sequence of zeroes to the left end).

Example 2.5 *(Fractional multiplication CA).* Let $n > 1$. If $\xi > 0$ is a real number and $\xi = \sum_{i=-\infty}^{\infty} \xi_i n^i$ is the unique base n expansion of ξ such that $\xi_i \neq n-1$ for infinitely many $i < 0$, we define $\mathrm{config}_n(\xi) \in \mathcal{L}_0(\Sigma_n)$ by

$$\mathrm{config}_n(\xi)[i] = \xi_{-i-1}$$

for all $i \in \mathbb{Z}$. In reverse, for $x \in \mathcal{L}_0(\Sigma_n)$ we define

$$\mathrm{real}_n(x) = \sum_{i=-\infty}^{\infty} x[-i] n^{i-1}.$$

Clearly $\mathrm{real}_n(\mathrm{config}_n(\xi)) = \xi$ and $\mathrm{config}_n(\mathrm{real}_n(x)) = x$ for every $\xi > 0$ and every $x \in \mathcal{L}_0(\Sigma_n)$ such that $x[i] \neq n-1$ for infinitely many $i > 0$.

For coprime $p > q > 1$ we define a $(0,1)$ local rule $g_{p,pq} : \Sigma_{pq} \times \Sigma_{pq} \to \Sigma_{pq}$ for a CA $\Pi_{p,pq}$, which performs multiplication by p in base pq in the sense that $\mathrm{real}_n(\Pi_{p,pq}(\mathrm{config}_n(\xi))) = p\xi$ for all $\xi > 0$. Digits $a, b \in \Sigma_{pq}$ are represented as $a = a_1 q + a_0$ and $b = b_1 q + b_0$, where $a_0, b_0 \in \Sigma_q$ and $a_1, b_1 \in \Sigma_p$: such representations always exist and they are unique. Then

$$g_{p,pq}(a,b) = g_{p,pq}(a_1 q + a_0, b_1 q + b_0) = a_0 p + b_1.$$

The map $g_{p,pq}$ encodes the usual algorithm for long multiplication by p in base pq (for more details, see e.g. [9,11]). It is also possible to define a *fractional multiplication automaton* $\Pi_{p/q,pq}$ that multiplies by p/q in base pq as the composition $\sigma^{-1} \circ \Pi_{p,pq} \circ \Pi_{p,pq}$. This CA has a $(1,1)$ local rule.

Definition 2.6. A space-time diagram $\theta \in \Sigma^{\mathbb{Z} \times (-\mathbb{N})}$ (of a configuration $x \in \Sigma^{\mathbb{Z}}$ with respect to a CA $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$) is defined by $\theta[(i,j)] = F^{-j}(x)[i]$ for $(i,j) \in \mathbb{Z} \times (-\mathbb{N})$.

The space-time diagram of x with respect to F is usually depicted by drawing the configurations $x, F(x), F^2(x), \ldots$ on consecutive rows as in Fig. 1.

For a CA $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, a configuration $x \in X$ and an interval $I = [i,j]$ with $i \leq j \in \mathbb{Z}$, the I-trace of x (with respect to F) is the one-sided sequence $\mathrm{Tr}_{F,I}(x)$ over the alphabet Σ^{j-i+1} (i.e. the symbols of the alphabet are words over Σ) defined

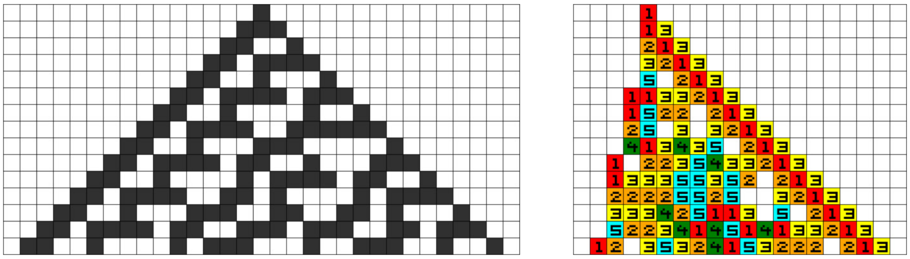

Fig. 1. The space-time diagrams of the configuration $\cdots 0001000 \cdots$ with respect to the CA $W_{30} : \Sigma_2^{\mathbb{Z}} \to \Sigma_2^{\mathbb{Z}}$ and $\Pi_{3/2,6} : \Sigma_6^{\mathbb{Z}} \to \Sigma_6^{\mathbb{Z}}$. White and black squares correspond to digits 0 and 1 respectively.

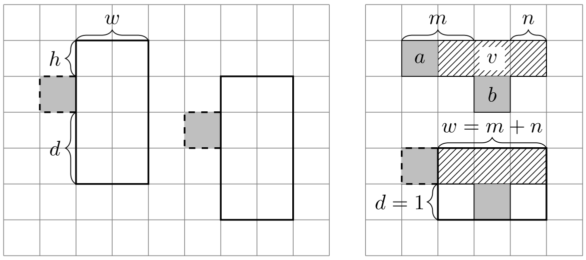

Fig. 2. Left: A pair of translations of $R(h,d,w)$ enclosed by thick lines. Assuming these translated rectangles have identical contents in a space-time diagram (or even if they are in different space-time diagrams) of a left expansive CA with dimensions (h,d,w), then also the symbols contained in the gray squares are identical.
Right: An (m,n) left permutive CA is left expansive.

by $\mathrm{Tr}_{F,I}(x)[t] = F^t(x)[I]$ for $t \in \mathbb{N}$. These correspond to columns of various width and position in the space-time diagram of x with respect to F. If $I = \{i\}$ is the degenerate interval, we may write $\mathrm{Tr}_{F,i}(x)$ and if $i = 0$, we may write $\mathrm{Tr}_F(x)$. If the CA F is clear from the context, we may write $\mathrm{Tr}_I(x)$.

3. Rapidly left expansive cellular automata

In this section we define the class of rapidly left expansive cellular automata. This definition is strongly guided by the proof of Theorem 3.5, which we also give in this section. One main component of the definition is the notion of left expansivity. This is a special case of the notions of expansive and one-sided expansive directions [2,4] for more general dynamical systems, and has appeared earlier in the context of cellular automata e.g. in [6].

Definition 3.1. A set of the form $R(h,d,w) = \{(i,j) \in \mathbb{Z}^2 \mid -d \leq i \leq h, 0 \leq j < w\}$ with $h, d \in \mathbb{N}$, $w \in \mathbb{Z}_+$ is called a *rectangle* (the rectangle of height h, depth d and width w, or the rectangle of dimensions (h,d,w)). A CA F is *left expansive* (with dimensions (h,d,w)) if for $R = R(h,d,w)$, for any pair of space-time diagrams θ_1, θ_2 with respect to F and any pair of points $(i_1, j_1), (i_2, j_2) \in \mathbb{Z}^2$ satisfying $(i_1, j_1) + R, (i_2, j_2) + R \subseteq \mathbb{Z} \times (-\mathbb{N})$ the implication

$$\theta_1|_{(i_1,j_1)+R} = \theta_2|_{(i_2,j_2)+R} \implies \theta_1[(i_1 - 1, j_1)] = \theta_2[(i_2 - 1, j_2)]$$

holds, see the left hand side of Fig. 2.

Given a CA F with an (m,n) local rule f (upper right of Fig. 2) the contents of the left gray cell ($a \in \Sigma$) and the striped area ($v \in \Sigma^{m+n}$) in a space-time diagram determine the content of the bottom gray cell via f by $b = f(av)$. If F is additionally (m,n) left permutive, the content of the left gray cell can be expressed in terms of the content of the striped area and the bottom gray cell as $a = f_v^{-1}(b)$. In particular (lower right of Fig. 2), the contents of the area enclosed by thick lines determine the content of the gray cell enclosed by dashed lines, and therefore an (m,n) left permutive CA is left expansive with dimensions $(0, 1, m+n)$. A fractional multiplication automaton is left expansive with dimensions $(1,1,1)$ by Proposition 3.7 of [11].

Intuitively left expansivity of $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ with dimensions (h,d,w) means that there is a CA with an (h,d) local rule over the alphabet Σ^w that treats the columns of space-time diagrams of F as configurations. The following lemma demonstrates one way to make use of left expansivity: if a sufficiently wide column in a space-time diagram is eventually periodic, so are also columns to the left of it. This corresponds to the fact that the image of an eventually periodic configuration via a CA is also eventually periodic.

Lemma 3.2. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is left expansive with dimensions (h, d, w) and $x \in \Sigma^{\mathbb{Z}}$, $i \in \mathbb{Z}$ are such that $\mathrm{Tr}_{[i,i+w-1]}(x)$ is eventually p-periodic with preperiod c, then $\mathrm{Tr}_{[i-1,(i+w-1)-1]}(x)$ is eventually p-periodic with preperiod $c + h$.*

Proof. Let $R = R(h, d, w)$ and let θ be the space-time diagram of x with respect to F. Let $t \geq c + h$. By the assumption of eventual periodicity, for all $(r_1, r_2) \in R$ it holds that

$$\begin{aligned}\theta[(i, -t) + (r_1, r_2)] &= \mathrm{Tr}_{i+r_1}(x)[t - r_2] \\ &= \mathrm{Tr}_{i+r_1}(x)[t + p - r_2] = \theta[(i, -(t + p)) + (r_1, r_2)].\end{aligned}$$

Therefore $\theta|_{(i,-t)+R} = \theta|_{(i,-(t+p))+R}$ and it follows that

$$\mathrm{Tr}_{i-1}(x)[t] = \theta[(i - 1, -t)] = \theta[(i - 1, -(t + p)] = \mathrm{Tr}_{i-1}(x)[t + p].$$

Combining this with $\mathrm{Tr}_{[i,i+w-1]}(x)[t] = \mathrm{Tr}_{[i,i+w-1]}(x)[t + p]$ yields

$$\mathrm{Tr}_{[i-1,(i+w-1)-1]}(x)[t] = \mathrm{Tr}_{[i-1,(i+w-1)-1]}(x)[t + p].$$

Since $t \geq c + h$ is arbitrary, $\mathrm{Tr}_{[i-1,(i+w-1)-1]}(x)$ is eventually p-periodic with preperiod $c + h$. $\square$

Another main component for the definition of rapidly left expansive CA is the notion of a spreading speed. This is similar to a Lyapunov exponent [16,17], but depends only on the action of the CA on the elements of $\mathcal{L}_0(\Sigma)$.

Definition 3.3. A CA $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is *left spreading* (on $\mathcal{L}_0(\Sigma)$) if $F(0^{\mathbb{Z}}) = 0^{\mathbb{Z}}$ and for each $x \in \mathcal{L}_0(\Sigma)$ there is a $t \in \mathbb{Z}_+$ such that $\ell(F^t(x)) < \ell(x)$. The *spreading speed* of a left spreading F is

$$\sup_{x\in\mathcal{L}_0(\Sigma)} \limsup_{t\in\mathbb{Z}_+} \frac{\ell(x) - \ell(F^t(x))}{t} = \sup_{x\in\mathcal{L}_0(\Sigma)} \limsup_{t\in\mathbb{Z}_+} \frac{-\ell(F^t(x))}{t}.$$

Multiplying a positive real number by a fraction p/q repeatedly t times causes the base pq representation to lengthen by approximately $t \log_{pq}(p/q)$ symbols to the left. Due to this fractional multiplication automata are left spreading with spreading speed $\log_{pq}(p/q)$. An elementary CA is left spreading if and only if its local rule maps the triplet 001 to 1, and then its spreading speed is equal to 1.

We are now ready to present the main definition of this paper.

Definition 3.4. A CA $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is *rapidly left expansive* (with width $w \in \mathbb{Z}_+$) if

- F is left expansive with dimensions (h, d, w),
- F is left spreading with spreading speed s,
- the inequality $s < 1/h$ is satisfied.

By the above discussion, this class of automata contains in particular all the fractional multiplication automata (because then $1/h = 1 > \log_{pq}(p/q) = s$) and all left permutive left spreading CA (because $1/0 = \infty > s$) such as Rule 30.

The following theorem has been proved earlier in Proposition 3 of [7] for left permutive left spreading elementary CA and in Proposition 3.8 of [11] for fractional multiplication automata. We reprove it (with essentially the same proof) for general rapidly left expansive CA.

Theorem 3.5. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is rapidly left expansive with width $w \in \mathbb{Z}_+$ and $x \in \mathcal{L}_0(\Sigma)$, then $\mathrm{Tr}_{[i,i+w-1]}(x)$ is not eventually periodic for any $i \in \mathbb{Z}$.*

Proof. Assume that F is left expansive with dimensions (h, d, w) and left spreading with speed s so that $s < 1/h$. Assume to the contrary and without loss of generality (by replacing x with $\sigma^i(x)$ if necessary) that $\mathrm{Tr}_{[0,w-1]}(x)$ is eventually p-periodic with preperiod $c \in \mathbb{N}$. A simple induction based on Lemma 3.2 shows that $\mathrm{Tr}_{[-i,-i+w-1]}(x)$ (and in particular $\mathrm{Tr}_{-i}(x)$) is eventually p-periodic with preperiod $c + ih$ for all $i \in \mathbb{N}$. By periodicity we have for all $i \in \mathbb{N}$ the implication

$$\mathrm{Tr}_{-i}(x)[t] = 0 \text{ for } 0 \leq t < c + ih + p \implies \mathrm{Tr}_{-i}(x) = 0^{\mathbb{N}}.$$

Because of this, if there exists an $N \in \mathbb{N}$ such that the left hand side of this implication is satisfied for all $i \geq N$, then $F^t(x)[-i] = 0$ whenever $t, i \in \mathbb{N}$, $i \geq N$, contradicting the left spreading property. We outline in Fig. 3 a visual proof for the existence of such an N and present more details below.

For an $N \in \mathbb{N}$ as in the previous paragraph to exist it is sufficient that for all sufficiently large $i \in \mathbb{N}$ the inequality $-i < \ell(F^t(x))$ is satisfied for $t \in \mathbb{N}$ such that $0 \leq t < c + ih + p$. By the left spreading property

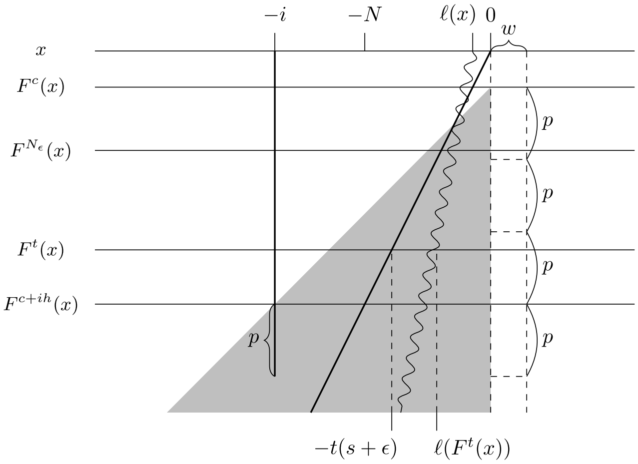

Fig. 3. The proof of Theorem 3.5. By left expansivity, all columns within the gray area are p-periodic. Find an $N \in \mathbb{N}$ such that, at all coordinates $-i$ to the left of $-N$, you can extend a line (the thick vertical line in the diagram) down to depth p within the gray area but which is fully above the wavy line following the left bound $\ell(F^t(x))$. To see that this is possible, construct a line (the thick line of slope $1/(s+\epsilon)$ in the diagram) that eventually stays above the wavy line but whose slope is greater than the slope h of the boundary of the gray area.

$$s \geq \limsup_{t \in \mathbb{Z}_+} \frac{-\ell(F^t(x))}{t}.$$

Thus for any $\epsilon > 0$ there is an $N_\epsilon \in \mathbb{N}$ such that for all $t \geq N_\epsilon$ it holds that

$$s + \epsilon \geq \frac{-\ell(F^t(x))}{t}$$

and $-t(s+\epsilon) \leq \ell(F^t(x))$. Since by assumption $s < 1/h$, we may fix ϵ so that $s+\epsilon < 1/h$. To conclude it is sufficient to show for all sufficiently large $i \in \mathbb{N}$ that $-i < -t(s+\epsilon)$ is satisfied for $N_\epsilon \leq t < c+ih+p$ and that $-i < \ell(F^t(x))$ is satisfied for $0 \leq t < N_\epsilon$. The latter condition is satisfied as long as i is sufficiently large and the former condition is satisfied when $(c+ih+p)(s+\epsilon) < i$. Dividing both sides by i results in

$$\frac{(c+p)(s+\epsilon)}{i} + (s+\epsilon)h < 1.$$

Letting i tend to infinity, we see that this inequality is satisfied for all sufficiently large i if $(s+\epsilon)h < 1$. This holds by our choice of ϵ. $\square$

The previously mentioned special cases can now be easily recovered.

Corollary 3.6. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is (m,n) left permutive and left spreading and $x \in \mathcal{L}_0(\Sigma)$, then $\mathrm{Tr}_{[i,i+m+n-1]}(x)$ is not eventually periodic for any $i \in \mathbb{Z}$.*

Corollary 3.7 *(Jen, [7], Proposition 3). If $F : \Sigma_2^{\mathbb{Z}} \to \Sigma_2^{\mathbb{Z}}$ is a left permutive and left spreading ECA and $x \in \mathcal{L}_0(\Sigma_2)$, then $\mathrm{Tr}_{[i,i+1]}(x)$ is not eventually periodic for any $i \in \mathbb{Z}$.*

Corollary 3.8 *(Kopra, [11], Proposition 3.8). If $\Pi_{p,q} : \Sigma_{pq}^{\mathbb{Z}} \to \Sigma_{pq}^{\mathbb{Z}}$ is the fractional multiplication automaton for coprime $p > q > 1$ and $x \in \mathcal{L}_0(\Sigma_{pq})$, then $\mathrm{Tr}_i(x)$ is not eventually periodic for any $i \in \mathbb{Z}$.*

Remark 3.9. The shift map $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is left expansive with dimensions $(h,d,w) = (1,0,1)$ and left spreading with spreading speed $s = 1$, but from $s = 1/h$ it follows that σ is not rapidly left expansive. The map σ does not satisfy the conclusion of Theorem 3.5, because $\mathrm{Tr}(x)$ is eventually periodic whenever $x \in \mathcal{L}_0(\Sigma)$ is eventually periodic.

To end this section with one more comment on rapidly left expansive CA, recall Wolfram's problem [18] from the introduction.

Problem 3.10. Let $x = \cdots 0001000 \cdots \in \Sigma_2^{\mathbb{Z}}$ be the configuration containing a single 1 at the origin. Is $\mathrm{Tr}_{W_{30},0}(x)$ eventually periodic?

It concerns the trace of width 1 of a single, very simple configuration (the case of trace of width 2 having been covered by Corollary 3.7), but it is probably equally difficult for all configurations of $\mathcal{L}_0(\Sigma_2)$. Ideally one could even solve this problem for some natural class of CA that contains Rule 30 and fractional multiplication automata (although Corollary 3.8 already covers the latter). Unfortunately the set of rapidly left expansive CA cannot be such a class, because it contains the additive ECA Rule 90 (with a $(1,1)$ local rule $f(abc) = a + c \pmod 2$) that produces a single eventually periodic column starting from the configuration with a single 1 at the origin.

4. Results

In this section we present two new results on rapidly left expansive cellular automata. The proofs of both of these results utilize Theorem 3.5 as their final step. The fractional part of a number $\xi \in \mathbb{R}$ is

$$\mathrm{frac}(\xi) = \xi - \lfloor \xi \rfloor \in [0,1).$$

Whenever $\xi > 0$ and $p > q > 1$, it is known that $\mathrm{frac}(\xi)$ can occur in the sequence $(\mathrm{frac}(\xi(p/q)^i))_{i\in\mathbb{N}}$ only finitely many times by Lemma 2.1 of [5]. Without loss of generality p and q are coprime, and then this is equivalent to the statement that $x[0,\infty]$ $(x \in \mathcal{L}_0(\Sigma_{pq}))$ can appear in the sequence $(\Pi^i_{p/q,pq}(x)[0,\infty])_{i\in\mathbb{N}}$ only finitely many times. We will now show that this result generalizes to the case where $\Pi_{p/q,pq}$ is replaced by an arbitrary rapidly left expansive cellular automaton. We first recall the Morse-Hedlund theorem.

Theorem 4.1 *(Morse and Hedlund, [13], Theorem 7.4). If $x \in \Sigma^{\mathbb{N}}$ and for some $n \in \mathbb{Z}_+$ at most n distinct words of length n occur in x, then x is eventually periodic.*

Lemma 4.2. *Let $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be left expansive with dimensions (h,d,w) and let $x \in \Sigma^{\mathbb{Z}}$. If $t, N \in \mathbb{Z}_+$ and $K = |\Sigma|^{tw}$ satisfy $(h+d)K + 1 \le tN$ and $F^{it}(x)[c,\infty] = x[c,\infty]$ for some $c \in \mathbb{Z}$ and all $0 \le i < N$, then x is eventually periodic.*

Proof. We may assume without loss of generality (by replacing x with $\sigma^c(x)$ if necessary) that $c = 0$. Assume that F is a radius-r CA and define

$$T_n = \{\mathrm{Tr}_{[j,j+w]}(x)[0,n-1] \mid j \ge r(t-1)\} \quad \text{for } n \in \mathbb{N},$$
$$U = \{F^{hK}(x)[j-K, j-1] \mid j \ge r(t-1)\}.$$

Observe that, since F has radius r, the map $\Sigma^{w+2r(t-1)} \to (\Sigma^w)^t$ defined by

$$F^i(x)[j - r(t-1), j + w - 1 + r(t-1)]$$
$$\mapsto \mathrm{Tr}_{[j,j+w-1]}(x)[i, i+t-1] \qquad \text{for } i \in \mathbb{N}, j \in \mathbb{Z}$$

is well defined. Let then $j \ge r(t-1)$: combining the previous observation with the fact that

$$F^{it}(x)[j - r(t-1), j + w - 1 + r(t-1)]$$
$$= x[j - r(t-1), j + w - 1 + r(t-1)] \qquad \text{for } 0 \le i < N$$

(which holds because $F^{it}(x)[0,\infty] = x[0,\infty]$ for $0 \le i < N$) implies that $\mathrm{Tr}_{[j,j+w-1]}(x)[it, (i+1)t-1] = \mathrm{Tr}_{[j,j+w-1]}(x)[0, t-1]$ for $0 \le i < N$. In other words, $\mathrm{Tr}_{[j,j+w-1]}(x)[0, tN-1]$ is a concatenation of N copies of $\mathrm{Tr}_{[j,j+w-1]}(x)[0,t-1]$ when $j \ge r(t-1)$. This together with the inequality $(h+d)K + 1 \le tN$ implies that

$$K \ge |T_t| = |T_{tN}| \ge \left|T_{(h+d)K+1}\right|.$$

By left expansivity the surjective mapping $T_{(h+d)K+1} \to U$ defined by

$$\mathrm{Tr}_{[j,j+w]}(x)[0,(h+d)K] \mapsto F^{hK}(x)[j-K, j-1]$$

is well defined, so U contains at most K words of length K. By Theorem 4.1 the sequence $F^{hK}(x)$ is eventually periodic. Sufficiently many applications of F transform $F^{hK}(x)$ to $F^{tN}(x)$ and $F^{tN}(x)[0,\infty] = x[0,\infty]$, so x is also eventually periodic. □

It turns out that the assumption of $x[0,\infty]$ repeating many times periodically in the sequence $(F^i(x)[0,\infty])_{i\in\mathbb{N}}$ can be significantly weakened. One special case of the following lemma is the simple observation that if $\mathrm{frac}(\xi(p/q)^i) = \mathrm{frac}(\xi)$ for some $i \in \mathbb{Z}_+$, then $\xi \in \mathbb{Q}$.

Lemma 4.3. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is left expansive, $x \in \Sigma^{\mathbb{Z}}$, and there exists a $t \in \mathbb{Z}_+$ such that $F^t(x)[0,\infty] = x[0,\infty]$, then x is eventually periodic.*

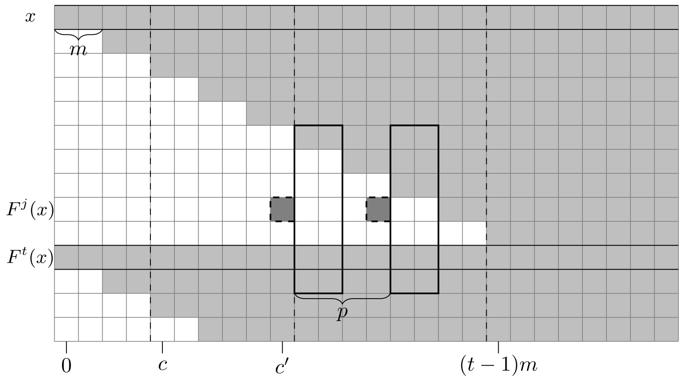

Fig. 4. A space-time diagram for the proof of Lemma 4.4. Initially p-periodicity is guaranteed within the light gray area. To show p-periodicity up to the horizontal coordinate c, slide two rectangles around the diagram so that they remain at horizontal distance p from each other. Identical contents within the rectangles imply identical contents within the dark gray squares. The starting point of the induction is at the horizontal coordinate $(t-1)m$ and proceeds leftwards.

Proof. This claim follows from Lemma 4.2 if we can show that for every $N \in \mathbb{Z}_+$ there is a $c \in \mathbb{N}$ such that $F^{it}(x)[c,\infty] = x[c,\infty]$ for all $0 \le i < N$. We show this by induction, so let $N \in \mathbb{Z}_+$ and $c \in \mathbb{N}$ be such that $F^{it}(x)[c,\infty] = x[c,\infty]$ holds for all $0 \le i < N$. Assuming that F^t has radius r, applying F^t to the configurations $F^{it}(x)$ yields $F^{(i+1)t}(x)[c+r,\infty] = F^t(x)[c+r,\infty]$ for all $0 \le i < N$, or equivalently $F^{it}(x)[c+r,\infty] = F^t(x)[c+r,\infty]$ for all $1 \le i < N+1$. It remains to show that $F^t(x)[c+r,\infty] = x[c+r,\infty]$, but this follows from the assumption $F^t(x)[0,\infty] = x[0,\infty]$. $\square$

Lemma 4.4. *Assume that $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ with memory m is left expansive with dimensions (e,e,w) (height and depth are equal) and let $c = m(e-1)$. Then for any $x \in \Sigma^{\mathbb{Z}}$ and $t \in \mathbb{N}$ with $x[0,\infty]$ and $F^t(x)[0,\infty]$ p-periodic, the sequence $F^i(x)[c,\infty]$ is p-periodic for all $0 \le i \le t$.*

Proof. Since $x[0,\infty]$ and $F^t(x)[0,\infty]$ are p-periodic and F has memory m, the sequences $F^i(x)[0,\infty]$ and $F^{t+i}(x)[0,\infty]$ are eventually p-periodic with preperiod im for $i \in \mathbb{N}$. In particular, for $0 \le i < e$, they are p-periodic with preperiod c. We may conclude the proof by using left expansivity as in Fig. 4.

More precisely, to prove the lemma assume to the contrary that there are $c' \ge c$ and $e \le j < t$ such that $F^j(x)[c',\infty]$ is not p-periodic. Furthermore, assume that the choice of c' is maximal, meaning that $F^i(x)[c'+1,\infty]$ is p-periodic for $0 \le i < t+e$. Let $R = R(e,e,w)$ and let θ be the space-time diagram of x with respect to F. Then $\theta|_{(c'+1,-j)+R} = \theta|_{(p,0)+(c'+1,-j)+R}$ and therefore $F^j(x)[c'] = \theta[(c',-j)] = \theta[(c'+p,-j)] = F^j(x)[c'+p]$. Thus $F^j(x)[c',\infty]$ is p-periodic, a contradiction. $\square$

Theorem 4.5. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is left expansive and $F^t(x)[0,\infty] = x[0,\infty]$ for infinitely many $t \in \mathbb{N}$, then for some $c \in \mathbb{N}$ the sequence $(F^t(x)[c,\infty])_{t\in\mathbb{N}}$ is periodic. Furthermore, if F is left expansive with height 0, then $(F^t(x))_{t\in\mathbb{N}}$ is periodic.*

Proof. By assumption there is a positive t such that $F^t(x)[0,\infty] = x[0,\infty]$ and thus by Lemma 4.3 we may assume (after replacing x with $\sigma^i(x)$ for a suitable $i \in \mathbb{N}$ if necessary) that $x[0,\infty]$ is periodic. Since there are infinitely many such t, by Lemma 4.4 there is a $c \in \mathbb{N}$ such that $F^i(x)[c,\infty]$ is periodic for all $i \in \mathbb{N}$.

Fix some $p \in \mathbb{Z}_+$ such that $F^p(x)[0,\infty] = x[0,\infty]$. For each $i \in \mathbb{N}$ let x_i be the unique periodic configuration that satisfies $x_i[c,\infty] = F^i(x)[c,\infty]$. Clearly $x_{i+1} = F(x_i)$ for each $i \in \mathbb{N}$, from which it follows that $F^p(x_0) = x_p = x_0$. This implies that $(F^t(x)[c,\infty])_{t\in\mathbb{N}}$ is p-periodic. In the special case when F is left expansive with height 0, an induction based on Lemma 3.2 shows that $(F^t(x)[c',\infty])_{t\in\mathbb{N}}$ is p-periodic also for all $c' < c$, meaning that $(F^t(x))_{t\in\mathbb{N}}$ is p-periodic. $\square$

Corollary 4.6. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is rapidly left expansive and $x \in \mathcal{L}_0(\Sigma)$, then $F^t(x)[0,\infty] = x[0,\infty]$ for finitely many $t \in \mathbb{N}$.*

Proof. Assume to the contrary that $F^t(x)[0,\infty] = x[0,\infty]$ for infinitely many $t \in \mathbb{N}$. By the previous theorem there exists a $c \in \mathbb{N}$ such that $\mathrm{Tr}_{[c,c+n]}(x)$ is periodic for all $n \in \mathbb{N}$. For a sufficiently large $n \in \mathbb{N}$ this contradicts Theorem 3.5. $\square$

We proceed to the second main result. It is a special case of Theorem 2 in [14] that, whenever $\xi > 0$ and $p > q > 1$, the sequence $(\mathrm{frac}(\xi(p/q)^i))_{i\in\mathbb{N}}$ has infinitely many limit points in the interval $[0,1]$. Again without loss of generality p and q are coprime, and then this is equivalent to the statement that for any $x \in \mathcal{L}_0(\Sigma_{pq})$ the sequence $(\Pi^i_{p/q,pq}(x)[0,\infty])_{i\in\mathbb{N}}$ has infinitely many limit points in $\Sigma_{pq}^{\mathbb{N}}$. This result also generalizes to the case where $\Pi_{p/q,pq}$ is replaced by any rapidly left expansive cellular automaton.

Theorem 4.7. *Let $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be a CA and let $x \in \Sigma^{\mathbb{Z}}$ be such that the sequence $(F^t(x)[0,\infty])_{t\in\mathbb{N}}$ has finitely many limit points in $\Sigma^{\mathbb{N}}$. Then for some $c \in \mathbb{N}$ the sequence $\mathrm{Tr}_{[c,c+n]}(x)$ is eventually periodic for all $n \in \mathbb{N}$.*

Proof. Assume that there are $k \in \mathbb{Z}_+$ different limit points and assume that F has radius $r \in \mathbb{N}$. It turns out that the choice $c = kr$ works. Let $n \in \mathbb{N}$ be arbitrary. For $0 \le i \le j$ let

$$W_{i,j} = \{v \in \Sigma^{j-i+1} \mid v = F^t(x)[i,j] \text{ for infinitely many } t \in \mathbb{N}\}, \quad f(i,j) = \left|W_{i,j}\right|.$$

Clearly $1 \le f(i,j) \le k$. This quantity is monotonous in the sense that if $i' \le i$ and $j \le j'$, then $f(i,j) \le f(i',j')$. It is possible to fix $i \le kr - r$ and $j \ge kr + n + r$ so that $f(i,j) = f(i+r, j-r)$, because otherwise

$$f(0, 2kr+n) > f(r, (2k-1)r+n) > f(2r, (2k-2)r+n) > \cdots > f(kr, kr+n) \ge 1$$

and $f(0, 2kr+n) \ge k+1$.

Since $i \le c$ and $j \ge c+n$, to show that $\mathrm{Tr}_{[c,c+n]}(x)$ is eventually periodic it is sufficient that $\mathrm{Tr}_{[i,j]}(x)$ is eventually periodic. We may assume without loss of generality (by replacing x with $F^t(x)$ for a sufficiently large $t \in \mathbb{N}$ if necessary) that $F^t(x)[i,j] \in W_{i,j}$ for all $t \in \mathbb{N}$. Let $p \in \mathbb{Z}_+$ be such that $F^p(x)[i,j] = x[i,j]$. We will prove by induction that $\mathrm{Tr}_{[i,j]}(x)$ is p-periodic, i.e. $F^{p+t}(x)[i,j] = F^t(x)[i,j]$ for all $t \in \mathbb{N}$. The base case $t = 0$ follows by the choice of p, so let $t \in \mathbb{N}$ be such that $F^{p+t}(x)[i,j] = F^t(x)[i,j]$. Since F has radius r, it follows that $F^{p+t+1}(x)[i+r, j-r] = F^{t+1}(x)[i+r, j-r]$. Since $f(i,j) = f(i+r, j-r)$, every word in $W_{i+r,j-r}$ has a unique extension in $W_{i,j}$, so it follows that $F^{p+t+1}(x)[i,j] = F^{t+1}(x)[i,j]$, which proves the induction step. □

Remark 4.8. It may be of interest to note that if Theorem 4.1 is viewed as a result concerning the dynamics of the map $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, then Theorem 4.7 can be interpreted as its generalization to other cellular automata. To recover Theorem 4.1 from Theorem 4.7, let $x \in \Sigma^{\mathbb{Z}}$ and for $i \in \mathbb{Z}_+$ let $p(i)$ be the number of distinct words of length i in $x[0,\infty]$. Let $n \in \mathbb{Z}_+$ be such that $p(n) = n$. If $p(1) = 1$, then x is eventually 1-periodic, so it remains to consider the case $p(1) > 1$. Then there is some $m \in \mathbb{Z}_+$ with $1 \le m < n$ such that $p(m) = p(m+1)$, so every word of length m occurring in $x[0,\infty]$ has a unique extension to the right. Necessarily also longer words occurring in $x[0,\infty]$ have unique extensions to the right, so $p(m) = p(m+1) = p(m+2) = \cdots$ and $(\sigma^t(x)[0,\infty])_{t\in\mathbb{N}}$ has at most $p(m)$ limit points. By Theorem 4.7 $\mathrm{Tr}_{\sigma,c}(x)$ is eventually periodic for some $c \in \mathbb{N}$, which is equivalent to x being eventually periodic.

Corollary 4.9. *If $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is rapidly left expansive and $x \in \mathcal{L}_0(\Sigma)$, then the sequence $(F^t(x)[0,\infty])_{t\in\mathbb{N}}$ has infinitely many limit points in $\Sigma^{\mathbb{N}}$.*

Proof. Assume to the contrary that $(F^t(x)[0,\infty])_{t\in\mathbb{N}}$ has finitely many limit points. By Theorem 4.7 there is a $c \in \mathbb{N}$ such that $\mathrm{Tr}_{[c,c+n]}(x)$ is eventually periodic for all $n \in \mathbb{N}$. For a sufficiently large n this contradicts Theorem 3.5. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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