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EXTREME OBSERVABLES AND OPTIMAL MEASUREMENTS IN QUANTUM THEORY

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[1]That is, much more than it does now.

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Quod neque sum cedro flavus nec pumice levis,
erubui domino cultior esse meo;
littera suffusas quod habet maculosa lituras,
laesit opus lacrimis ipse poeta suum.
Siqua videbuntur casu non dicta Latine,
in qua scribebat, barbara terra fuit.[2]
Dicite, lectores, si non grave, qua sit eundum,
quasque petam sedes hospes in urbe liber.

Publius Ovidius Naso, *Tristia* 3, 1, 13-20

[2] Auctor sua Latinitate insufficiente hunc librum lingua Anglica in barbara Finlandia scilicet scripsit.

Contents

Abstract

Optimization of quantum measurement processes has a pivotal role in carrying out better, more accurate or less disrupting, measurements and experiments on a quantum system. Especially, convex optimization, i.e., identifying the extreme points of the convex sets and subsets of quantum measuring devices plays an important part in quantum optimization since the typical figures of merit for measuring processes are affine functionals. In this thesis, we discuss results determining the extreme quantum devices and their relevance, e.g., in quantum-compatibility-related questions. Especially, we see that a compatible device pair where one device is extreme can be joined into a single apparatus essentially in a unique way. Moreover, we show that the question whether a pair of quantum observables can be measured jointly can often be formulated in a weaker form when some of the observables involved are extreme.

Another major line of research treated in this thesis deals with convex analysis of special restricted quantum device sets, covariance structures or, in particular, generalized imprimitivity systems. Some results on the structure of covariant observables and instruments are listed as well as results identifying the extreme points of covariance structures in quantum theory. As a special case study, not published anywhere before, we study the structure of Euclidean-covariant localization observables for spin-0-particles. We also discuss the general form of Weyl-covariant phase-space instruments.

Finally, certain optimality measures originating from convex geometry are introduced for quantum devices, namely, boundariness measuring how 'close' to the algebraic boundary of the device set a quantum apparatus is and the robustness of incompatibility quantifying the level of incompatibility for a quantum device pair by measuring the highest amount of noise the pair tolerates without becoming compatible. Boundariness is further associated to minimum-error discrimination of quantum devices, and robustness of incompatibility is shown to behave monotonically under certain compatibility-non-decreasing operations. Moreover, the value of robustness of incompatibility is given for a few special device pairs.

Tiivistelmä

Kvanttimittausten optimoinnilla on keskeinen rooli suoritettaessa parempia, tarkempia ja häiriöttömämpiä mittauksia ja kokeita kvanttisysteemeillä. Erityisesti konveksi optimointi eli kvanttimittausprosessien konveksien joukkojen ja osajoukkojen ääripisteiden määrääminen on erityisen tärkeä osa kvanttioptimointia, sillä kvanttiprosessien optimaalisuutta mitataan tyypillisesti affiineilla funktionaaleilla. Tässä väitöskirjassa keskitymme kvanttiteorian äärimittauksiin, niiden karakterisointiin ja merkitykseen esimerkiksi kvanttiyhteensopimattomuuden tutkimisessa. Erään tässä kirjasessa esitettävän tuloksen mukaan kaksi keskenään yhteensopivaa mittausprosessia, joista toinen on ääripiste, voidaan suorittaa yhdessä oleellisesti yksikäsitteisellä tavalla. Lisäksi näytämme, että kahden kvanttisuureen yhteismitattavuuden karakterisointia voidaan usein yksinkertaistaa, jos jokin tarkastelluista suureista on äärisuure.

Toinen tässä väitöskirjassa tarkasteltava tutkimusongelma on erityisten rajoitettujen kvanttisuure- ja instrumenttiosajoukkojen, niinsanottujen kovarianssirakenteiden tai yleistettyjen imprimitiivisysteemien konveksi analyysi. Esittelemme kovarianttien suureiden ja instrumenttien rakennetta ja kovarianssirakenteiden ääripisteiden karakterisointia koskevia tuloksia. Tutkailemme myös spin-0-objektin euklidikovarianttien lokalisointisuureiden rakennetta erityisenä aiemmin julkaisemattomana esimerkkinä kovarianssisysteemien teoriasta. Lisäksi määräämme Weyl-kovarianttien faasiavaruusinstrumenttien yleisen muodon.

Lopuksi tarkastelemme paria tapaa hyödyntää kvanttiteorian konveksien rakenteiden geometriaa mittausprosessien tutkimisessa. Ensiksi määrittelemme tavan mitata mittausprosessin etäisyyden konveksin kvanttirakenteen reunasta ja näytämme, että tämä etäisyysmitta on voimakkaasti kytköksissä kvanttimittausten minimivirhe-erotteluun. Toiseksi tutkimme, kuinka paljon hälyä mittausprosessipari sietää tulematta yhteensopivaksi, ja näytämme, että tämä hälynsieto käyttäytyy monotonisesti yhteensopivuutta vähentämättömissä prosesseissa. Lopuksi määritämme muutaman erityisen yhteensopimattoman mittausprosessiparin hälynsiedon.

List of papers

This thesis consists of a review of the subject and the following original research articles:

I **Extreme covariant quantum observables in the case of an Abelian symmetry group and a transitive value space**,
E. Haapasalo, and J.-P. Pellonpää,
J. Math. Phys. **52**, 122102 (2011).

II **Quantum measurements on finite dimensional systems: relabeling and mixing**,
E. Haapasalo, T. Heinosaari, and J.-P. Pellonpää,
Quantum Inf. Process. **11**, 1751-1763 (2012)

III **When do pieces determine the whole? Extreme marginals of a completely positive map**,
E. Haapasalo, T. Heinosaari, and J.-P. Pellonpää,
Rev. Math. Phys. **26**, 1450002 (2014)

IV **Distance to boundary and minimum-error discrimination**,
E. Haapasalo, M. Sedlák, and M. Ziman,
Phys. Rev. A **89**, 062303 (2014).

V **Compatibility properties of extreme quantum observables**,
E. Haapasalo, J.-P. Pellonpää, and R. Uola,
Lett. Math. Phys. **105**, 661-673 (2015)

VI **Robustness of incompatibility for quantum devices**,
E. Haapasalo,
J. Phys. A: Math. Theor. **48**, 255303 (2015)

VII **Covariant KSGNS construction and quantum instruments**,
E. Haapasalo, and J.-P. Pellonpää, [`arXiv:1506.01218`].

Chapter 1

Introduction

The quantum world is wider and deeper than the world of classical physics. The shape and geometry of the essential quantum structures exhibit peculiar features that are absent in the classical descriptions of physical systems. An especially hot topic in quantum theory is the study of the set of quantum states and the exotic properties that the states possess, particularly entanglement. Another non-classical feature of quantum structures is that quantum devices are typically not compatible. Within classical framework, measurements can be joined and they can be carried out without affecting the system under study. This does not apply in quantum physics: quantum measurement devices often cannot be combined into a single apparatus and a meaningful physical measurement always affects the system, i.e., measuring an observable is not compatible with the identity transformation of the system. Quantum incompatibility is behind many of the well-known oddities of quantum theory: the quantum mechanical quantities for position and momentum are a prime example of a pair of observables that cannot be measured jointly giving rise to the uncertainty relations of preparation and measurement.

To better understand these properties that make quantum theory so rich compared to the classical theories, it is of great importance to study the geometry of the relevant quantum structures, the sets of quantum states, observables, state transformations, and measurements. Especially, the convex optimization of these structures and their extreme points has an important role in finding those devices that exhibit the non-classical quantum peculiarities in their purest form. Another motivation for determining extreme quantum devices is that extreme points naturally maximize many figures of merit that quantify, e.g., the noiselessness, information gain, and other optimality criteria of the types of measuring schemes in our disposal. In addition to studying the extreme devices within the set of all devices, it is often physically motivated to study the geometry of restricted convex device subsets characterized by some extra conditions. On one hand, we might be able to carry out very particular types of measurements in the lab that exhibit certain regularity properties. On the other hand, we are often interested in very particular types of measurements and quantities, such as position, momentum, or energy measurements, which we require to posses special features, e.g., covariance properties.

We start by introducing the basic measurement devices, or rather their mathematical representations, in Chapter 2. We also discuss some of the general non-classical features of these devices, mainly the concept of incompatibility. This chapter does not contain much novel information but is rather just an invitation to the study of quantum structures fixing the basic notations that will be used throughout this treatise.

The convex extreme points of the global (i.e., non-restricted) quantum theoretic structures are determined in Chapter 3, and the importance of these extreme devices is further illustrated. Particularly, we discuss their relevance in quantum compatibility related questions. One of the main findings states roughly that a compatible device pair where one device is extreme can be joined into a single apparatus essentially in a single way or, in other words, a joint device with an extreme component is completely characterized by the extreme component and the remaining sub-device. Furthermore, connections between joint measurability and another compatibility property, coexistence, are studied in the presence of extreme observables. It is also emphasized through a couple of examples that an extreme observable needs not be sharp (i.e., projection-valued).

In Chapter 4, we concentrate on the structure and extreme points of the sets of particular restricted classes of quantum devices, covariance structures. The structure and extreme points of sets of covariant instruments and observables is in our main focus. We concentrate on a few particular cases: observables that are covariant with respect to an Abelian group with a value space that is a transitive space for the group and covariant observables the value space of which is a transitive space of their unimodular type-I symmetry group (with a compact stability subgroup). As an example of the latter case we study Euclidean-covariant position observables of an elementary spin-0 quantum object. We also give a structure theorem for covariant instruments which is valid when certain conditions on the observable marginals of such instruments are met. Particularly, this result holds in the case when the associated observables belong to either one of the two classes studied earlier in this chapter, but the theorem is valid in other exemplary cases as mentioned in Section 4.3.

Finally, in Chapter 5, we approach certain properties of the quantum structures using particular measures that arise directly from the convex geometry of these sets. We show that, in particular, a measure coined as boundariness is associated with minimum-error discrimination of quantum devices and we set up the robustness measure for quantum incompatibility that can be used to quantify how incompatible a given pair of quantum devices is.

Chapter 2

Quantum theoretic structures and their properties

In this treatise, we restrict our study to the standard Hilbert space quantum theory, for the basics of which we refer to [5, 7, 22, 39, 41, 54, 57, 78, 80]. Some of the results presented can be generalized to a wider context, though. However, to limit the technicality of this discussion we only briefly mention the possible generalization of each result.

2.0.1 Hilbert spaces and linear operators

Throughout this review, by simply Hilbert space we mean a complex and separable Hilbert space; it is separately noted, however, if the assumption on separability can be lifted. For any (possibly non-separable) Hilbert space $\mathscr{H}$, we denote the inner product associated with $\mathscr{H}$ by $\langle\cdot|\cdot\rangle_{\mathscr{H}}$, the norm $\mathscr{H}\ni\varphi\mapsto\sqrt{\langle\varphi|\varphi\rangle}\in\mathbb{R}$ by $\|\cdot\|_{\mathscr{H}}$, the von Neumann algebra of bounded linear operators on $\mathscr{H}$ by $\mathscr{L}(\mathscr{H})$, and the unit element, the identity operator, in $\mathscr{L}(\mathscr{H})$ by $\mathbb{1}_{\mathscr{H}}$; the subscripts are usually omitted if there is no risk of confusion. For two (not necessarily separable) Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $\mathscr{L}(\mathscr{H},\mathscr{K})$ stands for the set of bounded linear operators $A:\mathscr{H}\to\mathscr{K}$. The two-sided ideal of trace-class operators on $\mathscr{H}$ within $\mathscr{L}(\mathscr{H})$ is denoted by $\mathscr{T}(\mathscr{H})$ the trace functional on which is denoted by $\mathscr{T}(\mathscr{H})\ni T\mapsto\mathrm{tr}[T]\in\mathbb{C}$. The trace norm $\|\cdot\|_{\mathrm{tr}}:\mathscr{T}(\mathscr{H})\to\mathbb{R}$ is defined through $\|T\|_{\mathrm{tr}}=\mathrm{tr}[|T|]$ for any $T\in\mathscr{T}(\mathscr{H})$.

Let us make some additional definitions about linear operators on Hilbert spaces. The algebra $\mathscr{L}(\mathscr{H})$ contains the group $\mathscr{U}(\mathscr{H})$ of unitary operators on $\mathscr{H}$, i.e., $U^*=U^{-1}$ for all $U\in\mathscr{U}(\mathscr{H})$. Let $\mathscr{K}$ be another Hilbert space and $\eta\in\mathscr{H}$ and $\xi\in\mathscr{K}$. We occasionally encounter rank-1 operators $|\xi\rangle\langle\eta|:\mathscr{H}\to\mathscr{K}$ of the form $|\xi\rangle\langle\eta|\varphi=\langle\eta|\varphi\rangle\xi$ for all $\varphi\in\mathscr{H}$. Given two von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$, we denote their von Neumann tensor product (the σ-weak closure of their algebraic tensor product) simply by $\mathscr{A} \otimes \mathscr{B}$. Similarly, for two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $\mathscr{H} \otimes \mathscr{K}$ always stands for the Hilbert tensor product.

2.0.2 Measures, L^p-spaces, and direct integrals

Throughout this thesis, whenever we say that μ is a measure on a measurable space (Ω, Σ), where Ω is a non-empty set and Σ is a σ-algebra of subsets of Ω, what is meant is that μ is a positive scalar measure and in most cases μ is also σ-finite, although this is always separately mentioned. It should be pointed out, however, that we also deal with operator measures, but it is always clearly stated whether the measure in question is a scalar measure (which hereafter are by default positive as stated above) or an operator measure. When the basis set Ω is endowed with a topology, we denote the corresponding Borel σ-algebra by $\mathscr{B}(\Omega)$; in all the cases dealt with in this thesis the topology involved is clear from the context and is thus not indicated in the notation.

Given a measurable space (Ω, Σ) and a measure μ on (Ω, Σ), the associated L^p spaces, $1 \leq p < \infty$, are denoted $L^p(\mu)$, i.e., $f \in L^p(\mu)$ is a μ-equivalence class of functions $f' : \Omega \to \mathbb{C}$ such that $|f'|^p$ is μ-integrable. The space $L^\infty(\mu)$ is defined as usual and we typically consider the L^∞-spaces as von Neumann algebras with respect to the obvious algebraic operations. The normal misuse of notations apply, i.e., a member of an L^p-space is identified with a random representative of the equivalence class. When we also fix a Hilbert space $\mathscr{K}$, we define $L^2_{\mathscr{K}}(\mu)$ to be the Hilbert space of (equivalence classes of) functions $f : \Omega \to \mathscr{K}$ such that $\omega \mapsto \|f(\omega)\|^2$ is μ-integrable. Also the notation $L^2(\mu; \mathscr{K})$ is used when the subscript would look too bulky. We also use the notations $L^p(\Omega) := L^p(\mu)$, $L^2_{\mathscr{K}}(\Omega) = L^2(\Omega; \mathscr{K}) := L^2_{\mathscr{K}}(\mu)$ when the σ-algebra and the measure are clear from the context, e.g., $L^2(\mathbb{R}^n)$ is the space of (equivalence classes of) Lebesgue-square-integrable functions on $\mathbb{R}^n$.

The generalization of $L^2_{\mathscr{K}}(\mu)$ is a direct-integral Hilbert space $\int_\Omega^\oplus \mathscr{H}(\omega)\, d\mu(\omega)$ which, roughly speaking, is a space of (equivalence classes of) vector fields $\Omega \ni \omega \mapsto \varphi(\omega) \in \mathscr{H}(\omega)$ which are measurable with respect to fields of generating vectors and the associated function $\omega \mapsto \|\varphi(\omega)\|^2$ is μ-integrable. The inner product for φ, $\psi \in \int_\Omega^\oplus \mathscr{H}(\omega)\, d\mu(\omega)$ is given by

$$\langle \varphi | \psi \rangle = \int_\Omega \langle \varphi(\omega) | \psi(\omega) \rangle_{\mathscr{H}(\omega)}\, d\mu(\omega).$$

We encounter these spaces in Chapters 3 and 4. A special class of linear operators on a direct integral Hilbert space consists of the decomposable operators $A = \int_\Omega^\oplus A(\omega)\, d\mu(\omega)$, where $\Omega \ni \omega \mapsto A(\omega) \in \mathscr{L}\big(\mathscr{H}(\omega)\big)$ is a weakly μ-measurable field of operators such that $(A\varphi)(\omega) = A(\omega)\varphi(\omega)$ for almost all $\omega \in \Omega$ with respect to μ and all $\varphi \in \int_\Omega^\oplus \mathscr{H}(\omega)\, d\mu(\omega)$. Such a decomposable operator is bounded if and only if $\mu - \text{ess sup}_{\omega\in\Omega} \|A(\omega)\| < \infty$. For more on details on direct-integral spaces, see [23, Part II].

Let us finally fix some notations dealing with measure theory and functions. Suppose that (Ω, Σ) is a measurable space and μ is a positive scalar measure on (Ω, Σ). We generally shorten the recurrent phrase "for almost all $\omega \in \Omega$ with respect to μ" as "for μ-a.a. $\omega \in \Omega$". For any $X \in \Sigma$, we denote by $\chi_X : \Omega \to \{1, 0\}$ the *characteristic* or *indicator function of* X, i.e., $\chi_X(\omega) = 1$ if $\omega \in X$ and otherwise $\chi_X(\omega) = 0$. For scalar measures μ and ν, we denote $\nu \ll \mu$ when ν is absolutely continuous with respect to μ. The measure ν in this situation can also be an operator measure instead of a scalar measure.

2.0.3 Unitary group representations

Let us fix a group G and a Hilbert space $\mathscr{H}$. A unitary-operator-valued map $U : G \to \mathscr{U}(\mathscr{H})$ which is a group homomorphism, i.e., $U(gh) = U(g)U(h)$, $U(e) = \mathbb{1}_{\mathscr{H}}$, where e is the unit of G is called a *unitary representation*. The representation U is *irreducible* if the only closed subspaces $\mathscr{M} \subset \mathscr{H}$ that are invariant under U, i.e., $U(g)\varphi \in \mathscr{M}$ for all $g \in G$ and $\varphi \in \mathscr{M}$, are $\{0\}$ and $\mathscr{H}$. Usually, we assume that G is a topological group, i.e., endowed with a topology according to which the basic group operations $G \times G \ni (g, h) \mapsto gh \in G$ (multiplication) and $G \ni g \mapsto g^{-1} \in G$ (inversion) are continuous. We usually assume that topological groups are Hausdorff. In a typical physical situation, the group is, additionally, locally compact and second countable. Local compactness ensures the existence of left (right) Haar measures on G, i.e., Radon measures $\mu : \mathscr{B}(G) \to [0, \infty]$ such that $\mu(gX) = \mu(X)$ for all $g \in G$ and $X \in \mathscr{B}(G)$ (or $\mu(Xg) = \mu(X)$ for right Haar measures). Left Haar measures can differ from each other only up to a constant positive factor. The same applies to right Haar measures. The group is said to be *unimodular* if the left and right Haar measures coincide. The representation U is called *strongly continuous* when it is continuous with respect to the topology of G and the strong or, in this case equivalently, the weak operator topology of $\mathscr{L}(\mathscr{H})$.

For a locally compact group G, we may define the set $\hat{G}$ of unitary equivalence classes $\gamma = [\pi]$ of irreducible unitary representations of G. Here unitary equivalence of two (irreducible) representations $\pi : G \to \mathscr{U}(\mathscr{H})$ and $\rho : G \to \mathscr{U}(\mathscr{K})$ means that there is a unitary operator $W : \mathscr{H} \to \mathscr{K}$ such that $\rho(g)W = W\pi(g)$ for all $g \in G$. The set $\hat{G}$ is called as the *representation dual of* G and it is equipped with the Fell topology; see [25, Chapter 7] for details. If G is Abelian, all its irreducible representations are one-dimensional and $\hat{G}$ is a group consisting of group homomorphisms of G into the torus $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. With this identification, it is usually denoted $\gamma(g) = \langle g, \gamma \rangle$ for all $g \in G$ and $\gamma \in \hat{G}$. The Fell topology in this case coincides with the topology of compact convergence which makes $\hat{G}$ into a locally compact group as well.

In Chapter 4, we usually assume that the symmetry group G is locally compact and second countable and of *type I*. The latter property means that any primary representation of G is a direct sum of some irreducible representation. Primarity of a representation U means that the von Neumann algebra of operators commuting with $U(g)$ for all $g \in G$ consists only of scalar multiples of the identity operator. The reason we evoke this condition is that the strongly continuous unitary representations $U : G \to \mathscr{U}(\mathscr{H})$ of locally compact second countable type-I groups in separable Hilbert spaces $\mathscr{H}$ have a particular structure: for any such U, there is a (standard) measure $\mu : \mathscr{B}(\hat{G}) \to [0, \infty]$ on the Borel σ-algebra defined by the Fell topology and a measurable field $\hat{G} \ni \gamma \mapsto \mathscr{L}(\gamma)$ of separable Hilbert spaces such that $\mathscr{H}$ is unitarily equivalent with $\int_{\hat{G}}^{\oplus} \mathscr{K}(\gamma) \otimes \mathscr{L}(\gamma)\, d\mu(\gamma)$, where $\mathscr{K}(\gamma)$ is the representation space of some representative $\pi_\gamma \in \gamma$ and, when we identify $\mathscr{H}$ with this direct-integral space, one can write

$$\big(U(g)\varphi\big)(\gamma) = \big(\pi_\gamma(g) \otimes \mathbb{1}_{\mathscr{L}(\gamma)}\big)\varphi(\gamma)$$

for all $g \in G$, $\varphi \in \mathscr{H}$, and for μ-a.a. $\gamma \in \hat{G}$.

In the case of an Abelian locally compact second countable group G which is automatically of type I, the above decomposition result implies the SNAG-theorem: for any strongly continuous unitary representation $U : G \to \mathscr{U}(\mathscr{H})$ of G in a separable Hilbert space $\mathscr{H}$, there is a spectral measure $\mathsf{P} : \mathscr{B}(\hat{G}) \to \mathscr{L}(\mathscr{H})$, i.e., a map such that $\mathrm{tr}[T\mathsf{P}(\cdot)]$ is a probability measure for all positive $T \in \mathscr{T}(\mathscr{H})$ with $\mathrm{tr}[T] = 1$ and $\mathsf{P}(X)^2 = \mathsf{P}(X)$ for all $X \in \mathscr{B}(\hat{G})$, such that

$$U(g) = \int_{\hat{G}} \langle g, \gamma \rangle \, d\mathsf{P}(\gamma), \qquad g \in G.$$

Operator measures are discussed more in depth later in this chapter. Also the Peter-Weyl theorem for compact (consequently type-I) groups follows from the general decomposition result for type-I groups.

Whenever G is unimodular, locally compact, second countable, and Hausdorff and a Haar measure μ_G is fixed for G, the dual $\hat{G}$ possesses a unique measure $\mu_{\hat{G}} : \mathscr{B}(\hat{G}) \to [0, \infty]$ called as the *Plancherel measure* such that the map $(L^1 \cap L^2)(\mu_G) \ni f \mapsto \hat{f} \in \int_{\hat{G}}^{\oplus} \mathscr{K}_{\mathrm{HS}}(\gamma)\, d\mu_{\hat{G}}(\gamma)$,

$$\hat{f}(\gamma) = \int_G f(g)\pi_\gamma(g)\, d\mu_G(g), \qquad \gamma \in \hat{G},$$

where $\mathscr{K}_{\mathrm{HS}}(\gamma)$ is the Hilbert space of Hilbert-Schmidt operators on the representation space $\mathscr{K}(\gamma)$ of a representative $\pi_\gamma \in \gamma$, extends into a unitary map from $L^2(\mu_G)$ onto $\int_{\hat{G}}^{\oplus} \mathscr{K}_{\mathrm{HS}}(\gamma)\, d\mu_{\hat{G}}(\gamma)$. This unitary operator (defined by μ_G) is called as the *Fourier-Plancherel operator*. The Fourier-Plancherel operator is needed in the essential proofs of sections 4.2.1 and 4.2.2, and this is why the symmetry group is assumed to be unimodular in Section 4.2.2. If G is Abelian, the Plancherel measure is simply a Haar measure of the dual group $\hat{G}$.

Many of the results of Chapter 4 can be stated using projective unitary representations instead of ordinary unitary representations. When G is a group and $\mathscr{H}$ is a Hilbert space, a unitary-operator-valued map $U : G \to \mathscr{U}(\mathscr{H})$ is a *projective unitary representation* when the map, the *action*, $G \ni g \mapsto \beta_g \in \mathrm{Aut}\big(\mathscr{L}(\mathscr{H})\big)$, $\beta_g(A) = U(g)AU(g)^*$, is a group homomorphism from G into the group of automorphisms on $\mathscr{L}(\mathscr{H})$. It follows that for such a projective representation, there is a *multiplier* or *2-cocycle* $m : G \times G \mapsto \mathbb{T}$, i.e., $m(e,g) = m(g,e) = 1$ and $m(g,h)m(gh,k) = m(g,hk)m(h,k)$ for all g, h, $k \in G$ such that $U(gh) = m(g,h)U(g)U(h)$ for all g, $h \in G$. Irreducibility and strong continuity are defined for projective representations in the same way as for ordinary representations. Two strongly continuous projective representations U, $U' : G \to \mathscr{U}(\mathscr{H})$ with the associated multipliers m and m' are equivalent in the sense that they determine the same action $g \mapsto \beta_g$ if and only if there is a measurable function $a : G \to \mathbb{T}$ such that $U'(g) = a(g)U(g)$ and $m'(g,h) = a(gh)\overline{a(g)a(h)}m(g,h)$ for all g, $h \in G$. In particular, a projective representation is equivalent with an ordinary representation if and only if its multiplier is of the form $(g,h) \mapsto a(gh)\overline{a(g)a(h)}$ for some measurable $a : G \to \mathbb{T}$. In most cases, we may restrict to studying ordinary representations, since, given a projective unitary representation $U : G \to \mathscr{U}(\mathscr{H})$ associated with the multiplier m, we may define the group $\overline{G} := G \times \mathbb{T}$ with the group law $(g,s)(h,t) = \big(gh, stm(g,h)\big)$ for all (g,s), $(h,t) \in \overline{G}$ and the ordinary representation $\overline{U} : \overline{G} \to \mathscr{U}(\mathscr{H})$, $\overline{U}(g,s) = \overline{s}U(g)$ for all $(g,s) \in \overline{G}$ and view U as the subrepresentation of $\overline{U}$.

2.1 Quantum devices

Four relevant structures in quantum theory are the sets of *states, observables, channels,* and *instruments*. Below, we give the mathematical descriptions of these basic devices.

2.1.1 Quantum probabilities: states and effects

The state of a physical system is the full description of the observable properties of the system. In the standard Hilbert-space quantum theory, a state is represented by a trace-1 positive trace-class operator on the Hilbert space $\mathscr{H}$ associated with the system. Hence, the set of states is

$$\mathscr{S}(\mathscr{H}) = \{\rho \in \mathscr{T}(\mathscr{H}) \,|\, \rho \geq 0,\ \mathrm{tr}[\rho] = 1\}.$$

The topological dual of the Banach space (with respect to the trace norm) of trace-class operators $\mathscr{T}(\mathscr{H})$ is $\mathscr{L}(\mathscr{H})$. This means that any trace-norm continuous affine map f on $\mathscr{S}(\mathscr{H})$ has to be of the form $f(\rho) = \mathrm{tr}[\rho E]$, $\rho \in \mathscr{S}(\mathscr{H})$, for some $E \in \mathscr{L}(\mathscr{H})$. Simple properties of a quantum system can be seen as two-valued measurements or yes-no questions; either the system has the property (yes) or not (no). The system has the property with a certain probability depending on its state and the attribution of these probabilities to states $\rho \in \mathscr{S}(\mathscr{H})$ can be naturally required to be a continuous positive affine map on the state space bounded from above by 1. Hence the basic properties of the system can be associated to the *effect operators* $E \in \mathscr{E}(\mathscr{H}) \subset \mathscr{L}(\mathscr{H})$, i.e., $0 \leq E \leq \mathbb{1}_{\mathscr{H}}$ and the probability that the system is found to have this property in a measurement is $p_\rho^E = \mathrm{tr}[\rho E]$.

The above discussion means that the set of states and effects form a *statistical duality* $\langle \mathscr{S}(\mathscr{H}), \mathscr{E}(\mathscr{H}) \rangle$ where the biaffine map $\langle \cdot, \cdot \rangle : \mathscr{S}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}) \to [0,1]$, $(\rho, E) \mapsto \langle \rho, E \rangle = \mathrm{tr}[\rho E]$, associates probabilities of detecting basic properties when the system is in a particular state to each state-effect pair [54]. More complicated quantum measurements build upon this basic duality of the quantum theory.

2.1.2 State transformations: operations and channels

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. For any trace-norm-bounded linear map $\mathscr{F} : \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{K})$, one may define the dual $\mathscr{F}^* : \mathscr{L}(\mathscr{K}) \to \mathscr{L}(\mathscr{H})$ through $\mathrm{tr}[\mathscr{F}(T)B] = \mathrm{tr}[T\mathscr{F}^*(B)]$ for any $T \in \mathscr{T}(\mathscr{H})$ and $B \in \mathscr{L}(\mathscr{K})$. We say that the bounded linear map $\mathscr{F} : \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{K})$ is *completely positive* (CP) if the dual $\mathscr{F}^*$ is completely positive, i.e., for any $n = 1, 2, \ldots, \varphi_1, \ldots, \varphi_n \in \mathscr{H}$, and $B_1, \ldots, B_n \in \mathscr{L}(\mathscr{K})$

$$\sum_{j,k=1}^{n} \langle \varphi_j | \mathscr{F}^*(B_j^* B_k) \varphi_k \rangle \geq 0.$$

Such a completely positive map $\mathscr{F} : \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{K})$ is called an *operation* if it is trace non-increasing on positive operators, i.e., $\mathrm{tr}[\mathscr{F}(\rho)] \leq 1$ for all $\rho \in \mathscr{S}(\mathscr{H})$. Operations are needed in detailed descriptions of quantum measurements, as we will see shortly.

An operation $\mathscr{E} : \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{K})$ that is trace-preserving, i.e., $\mathrm{tr}[\mathscr{E}(\rho)] = 1$ for all $\rho \in \mathscr{S}(\mathscr{H})$, or equivalently unital, $\mathscr{E}^*(\mathbb{1}_{\mathscr{K}}) = \mathbb{1}_{\mathscr{H}}$, in the dual picture, is called a *channel*. We denote the set of channels $\mathscr{E} : \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{K})$ by $\mathbf{Ch}(\mathscr{H}, \mathscr{K})$. Note that a channel maps a quantum state to another quantum state. Channels are mathematical representations of transformations of quantum systems. Complete positivity of such a transformation means that if we join the input and output systems with any ancillary quantum system described by a (finite-dimensional) Hilbert space $\mathscr{M}$ and trivially extend the channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$ into a map $\overline{\mathscr{E}} : \mathscr{S}(\mathscr{H} \otimes \mathscr{M}) \to \mathscr{S}(\mathscr{K} \otimes \mathscr{M})$, $\overline{\mathscr{E}}(\rho \otimes \sigma) = \mathscr{E}(\rho) \otimes \sigma$ for all $\rho \in \mathscr{S}(\mathscr{H})$ and $\sigma \in \mathscr{S}(\mathscr{M})$, the extension $\overline{\mathscr{E}}$ is also a channel. Thus complete positivity can be seen as a minimal requirement for a state transformation to be extendable on larger systems.

2.1.3 Quantum measurements: observables and instruments

In a measurement of a quantum system, the probability of detecting an outcome from a subset X of the entire outcome set Ω depends on the initial state of the system being measured. From the statistical duality $\langle \mathscr{S}(\mathscr{H}), \mathscr{E}(\mathscr{H})\rangle$, it follows that the part M of the measurement that assigns these measurement outcome probability distributions p_ρ^{M} to the states $\rho \in \mathscr{S}(\mathscr{H})$ is a map that turns subsets X of outcomes into effects $\mathsf{M}(X) \in \mathscr{E}(\mathscr{H})$ so that $p_\rho^{\mathsf{M}}(X) = \mathrm{tr}[\rho \mathsf{M}(X)]$. This branch of the measurement, i.e., the map M, is called as an *observable*. For any $\rho \in \mathscr{S}(\mathscr{H})$ and any subset X of outcomes, the number $p_\rho^{\mathsf{M}}(X)$ is the probability for the event that any measurement of M yields a result in X when the system being measured is in the state ρ.

Let us express the above in a more formal manner. Let us pick a measurable space (Ω, Σ) that models the measurement outcome set of the quantum measurement, i.e., Ω is a non-empty set and Σ is a σ-algebra of subsets of Ω.The natural mathematical description for an (Ω, Σ)-valued observable on a system associated with the Hilbert space $\mathscr{H}$ is a *normalized positive-operator-valued measure* (POVM) [41], i.e., a map $\mathsf{M} : \Sigma \to \mathscr{E}(\mathscr{H})$ that is weakly σ-additive, i.e., $\mathsf{M}\big(\cup_j X_j\big) = \sum_j \mathsf{M}(X_j)$ with respect to the weak operator topology for any disjoint sequence $(X_j) \subset \Sigma$, for any $X \in \Sigma$ the operator $\mathsf{M}(X)$ is positive, and $\mathsf{M}(\Omega) = \mathbb{1}_{\mathscr{H}}$ such that $p_\rho^{\mathsf{M}}(X) = \mathrm{tr}[\rho \mathsf{M}(X)]$ for any $\rho \in \mathscr{S}(\mathscr{H})$ and $X \in \Sigma$. We denote the set of (Ω, Σ)-valued observables on a system described by $\mathscr{H}$ by $\mathbf{Obs}(\Sigma, \mathscr{H})$ and, for simplicity, we identify the observables of $\mathbf{Obs}(\Sigma, \mathscr{H})$ with the POVMs $\Sigma \to \mathscr{L}(\mathscr{H})$. We say that an observable $\mathsf{P} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ is *sharp*, if it is represented by a *projection-valued measure* (PVM), i.e., the range $\mathrm{ran}\, \mathsf{P} = \{\mathsf{P}(X) \,|\, X \in \Sigma\}$ consists of projections or, equivalently, $\mathsf{P}(X) = \mathsf{P}(X)^2$ for any $X \in \Sigma$.

An observable $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ is said to be *discrete*, if there is a countable set $\{\omega_j\}_j \subset \Omega$ such that all the singletons $\{\omega_j\}$ are Σ-measurable and M is supported by $\{\omega_j\}_j$ or, in other words, $\mathsf{M}(\{\omega_j\}_j) = \sum_j \mathsf{M}(\{\omega_j\}) = \mathbb{1}_{\mathscr{H}}$. For such a discrete observable, we usually define $\mathsf{M}_j := \mathsf{M}(\{\omega_j\})$ for all j.

In a measurement, the system typically transforms conditioned by registering an outcome $\omega \in X \in \Sigma$. The conditional state transformation is described by an operation $\Gamma(X)$ satisfying $\mathrm{tr}[[\Gamma(X)](\rho)] = p_\rho^{\mathsf{M}}(X)$ where M is the observable being measured in the measurement. If the probability $p_\rho^{\mathsf{M}}(X)$ is non-zero, the conditional state entering the post-measurement processes is $p_\rho^{\mathsf{M}}(X)^{-1}[\Gamma(X)](\rho)$ when the initial state is ρ.

Again, let us formalize the above discussion. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Let us denote by $\mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ the set of $\mathscr{H} \to \mathscr{K}$ -operation-valued maps Γ defined on Σ that are weakly σ-additive and $\Gamma(\Omega)$ is a channel. From now on, we denote $[\Gamma(X)](\rho) = \Gamma(X, \rho)$ for any $X \in \Sigma$ and $\rho \in \mathscr{S}(\mathscr{H})$. Weak σ-additivity here means that for any disjoint sequence $(X_j)_j \subset \Sigma$, any $\rho \in \mathscr{S}(\mathscr{H})$,

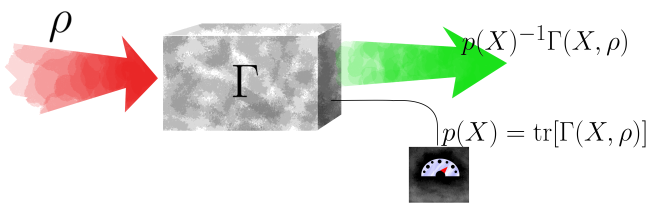

Figure 2.1: Illustration of an instrument. The state entering the measurement device represented by the instrument Γ is ρ. The instrument has the statistics arm (the lower branch right of the instrument box in the illustration) and the state change arm (the upper branch right of the instrument box). When a value is detected with certainty in the set X in the statistics arm, which happens with probability $p(X)$, the state change arm gives the conditional state $\rho_X = p(X)^{-1}\Gamma(X,\rho)$. When the state changes are neglected, the statistics arm reduces to the associated observable, and when the statistics are ignored, the state-change arm reduces to the channel associated with Γ unconditioned by registering an outcome.

and $B \in \mathscr{L}(\mathscr{K})$, one has $\mathrm{tr}[\Gamma(\cup_j X_j,\rho)B] = \sum_j \mathrm{tr}[\Gamma(X_j,\rho)B]$. We call elements $\Gamma \in \mathbf{Ins}(\Sigma,\mathscr{H},\mathscr{K})$ as *instruments*. Instrument is an effective mathematical description for a measurement process: An instrument $\Gamma \in \mathbf{Ins}(\Sigma,\mathscr{H},\mathscr{K})$ is a representative of the different measuring schemes for the associated observable $\mathsf{M}_\Gamma \in \mathbf{Obs}(\Sigma,\mathscr{H})$ such that $p_\rho^{\mathsf{M}_\Gamma}(X) = \mathrm{tr}[\Gamma(X,\rho)]$ for all $\rho \in \mathscr{S}(\mathscr{H})$ and $X \in \Sigma$. The conditional non-normalized state exiting the instrument when the input state is ρ conditioned on an outcome being measured within the set X is $\Gamma(X,\rho)$. The relevance of these conditional states is in the possibility of defining the bimeasures $(X,Y) \mapsto \mathrm{tr}[\Gamma(X,\rho)\mathsf{N}(Y)]$ for events where an observable $\mathsf{N} \in \mathbf{Obs}(\Sigma',\mathscr{K})$ with a possibly different value space (Ω',Σ') is measured after a measurement of M_Γ corresponding to Γ. The associated operator bimeasure $(X,Y) \mapsto [\Gamma(X)]^*\big(\mathsf{N}(Y)\big)$ extends into an operator measure on the minimal product value space $(\Omega\times\Omega',\Sigma\otimes\Sigma')$ when the measurable spaces involved are standard Borel [58], i.e., the sequential measurement can be used to measure an observable on $(\Omega\times\Omega',\Sigma\otimes\Sigma')$. More on quantum instruments and possible post-measurement states can be read in [67, 68].

2.1.4 Convexity in quantum structures

All the sets $\mathscr{S}(\mathscr{H})$, $\mathbf{Obs}(\Sigma,\mathscr{H})$, $\mathbf{Ch}(\mathscr{H},\mathscr{K})$, and $\mathbf{Ins}(\Sigma,\mathscr{H},\mathscr{K})$ are convex in a natural way. Especially for instruments Γ, $\Gamma' \in \mathbf{Ins}(\Sigma,\mathscr{H},\mathscr{K})$ and $t \in [0,1]$, we may define the convex combination

$$\big(t\Gamma + (1-t)\Gamma'\big)(X,\rho) = t\Gamma(X,\rho) + (1-t)\Gamma'(X,\rho)$$

for all $X \in \Sigma$ and $\rho \in \mathscr{S}(\mathscr{H})$. A convex combination of quantum devices reflects classical fuzziness: e.g., the convex combination $t\rho + (1-t)\sigma$ of states ρ and σ can be seen as a result of using a state preparator that combines two preparation schemes, one producing ρ firing with probability t and the other producing σ with probability $1-t$.

The extreme points of the sets of quantum devices are of interest because they do not exhibit this classical randomness, i.e., in a sense, they exhibit the important quantum properties in the purest possible way. The extreme points are important in optimization tasks since they extremalize affine functionals and figures of merit. We will also find out that the extreme devices possess certain objective optimality properties that clarify, e.g., their compatibility properties. The extreme states are exactly the one-dimensional projections $|\varphi\rangle\langle\varphi|$ for unit vectors $\varphi \in \mathscr{H}$. Thus extreme or *pure states* are also called as *vector states*. The sharp observables are all extreme observables, but there are others. We will give necessary and sufficient characterizations for extremality for all the relevant classes of quantum devices also in the presence of particular restrictions on the devices being considered, namely covariance.

The convex sets of quantum measuring devices differ foundationally from the corresponding structures of classical physics. Despite any quantum state $\rho \in \mathscr{S}(\mathscr{H})$ can be expressed as a convex combination $\rho = \sum_j t_j |\varphi_j\rangle\langle\varphi_j|$, $t_j \geq 0$, $\sum_j t_j = 1$, of pure states φ_j, e.g., as the combination given by the spectral decomposition of the state operator, one cannot consider ρ as a state ensemble $\{(\varphi_j, t_j)\}_j$ where the actual state of the system is the pure state $|\varphi_j\rangle\langle\varphi_j|$ with the probability t_j, since each non-pure state has infinitely many decompositions into pure states [11, 47]. This means that, unlike the set of states for a classical physical system (the set of probability measures on the Lebesgue σ-algebra on the phase space), the quantum state space $\mathscr{S}(\mathscr{H})$ is not a simplex and the ignorance interpretation of a mixed quantum state is not valid; see, especially, [7, sections II.2.5 and II.2.6]. The same non-classical feature applies also to all the other quantum devices.

2.2 Compatibility

Classical measurements and processes can be joined freely and implemented without transforming the system under study. On the quantum side, however, this no longer applies and tight restrictions on joining devices have to be introduced. Moreover, any physically meaningful measurement disturbs the system. These facts are related to the inherent feature of *incompatibility* within quantum theory that is absent in classical theories. In this section the different notions of compatibility for the relevant quantum devices are presented.

2.2.1 Joint measurability and coexistence

Let us fix the measurable spaces (Ω, Σ) and $(\overline{\Omega}, \overline{\Sigma})$. For simplicity, let us assume that these value spaces are standard Borel. We say that a map $\beta : \Sigma \times \overline{\Omega} \to \mathbb{R}$ is a *Markov kernel*, if $\beta(X, \cdot)$ is $\overline{\Sigma}$-measurable for all $X \in \Sigma$ and $\beta(\cdot, \omega)$ is a probability measure for all $\omega \in \overline{\Omega}$. Let us pick an observable $\mathsf{G} \in \mathbf{Obs}(\overline{\Sigma}, \mathscr{H})$, and a Markov kernel $\beta : \Sigma \times \overline{\Omega} \to \mathbb{R}$. We define the observable $\mathsf{G}^\beta \in \mathbf{Obs}(\Sigma, \mathscr{H})$ through

$$\mathsf{G}^\beta(X) = \int_{\Omega} \beta(X, \omega)\mathsf{G}(d\omega), \qquad X \in \Sigma.$$

We call such a G^β as the *post-processing of* G *by* β. One way to effectively measure the post-processing G^β is to measure first G and then process the output data p_ρ^{G} of G with the 'classical channel' presented by the Markov kernel β. For more on post-processing in the case of a discrete value space, we refer to [62], and for further issues, also in the case including value spaces that are not standard Borel, one may consult [50, 51].

Post-processing defines a preorder among observables in $\mathbf{Obs}(\Sigma, \mathscr{H})$ with the fixed Hilbert space $\mathscr{H}$ but varying (standard Borel) value spaces (Ω, Σ). Namely, we denote $\mathsf{N} \leq_{\mathrm{post}} \mathsf{M}$ when there is a Markov kernel β such that $\mathsf{N} = \mathsf{M}^\beta$. The observables in the maximal equivalence classes with respect to this preorder are called as *post-processing maximal* and, in the discrete-outcome settings, these observables are those $\mathsf{M} = (\mathsf{M}_j)_j$ with rank-1 effects [3, 62], i.e., $\mathsf{M}_j = |\varphi_j\rangle\langle\varphi_j|$ with some vectors $\varphi_j \in \mathscr{H}$ for all j.

Let us fix two standard Borel spaces (Ω, Σ) and (Ω', Σ'). We say that the observables $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ and $\mathsf{N} \in \mathbf{Obs}(\Sigma', \mathscr{H})$ are *jointly measurable* if there is a third value space $(\overline{\Omega}, \overline{\Sigma})$ and an observable $\mathsf{G} \in \mathbf{Obs}(\overline{\Sigma}, \mathscr{H})$ such that M and N can be obtained from G through statistical means, i.e., there are Markov kernels β and γ such that $\mathsf{M} = \mathsf{G}^\beta$ and $\mathsf{N} = \mathsf{G}^\gamma$. The standard Borel property ensures that, when M and N are jointly measurable, there is an observable $\mathsf{G} \in \mathbf{Obs}(\Sigma \otimes \Sigma', \mathscr{H})$ such that

$$\mathsf{M}(X) = \mathsf{G}(X \times \Omega'), \quad \mathsf{N}(Y) = \mathsf{G}(\Omega \times Y), \qquad X \in \Sigma, \quad Y \in \Sigma',$$

otherwise, we would have to deal with operator bimeasures [58, 82]. Hence, we may take the latter simple characterization as the definition of joint measurability. An observable $\mathsf{G} \in \mathbf{Obs}(\Sigma \otimes \Sigma', \mathscr{H})$ like that above is called as a *joint observable for* M *and* N. Using dilation theory, it is easy to see that any observable (with standard Borel value space) is jointly measurable with itself.

If observables are not jointly measurable, they are *incompatible*. Incompatible observables cannot be measured with a single measurement apparatus simultaneously. Incompatibility of observables is operationally linked to steering of quantum states [72, 77], which goes to show that incompatibility is a similar non-classical resource as entanglement for quantum technologies.

A typical example of an incompatible pair of observables is the position-momentum pair (of a non-relativistic object confined to a line) where $(\Omega, \Sigma) = (\Omega', \Sigma') = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$, $\mathscr{H} = L^2(\mathbb{R})$, and $\mathsf{M} = \mathsf{Q}$ and $\mathsf{N} = \mathsf{P} = \mathscr{F}^*\mathsf{Q}(\cdot)\mathscr{F}$ where $\mathsf{Q}(X)\varphi = \chi_X\varphi$ for all $X \in \mathscr{B}(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, and $\mathscr{F}$ is the Fourier-Plancherel operator.

It has been shown [37] that a sharp observable $\mathsf{P} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ is compatible with $\mathsf{M} \in \mathbf{Obs}(\Sigma', \mathscr{H})$ if and only if P and M commute, i.e., $\mathsf{P}(X)\mathsf{M}(Y) = \mathsf{M}(Y)\mathsf{P}(X)$ for all $X \in \Sigma$ and $Y \in \Sigma'$, in which case P and M have a unique joint observable G defined by

$$\mathsf{G}(X \times Y) = \mathsf{P}(X)\mathsf{M}(Y), \qquad X \in \Sigma, \quad Y \in \Sigma'.$$

Also other definitions of compatibility for observables have been suggested in addition to joint measurability. One such is *coexistence* [55, 56]. Given a set $\mathfrak{E} \subset \mathscr{E}(\mathscr{H})$ of effects, one can naturally ask if there is a value space $(\overline{\Omega}, \overline{\Sigma})$ and an observable $\mathsf{G} \in \mathbf{Obs}(\overline{\Sigma}, \mathscr{H})$ such that $\mathfrak{E} \subset \operatorname{ran}\mathsf{G} = \{\mathsf{G}(Z) \mid Z \in \overline{\Sigma}\}$. Motivated by this general question, observables $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ and $\mathsf{N} \in \mathbf{Obs}(\Sigma', \mathscr{H})$ are defined to be *coexistent* if there is $\mathsf{G} \in \mathbf{Obs}(\overline{\Sigma}, \mathscr{H})$ such that $\operatorname{ran}\mathsf{M} \cup \operatorname{ran}\mathsf{N} \subset \operatorname{ran}\mathsf{G}$. An observable like G above is called as a *mother observable for* M *and* N. It follows that when observables are jointly measurable they are coexistent, but the converse does not hold [73]. However, if either one of the observables is sharp, this implication can be reversed, and we will see that in the case of more general extreme observables similar results hold.

2.2.2 Compatible observables and channels

Channels $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K}_1)$ and $\mathscr{F} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K}_2)$ are compatible if there is a third channel $\mathscr{G} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K}_1 \otimes \mathscr{K}_2)$ such that

$$\mathscr{E}(\rho) = \operatorname{tr}_{\mathscr{K}_2}[\mathscr{G}(\rho)], \quad \mathscr{F}(\rho) = \operatorname{tr}_{\mathscr{K}_1}[\mathscr{G}(\rho)]$$

for all $\rho \in \mathscr{S}(\mathscr{H})$ or, equivalently

$$\mathscr{E}^*(A) = \mathscr{G}^*(A \otimes \mathbb{1}_{\mathscr{K}_2}), \quad \mathscr{F}^*(B) = \mathscr{G}^*(\mathbb{1}_{\mathscr{K}_1} \otimes B)$$

for all $A \in \mathscr{L}(\mathscr{K}_1)$ and $B \in \mathscr{L}(\mathscr{K}_2)$. The compatibility of channels means thus that there is a channel from which the compatible channels can be obtained as reduced dynamics. The mother channel like $\mathscr{G}$ above is a *joint channel for* $\mathscr{E}$ *and* $\mathscr{F}$. If channels are not compatible, they are *incompatible*.

We say that an observable $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ and a channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$ are compatible if there is an instrument $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ such that

$$\operatorname{tr}[\rho\mathsf{M}(X)] = \operatorname{tr}[\Gamma(X, \rho)], \quad \mathscr{E}(\rho) = \Gamma(\Omega, \rho)$$

for all $\rho \in \mathscr{S}(\mathscr{H})$ and $X \in \Sigma$. The first condition can also be written in the form $\mathsf{M}(X) = \Gamma(X)^*(\mathbb{1}_{\mathscr{K}})$. Such an instrument is called as a *joint instrument for* M *and* $\mathscr{E}$. Again, if an observable and a channel are not compatible, they are *incompatible*. Compatibility of an observable and a channel means that there is a measuring process described by a joint instrument whose measurement statistics are given by the observable and whose unconditioned state change is the channel. It turns out that, in order to be compatible with the identity channel, an observable has to be trivial in the sense that $\mathsf{M}(X) = p(X)\mathbb{1}_{\mathscr{H}}$ for all $X \in \Sigma$ with some probability measure p. Moreover, it is known [63] that a sharp observable $\mathsf{P} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ and a channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$ are compatible if and only if they commute, i.e., $\mathsf{P}(X)\mathscr{E}^*(B) = \mathscr{E}^*(B)\mathsf{P}(X)$ for all $X \in \Sigma$ and $B \in \mathscr{L}(\mathscr{K})$, in which case P and $\mathscr{E}$ have a unique joint instrument $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ given by

$$\Gamma(X, \rho) = \mathscr{E}\big(\rho\mathsf{P}(X)\big) = \mathscr{E}\big(\mathsf{P}(X)\rho\big), \qquad X \in \Sigma, \quad \rho \in \mathscr{S}(\mathscr{H}).$$

Note that when the observable $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ and the channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$ are compatible and $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ is a joint channel for M and $\mathscr{E}$, for any $X \in \Sigma$, there is a channel $\mathscr{F}_X \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$ such that

$$\Gamma(X, \rho) = \mathscr{F}_X\big(\sqrt{\mathsf{M}(X)}\rho\sqrt{\mathsf{M}(X)}\big)$$

for all $\rho \in \mathscr{S}(\mathscr{H})$ [68] showing that the compatible pair of a sharp observable P and a channel $\mathscr{E}$ is a special case of this general result with $\mathscr{F}_X = \mathscr{E}$ for all $X \in \Sigma$.

Joint measurability of observables and the notion of compatibility between observables and channels are related, since they are reflections of a more general definition of compatibility of completely positive maps, as is explained later in Section 3.2. Note, however, that coexistence of observables is not directly related to these compatibility properties. No general necessary and sufficient conditions for quantum compatibility are known for device pairs where one of the devices is not a sharp observable. Compatibility can, however be detected with a semidefinite program [81]. Moreover, in the case of discrete measurements, the channels compatible with a fixed discrete observable can be characterized with a particular observable-dependent maximal observable yielding interesting results, e.g., in understanding the interplay of information gain and perturbativeness of a measurement [33]. One can also define a stronger incompatibility criterion, strong incompatibility, which requires that the two quantum devices cannot be obtained from a single device (instrument) even with intermediate modifications of the outputs of the joint device [35]. It is immediate that for observables strong and ordinary incompatibility are the same, but for channels situation is more complicated.

Chapter 3

Extremality

In this chapter, we present a complete characterization for extreme quantum devices and discuss the consequences of extremality in different problem settings within quantum theory. Let us briefly recall what extremality means. Let V be a real or complex vector space and $K \subset V$ a convex set. An element $z \in K$ is an extreme point of K, or $z \in \operatorname{ext} K$, if from $z = tx + (1-t)y$, where $x, y \in K$ and $t \in (0,1)$, it follows that $x = y = z$, i.e., z has no genuine convex decompositions within K. We often use the term 'mixing' for forming convex combinations out of a (finite) family of elements of a convex set. Thus extreme points cannot be obtained by mixing several elements of the convex set and, from general statistical physics point of view, this means that extreme measuring devices are free from randomness caused by mixing of different measurement selections. Theorem 4 and all theorems after that are original results whose proofs can be found from the articles included in this thesis.

3.1 General characterization for extreme quantum devices

In order to give an exhaustive characterization for extreme quantum devices, we present the relevant devices in a unified way. The following discussion is amenable to generalizations to accommodate for sesquilinear-form-valued maps on C^*-modules instead of Hilbert spaces [28, 69], but we restrict generality for the sake of brevity. We need some operator-algebraic tools: Some of the results are applicable also for C^*-algebras such as the existence of a minimal dilation for a CP map defined on a C^*-algebra, but for simplicity we restrict our attention on von Neumann algebras.

Let us fix a von Neumann algebra $\mathscr{A}$ whose unit we denote by $1_{\mathscr{A}}$ and a Hilbert space $\mathscr{H}$. We study linear maps $\mathscr{A} \to \mathscr{L}(\mathscr{H})$. We say that such a linear map Φ is *completely positive* (CP) if for any $n = 1, 2, \ldots, a_1, \ldots, a_n \in \mathscr{A}$,

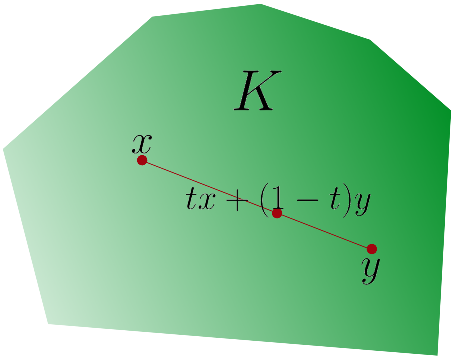

Figure 3.1: Within a convex set K of devices, a convex decomposition $z = tx + (1-t)y$ reflects the fact that z can be realized by statistically mixing the measurement procedures represented by x and y. The extreme devices, the edge points of the contour of K in this picture, cannot be considered as such non-trivial statistical mixtures meaning that these devices are free from added noise caused by mixing of different processes.

and $\varphi_1, \ldots, \varphi_n \in \mathscr{H}$

$$\sum_{j,k=1}^{n} \langle \varphi_j | \Phi(a_j^* a_k) \varphi_k \rangle \geq 0.$$

In addition the map Φ is *normal* if it is continuous with respect to the ultraweak operator topologies of $\mathscr{A}$ and $\mathscr{L}(\mathscr{H})$ or, equivalently, for any increasing net $(a_\lambda)_\lambda \subset \mathscr{A}$ of selfadjoint elements one has $\Phi\big(\sup_\lambda a_\lambda\big) = \sup_\lambda \Phi(a_\lambda)$.

Given a CP map $\Phi : \mathscr{A} \to \mathscr{L}(\mathscr{H})$, we say that a triple $(\mathscr{M}, \pi, J)$ consisting of a Hilbert space $\mathscr{M}$, a unital *-representation $\pi : \mathscr{A} \to \mathscr{L}(\mathscr{M})$ (i.e., $\pi(1_{\mathscr{A}}) = \mathbb{1}_{\mathscr{M}}$), and a linear map $J : \mathscr{H} \to \mathscr{M}$ is a *Stinespring dilation for* Φ if

$$\Phi(a) = J^* \pi(a) J, \qquad a \in \mathscr{A}.$$

The dilation $(\mathscr{M}, \pi, J)$ is called *minimal* if the vectors $\pi(a)J\varphi$, $a \in \mathscr{A}$, $\varphi \in \mathscr{M}$, span a dense linear subspace of $\mathscr{M}$. There exists a minimal Stinespring dilation for any CP map Φ and any two minimal dilations $(\mathscr{M}_1, \pi_1, J_1)$ and $(\mathscr{M}_2, \pi_2, J_2)$ for the same CP map Φ are unitarily equivalent in the sense that there is a unitary operator $U : \mathscr{M}_1 \to \mathscr{M}_2$ such that $U\pi_1(a) = \pi_2(a)U$ for all $a \in \mathscr{A}$ and $J_2 = UJ_1$ [75]. Moreover, if the map Φ is normal, so is the *-representation π in any of the minimal Stinespring dilations of Φ [22].

We define $\mathbf{CP}(\mathscr{A}; \mathscr{H})$ as the set of unital CP maps $\Phi : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ and denote the subset of normal maps within $\mathbf{CP}(\mathscr{A}; \mathscr{H})$ by $\mathbf{NCP}(\mathscr{A}; \mathscr{H})$. These sets are naturally convex and one easily sees that $\mathbf{NCP}(\mathscr{A};\mathscr{H})$ is a face of $\mathbf{CP}(\mathscr{A};\mathscr{H})$, i.e., whenever $t\Phi_1 + (1-t)\Phi_2 \in \mathbf{NCP}(\mathscr{A};\mathscr{H})$ with Φ_1, $\Phi_2 \in \mathbf{CP}(\mathscr{A};\mathscr{H})$ and $0 \leq t \leq 1$, then, in fact, Φ_1, $\Phi_2 \in \mathbf{NCP}(\mathscr{A};\mathscr{H})$. The latter property guarantees that the following extremality characterization proved in [1, 31] also holds for $\mathbf{NCP}(\mathscr{A};\mathscr{H})$. In the following theorem, the Hilbert space $\mathscr{H}$ need not be separable

Theorem 1. *Let $(\mathscr{M},\pi,J)$ be a minimal Stinespring dilation for a unital CP map $\Phi \in \mathbf{CP}(\mathscr{A};\mathscr{H})$. One has $\Phi \in \operatorname{ext}\mathbf{CP}(\mathscr{A};\mathscr{H})$ if and only if the map $\pi(\mathscr{A})' \ni D \mapsto J^*DJ \in \mathscr{L}(\mathscr{H})$ is an injection. Here $\pi(\mathscr{A})'$ denotes the commutant of the range of π.*

Using Theorem 1, one can give extremality characterizations for all quantum devices since all quantum structures can be expressed in the form $\mathbf{NCP}(\mathscr{A};\mathscr{H})$. For the state space $\mathscr{S}(\mathscr{K})$, $\mathscr{A} = \mathscr{L}(\mathscr{K})$ and $\mathscr{H} = \mathbb{C}$. For the set $\mathbf{Obs}(\Sigma,\mathscr{H})$ of observables, we may define for any σ-finite measure μ on (Ω,Σ) the restricted convex set $\mathbf{Obs}_\mu$ of those $\mathsf{M} \in \mathbf{Obs}(\Sigma,\mathscr{H})$ such that $p_\rho^{\mathsf{M}} \ll \mu$ for all $\rho \in \mathscr{S}(\mathscr{H})$ and, in the case of separable $\mathscr{H}$, $\mathbf{Obs}(\Sigma,\mathscr{H}) = \cup_\mu \mathbf{Obs}_\mu$, where the union runs over all σ-finite measures. One has $\mathbf{Obs}_\mu \simeq \mathbf{NCP}\big(L^\infty(\mu);\mathscr{H}\big)$, where a POVM $\mathsf{M} : \Sigma \to \mathscr{L}(\mathscr{H})$ is identified with the normal unital map $L^\infty(\mu) \ni f \mapsto \int_\Omega f\, d\mathsf{M} \in \mathscr{L}(\mathscr{H})$. Since each of the elements in the union are faces of $\mathbf{Obs}(\Sigma,\mathscr{H})$, Theorem 1 gives a characterization for extreme observables utilizing the minimal Naĭmark dilation of a POVM. Also the set $\mathbf{Ins}(\Sigma,\mathscr{H},\mathscr{K})$ consists of the faces

$$\mathbf{Ins}_\mu \simeq \mathbf{NCP}\big(L^\infty(\mu) \otimes \mathscr{L}(\mathscr{K});\mathscr{H}\big),$$

where μ runs over the set of σ-finite measures, that exhaust the total instrument set. Indeed, if $\operatorname{tr}[\Gamma(\cdot,\rho)] \ll \mu$ for all $\rho \in \mathscr{S}(\mathscr{H})$ (such a σ-finite measure existing for any instrument when $\mathscr{H}$ and $\mathscr{K}$ are separable), then we may identify Γ with a map $G \in \mathbf{NCP}\big(L^\infty(\mu) \otimes \mathscr{L}(\mathscr{K})\big)$ through

$$\operatorname{tr}[\rho G(f \otimes B)] = \int_\Omega f(\omega)\operatorname{tr}[\Gamma(d\omega,\rho)B] \tag{3.1}$$

for all $\rho \in \mathscr{S}(\mathscr{H})$, $f \in L^\infty(\mu)$, and $B \in \mathscr{L}(\mathscr{K})$. The extremality characterization for instruments and observables is given in Section 3.1.1.

A minimal dilation of a channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H},\mathscr{K})$ consists simply of a Hilbert space $\mathscr{M}_0$ and an isometry $J : \mathscr{H} \to \mathscr{K} \otimes \mathscr{M}_0$ such that $\mathscr{E}^*(A) = J^*(A \otimes \mathbb{1}_{\mathscr{M}_0})J$ for all $A \in \mathscr{L}(\mathscr{K})$ and the linear span of vectors $(A \otimes \mathbb{1}_{\mathscr{M}_0})J\varphi$, $A \in \mathscr{L}(\mathscr{K})$, $\varphi \in \mathscr{H}$, is dense in $\mathscr{K} \otimes \mathscr{M}_0$. When both $\mathscr{H}$ and $\mathscr{K}$ are separable, then $\mathscr{M}_0$ is separable, and we may pick an orthonormal basis $\{e_j\}_j$ of $\mathscr{M}_0$. With the operators $V_j : \mathscr{K} \to \mathscr{K} \otimes \mathscr{M}_0$, $V_j\psi = \psi \otimes e_j$, $\psi \in \mathscr{K}$, we may define the operators $K_j = V_j^* J$ such that

$$\mathscr{E}(\rho) = \sum_j K_j \rho K_j^*, \qquad \rho \in \mathscr{S}(\mathscr{H}),$$

i.e., the operators K_j are *Kraus operators of* $\mathscr{E}$. Moreover, the Kraus operators obtained as above from a minimal Stinespring dilation of a channel constitute a minimal set of Kraus operators for $\mathscr{E}$. Let now $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$ be associated with the minimal dilation $(\mathscr{M}_0, J)$ and the related minimal set $\{K_j\}_j$ of Kraus operators. It follows that $\mathscr{E} \in \mathrm{ext}\,\mathbf{Ch}(\mathscr{H}, \mathscr{K})$ if and only if from $J^*(\mathbb{1}_{\mathscr{K}} \otimes D)J = 0$ with $D \in \mathscr{L}(\mathscr{M}_0)$ it follows that $D = 0$ or, equivalently, the set $\{K_j^* K_k\}_{j,k}$ is *strongly independent*. This result and the definition of strong independence can be found, e.g., in [76]. Since strong independence is a generalization of linear independence of a finite set, this result generalizes the extremality characterization for channels between finite-dimensional systems presented in [19].

3.1.1 Extreme observables and instruments

In this section, we fix two separable Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ and any measurable space (Ω, Σ); note that the measurable space does not have to be countably generated. Let $\{e_j\}_j$ (respectively $\{f_m\}_m$) be an orthonormal basis for $\mathscr{H}$ (respectively for $\mathscr{K}$). When we consider a minimal dilation of an instrument $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$, it is immediate that one may assume the dilation space to be of the form $\mathscr{K} \otimes \mathscr{M}_0$ and the *-representation to be the identity map on $\mathscr{L}(\mathscr{K})$ tensored with a *-representation of $L^\infty(\mu)$ into $\mathscr{L}(\mathscr{M}_0)$ with a suitable σ-finite measure μ on (Ω, Σ) arising from a spectral measure. We can, however, say a little more. For the next result obtained in [67], whenever (Ω, Σ, μ) is a σ-finite measure space and $\mathscr{H}_\oplus = \int_\Omega^\oplus \mathscr{H}(\omega)\, d\mu(\omega)$, is a direct-integral Hilbert space, we define the *-representation $L^\infty(\mu) \ni f \mapsto M_f \in \mathscr{L}(\mathscr{H}_\oplus)$ such that $(M_f \eta)(\omega) = f(\omega)\eta(\omega)$ for all $f \in L^\infty(\mu)$, $\eta \in \mathscr{H}_\oplus$, and μ-a.a. $\omega \in \Omega$. Moreover, for any instrument $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$, we define the 'dual' instrument $\tilde{\Gamma} : \Sigma \times \mathscr{L}(\mathscr{K}) \to \mathscr{L}(\mathscr{H})$ through $\tilde{\Gamma}(X, B) = [\Gamma(X)]^*(B)$ for all $X \in \Sigma$ and $B \in \mathscr{L}(\mathscr{K})$: we use these notations throughout this treatise.

Theorem 2. *Let $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ and let $\mu : \Sigma \to \mathbb{R}$ be a σ-finite positive measure such that $\mathrm{tr}[\Gamma(\cdot, \rho)] \ll \mu$ for all $\rho \in \mathscr{S}(\mathscr{H})$. There is a direct-integral space $\mathscr{H}_\oplus = \int_\Omega^\oplus \mathscr{H}(\omega)\, d\mu(\omega)$ and vectors $\psi_{j,m} \in \mathscr{H}_\oplus$, $1 \le j < \dim \mathscr{H} + 1$, $1 \le m < \dim \mathscr{K} + 1$, such that, when we define the *-representation $\rho : L^\infty(\mu) \otimes \mathscr{L}(\mathscr{K}) \to \mathscr{L}(\mathscr{K} \otimes \mathscr{H}_\oplus)$ through $\rho(f \otimes B) = B \otimes M_f$ for all $f \in L^\infty(\mu)$ and $B \in \mathscr{L}(\mathscr{K})$ and the isometry $Y : \mathscr{H} \to \mathscr{K} \otimes \mathscr{H}_\oplus$ through $Y e_j = \sum_m f_m \otimes \psi_{j,m}$, the triple $(\mathscr{K} \otimes \mathscr{H}_\oplus, \rho, Y)$ is a minimal Stinespring dilation for Γ.*

We have $\Gamma \in \mathrm{ext}\,\mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ if and only if for any decomposable $D =$

$\int_\Omega^\oplus D(\omega)\,d\mu(\omega) \in \mathscr{L}(\mathscr{H}_\oplus)$,*one has*

$$\sum_m \int_\Omega \langle \psi_{j,m}(\omega)|D(\omega)\psi_{k,m}(\omega)\rangle \, d\mu(\omega) = 0$$

for all j, k *if and only if* $D = 0$.

From the preceding theorem, it follows that we may give a 'pointwise Kraus decomposition' for Γ consisting of fields $\omega \mapsto A_j(\omega)$ of operators $A_j(\omega) : \mathscr{D} \mapsto \mathscr{K}$ for all $1 \le j < \dim \mathscr{H}(\omega) + 1$, where $\mathscr{D} = \mathrm{lin}_j e_j$ so that

$$\langle \varphi|\tilde{\Gamma}(X, B)\psi\rangle = \int_X \sum_{j=1}^{\dim \mathscr{H}(\omega)} \langle A_j(\omega)\varphi|BA_j(\omega)\psi\rangle \, d\mu(\omega)$$

for all φ, $\psi \in \mathscr{D}$, $X \in \Sigma$, and $B \in \mathscr{L}(\mathscr{K})$. It also follows that the set of structure vectors $\{\psi_{j,m}\}_j$ is orthonormal for all m. The number $\dim \mathscr{H}(\omega)$ is called as the *rank* of the instrument Γ associated with the outcome.[1]

A similar result holds also for observables [46, 66, 67]:

Theorem 3. *Fix* $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ *and pick a* σ*-finite measure* μ *on* (Ω, Σ) *such that* $p_\rho^{\mathsf{M}} \ll \mu$ *for all* $\rho \in \mathscr{S}(\mathscr{H})$. *There is a direct-integral space* $\mathscr{H}_\oplus = \int_\Omega^\oplus \mathscr{H}(\omega)\,d\mu(\omega)$ *and an orthonormal system* $\{\psi_j \in \mathscr{H}_\oplus \,|\, 1 \le j < \dim \mathscr{H} + 1\}$ *such that, when one defines the spectral measure* $\mathsf{P} : \Sigma \to \mathscr{L}(\mathscr{H}_\oplus)$, $\mathsf{P}(X) = M_{\chi_X}$ *for all* $X \in \Sigma$, *and the isometry* $J : \mathscr{H} \to \mathscr{H}_\oplus$ *through* $Je_j = \psi_j$, *the triple* $(\mathscr{H}_\oplus, \mathsf{P}, J)$ *is a minimal Naĭmark dilation for* M. *Furthermore,* $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}(\Sigma, \mathscr{H})$ *if and only if, for any decomposable* $D \in \mathscr{L}(\mathscr{H}_\oplus)$, *one has*

$$\int_\Omega \langle \psi_j(\omega)|D(\omega)\psi_k(\omega)\rangle \, d\mu(\omega) = 0$$

for all j, k *if and only if* $D = 0$.

From the preceding theorem, it follows that we may write

$$\langle e_j|\mathsf{M}(X)e_k\rangle = \int_X \langle \psi_j(\omega)|\psi_k(\omega)\rangle \, d\mu(\omega), \qquad X \in \Sigma, \quad 1 \le j,\, k < \dim \mathscr{H} + 1.$$

Again, we may define the rank (or rank function) $r(\omega)$ as the dimension of the component space $\mathscr{H}(\omega)$.

As already mentioned, the sharp observables are extreme but there are others. Examples on unsharp extreme observables include the canonical time observable introduced in Section 4.2.1 in Equation (4.4) and the exemplary ob-

[1] Rather than talking about individual ranks associated to different outcomes, it is mathematically more rigorous to define the rank as the measurable map $r : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$, $r(\omega) = \dim \mathscr{H}(\omega)$ for μ-a.a. $\omega \in \Omega$.

servable discussed below. Other unsharp extreme observables are the canonical phase [36] and the phase space observable generated by the vacuum [66].

As an example of utilizing Theorem 3, let us take a look at a particular $\big([0,1],\mathscr{B}[0,1]\big)$-valued observable operating in an infinite-dimensional Hilbert space $\mathscr{H}$ with the orthonormal basis $\{e_0,\, e_1,\, e_2, \ldots\}$. For any $s \in [0,1]$ and $n \in \mathbb{Z}$, we define $\langle s, n\rangle = e^{i2\pi sn}$. Our observable $\mathsf{M} \in \mathbf{Obs}\big(\mathscr{B}[0,1],\mathscr{H}\big)$ is given by

$$\mathsf{M}(X) = \sum_{m,n=0}^{\infty} \int_X \langle s, m-n\rangle\, ds\, \big(|e_{2m}\rangle\langle e_{2n}| + |e_{2m+1}\rangle\langle e_{2n+1}|\big), \qquad X \in \mathscr{B}[0,1].$$

Note that M is unsharp. Let us fix the orthonormal basis $\{|0\rangle,\, |1\rangle\}$ in the 2-dimensional Hilbert space $\mathbb{C}^2$ and define the linear map $J : \mathscr{H} \to L^2_{\mathbb{C}^2}[0,1]$ through

$$(Je_{2m+j})(s) = \overline{\langle s, m\rangle}|j\rangle, \qquad m = 0,\, 1,\, 2, \ldots, \quad j = 0,\, 1, \quad 0 \le s \le 1.$$

Moreover, we denote by P the spectral measure on $\mathscr{B}[0,1]$ operating in $L^2_{\mathbb{C}^2}[0,1]$ defined by $\big(\mathsf{P}(X)\eta\big)(s) = \chi_X(s)\eta(s)$ for all $X \in \mathscr{B}[0,1]$, $\eta \in L^2_{\mathbb{C}^2}[0,1]$, and a.a. $s \in [0,1]$. It is simple to verify that $(L^2_{\mathbb{C}^2}[0,1], \mathsf{P}, J)$ is a minimal Naĭmark dilation for M. From this we see that M is of rank 2, i.e., the rank function associated with M has the constant value 2.

We may now prove that $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}(\mathscr{B}[0,1],\mathscr{H})$ using the criterion given in Theorem 3. Let $D \in \mathscr{L}\big(L^2_{\mathbb{C}^2}[0,1]\big)$ be decomposable so that $(D\eta)(s) = D(s)\eta(s)$ for all $\eta \in L^2_{\mathbb{C}^2}[0,1]$ and a.a. $s \in [0,1]$ where $D(s)$ are bounded operators on $\mathbb{C}^2$. Denote $\langle j|D(\cdot)|k\rangle = d_{j,k} \in L^\infty[0,1]$ for $j,\, k = 0,\, 1$. One has

$$\langle Je_{2n+j}|DJe_{2m+k}\rangle = \int_0^1 \langle s, m-n\rangle d_{j,k}(s)\, ds = \hat{d}_{j,k}(m-n)$$

for all $m,\, n = 0,\, 1,\, 2, \ldots$ and $j,\, k = 0,\, 1$, where $\hat{d}_{j,k}$ is the Fourier transform (which, in this case, is a sequence on $\mathbb{Z}$) of $d_{j,k}$. It follows immediately that $J^*DJ = 0$ if and only if $\hat{d}_{j,k} = 0$ for $j,\, k = 0,\, 1$, i.e., $D = 0$. Thus, $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}(\mathscr{B}[0,1],\mathscr{H})$.

Let us briefly discuss the structure of extreme discrete instruments. Let $N \in \{1,\, 2, \ldots\} \cup \{\infty\}$ and denote by $\mathbf{Ins}_N(\mathscr{H},\mathscr{K})$ the set of N-outcome instruments with the input Hilbert space $\mathscr{H}$ and output Hilbert space $\mathscr{K}$. Whenever $\Gamma \in \mathbf{Ins}_N(\mathscr{H},\mathscr{K})$, we denote $\Gamma = (\Gamma_j)_{j=1}^N$ where $\Gamma_j = \Gamma(\{j\},\cdot)$. Assume that $\Gamma \in \mathbf{Ins}_N(\mathscr{H},\mathscr{K})$ and, for each j, the operation Γ_j has the minimal set $\{K_{jn}\}_{n=1}^{r_j}$ of Kraus operators; here r_j is the rank of the outcome j. It follows that $\Gamma \in \mathrm{ext}\,\mathbf{Ins}_N(\mathscr{H},\mathscr{K})$ if and only if the set

$$\{K_{jm}^* K_{jn} \in \mathscr{L}(\mathscr{H}) \,|\, 1 \le m,\, n < r_j + 1,\ 1 \le j < N+1\}$$

is strongly independent.

3.1.2 Extreme observables in finite dimensions

Let us concentrate on N-outcome observables, $N < \infty$, operating in a Hilbert space $\mathscr{H}$, $\dim \mathscr{H} =: d < \infty$, the set of which is here denoted by $\mathbf{Obs}_N(\mathscr{H})$. For each $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$, we write $\mathsf{M} = (\mathsf{M}_j)_{j=1}^N$ where $\mathsf{M}_j = \mathsf{M}(\{j\})$. In fact, as pointed out, e.g., in [17, 18, 30], an extreme observable operating on a d-dimensional ($d < \infty$) Hilbert space is always supported by a subset of values of cardinality lower than or equal to d^2, and hence observables in finite dimensions are essentially finite valued. Suppose that $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$ and

$$\mathsf{M}_j = \sum_{n=1}^{r_j} |\varphi_{jn}\rangle\langle\varphi_{jn}|, \qquad j = 1, \ldots, N,$$

where φ_{jn}, $n = 1, \ldots, r_j$, are the non-normalized eigenvectors of M_j and r_j is the rank of M_j. Theorem 3 implies that $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}_N(\mathscr{H})$ if and only if the set

$$\{|\varphi_{jm}\rangle\langle\varphi_{jn}| \,|\, m, n = 1, \ldots, r_j,\ j = 1, \ldots, N\}$$

is linearly independent, a result found earlier also in [21, 64]. It follows that a rank-1 observable M is extreme if and only if its effects $\mathsf{M}_1, \ldots, \mathsf{M}_N$ are linearly independent.

Let us consider an example of a finite-outcome extreme observable in the case where $d = 4$ and $N = 3$. We define the observable $\mathsf{M} \in \mathbf{Obs}_3(\mathbb{C}^4)$ through

$$\mathsf{M}_j = \frac{1}{3}\big(\mathbb{1} + \omega_3^j(|1\rangle\langle 3| + |2\rangle\langle 4|) + \overline{\omega}_3^j(|3\rangle\langle 1| + |4\rangle\langle 2|)\big), \quad j = 1, 2, 3,$$

where $\omega_3 = e^{i2\pi/3}$ and $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ is an orthonormal basis of $\mathbb{C}^4$. Using the above extremality condition, it can be shown that M is extreme [30]. M is neither sharp nor rank-1; in fact, all the effects M_j of M are of rank 2.

Furthermore, in [30], a way of implementing any $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$ from a restricted subset of observables was established. Certainly, the extreme observables determine the rest in the way implied by the Choquet theorem, but we can say something more. Let us consider a particular kind of post-processing. We fix positive integers M and N and a function $f : \{1, \ldots, N\} \to \{1, \ldots, M\}$. Now, pick $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$ so that we may define the observable $\mathsf{M}^f \in \mathbf{Obs}_M(\mathscr{H})$ through

$$\mathsf{M}^f_k = \sum_{j \in f^{-1}(k)} \mathsf{M}_j, \qquad k = 1, \ldots, M.$$

We call such an observable M^f a *relabeling of* M *with* f; M^f is obtained by switching and possibly combining the outcomes of M. The following result from [30] says that the rank-1 extreme observables are essentially enough to generate the other observables through simple classical manipulation of their outcome statistics.

Theorem 4. *Any* $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$ *can be obtained by relabeling and mixing rank-1 extreme observables from* $\mathbf{Obs}_{d^2}(\mathscr{H})$.

This result is rather intuitive since all other observables can be obtained by classical post-processing from the rank-1 elements and the mixtures of extreme observables generate the whole set of observables. Thus extreme rank-1 observables can be expected to be the most informative class of observables.

For any $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$, we denote by $r_j(\mathsf{M})$ the rank of the jth atomary effect M_j. In addition to the baseline requirement $r_1(\mathsf{M}) + \cdots + r_N(\mathsf{M}) \geq d$ satisfied by all observables, an extreme observable $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}_N(\mathscr{H})$ must conform at least to the two additional conditions:

(i) $r_1(\mathsf{M})^2 + \cdots + r_N(\mathsf{M})^2 \leq d^2$,

(ii) whenever $j \neq k$, then $r_j(\mathsf{M}) + r_k(\mathsf{M}) \leq d$.

Note that the condition (i) above is a direct corollary of the extremality criterion for observables in $\mathbf{Obs}_N(\mathscr{H})$ stated in the beginning of this section; the number of operators required to be linearly independent is $r_1(\mathsf{M})^2 + \cdots + r_N(\mathsf{M})^2$. The condition (ii) is proven in [21, Corollary 3]. However, it is yet unclear whether for any rank combination $(r_j)_{j=1}^N$ such that $r_1 + \cdots + r_N \geq d$, $r_1^2 + \cdots + r_N^2 \leq d^2$, and $r_j + r_k \leq d$ for any $j,\, k = 1, \ldots, N$, $j \neq k$, there exists an extreme observable $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}_N(\mathscr{H})$ such that $r_j = r_j(\mathsf{M})$, $j = 1, \ldots, N$. However, in [30], it was shown that, for any $N = d, \ldots, d^2$, there exist extreme rank-1 observables $\mathsf{M} \in \mathrm{ext}\,\mathbf{Obs}_N(\mathscr{H})$ with no 0-outcomes.

3.2 Extremality and compatibility

In this section we elucidate a link between extremality and compatibility properties of quantum devices. We first express the essential result obtained in [31] in the general picture involving the generalized device sets $\mathbf{CP}(\mathscr{A};\mathscr{H})$. The same results apply to the set $\mathbf{NCP}(\mathscr{A};\mathscr{H})$. We fix the von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ and the Hilbert space $\mathscr{H}$ that does not have to be separable.

When $\Psi \in \mathbf{CP}(\mathscr{A} \otimes \mathscr{B};\mathscr{H})$, we may define the *margins of* Ψ, $\Psi_{(1)} \in \mathbf{CP}(\mathscr{A};\mathscr{H})$ and $\Psi_{(2)} \in \mathbf{CP}(\mathscr{B};\mathscr{H})$, through

$$\Psi_{(1)}(a) = \Psi(a \otimes 1_{\mathscr{B}}), \quad \Psi_{(2)}(b) = \Psi(1_{\mathscr{A}} \otimes b) \tag{3.2}$$

for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$. On the other hand, when $\Phi_1 \in \mathbf{CP}(\mathscr{A};\mathscr{H})$ and $\Phi_2 \in \mathbf{CP}(\mathscr{B};\mathscr{H})$ are margins of a map $\Psi \in \mathbf{CP}(\mathscr{A} \otimes \mathscr{B};\mathscr{H})$, i.e., $\Phi_1 = \Psi_{(1)}$ and $\Phi_2 = \Psi_{(2)}$, we say that Φ_1 and Φ_2 are *compatible* and Ψ is a *joint map for* Φ_1 *and* Φ_2. If maps are not compatible, they are *incompatible*. One can show the following [31]:

Theorem 5. *Let $\Phi_1 \in \mathbf{CP}(\mathscr{A};\mathscr{H})$ and $\Phi_2 \in \mathbf{CP}(\mathscr{B};\mathscr{H})$ be compatible.*

(a) *If $\Phi_1 \in \operatorname{ext}\mathbf{CP}(\mathscr{A};\mathscr{H})$ or $\Phi_2 \in \operatorname{ext}\mathbf{CP}(\mathscr{B};\mathscr{H})$, the joint map for Φ_1 and Φ_2 is unique.*

(b) *If both $\Phi_1 \in \operatorname{ext}\mathbf{CP}(\mathscr{A};\mathscr{H})$ and $\Phi_2 \in \operatorname{ext}\mathbf{CP}(\mathscr{B};\mathscr{H})$, the unique joint map for Φ_1 and Φ_2 is extreme in $\mathbf{CP}(\mathscr{A}\otimes\mathscr{B};\mathscr{H})$.*

(c) *If Φ_1 or Φ_2 is a *-representation, then Φ_1 and Φ_2 have to commute ($\Phi_1(a)\Phi_2(b) = \Phi_2(b)\Phi_1(a)$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$) and the unique joint map Ψ for Φ_1 and Φ_2 is given by*

$$\Psi(a\otimes b) = \Phi_1(a)\Phi_2(b), \qquad a \in \mathscr{A}, \quad b \in \mathscr{B}.$$

The definition of compatibility for CP maps given above generalizes the notion of compatibility (but not coexistence of observables) for quantum devices presented separately for different quantum structures in Section 2.2. For an instrument $\Gamma \in \mathbf{Ins}(\Sigma,\mathscr{H},\mathscr{K})$ associated with the map $G \in \mathbf{NCP}\big(L^\infty(\mu)\otimes\mathscr{L}(\mathscr{K});\mathscr{H}\big)$ according to (3.1), e.g., the margins are given by the conditions

$$\begin{aligned}
\operatorname{tr}[\rho G(\chi_X\otimes\mathbb{1}_{\mathscr{K}})] &= \operatorname{tr}[\Gamma(X,\rho)] = \operatorname{tr}[\rho\mathsf{M}_\Gamma(X)],\\
\operatorname{tr}[\rho G(1\otimes B)] &= \operatorname{tr}[\Gamma(\Omega,\rho)B] = \operatorname{tr}[\mathscr{E}_\Gamma(\rho)B]
\end{aligned}$$

for all $\rho \in \mathscr{S}(\mathscr{H})$, $X \in \Sigma$, and $B \in \mathscr{L}(\mathscr{K})$, where $\mathsf{M}_\Gamma \in \mathbf{Obs}(\Sigma,\mathscr{H})$ and $\mathscr{E}_\Gamma \in \mathbf{Ch}(\mathscr{H},\mathscr{K})$ are the observable and channel associated with the instrument Γ as explained in Section 2.2 and 1 is the constant function $\omega \mapsto 1$. Thus Theorem 5 yields immediate results for quantum compatibility related questions. Some of the corollaries are well known; e.g., the results concerning compatibility properties of sharp observables with other quantum devices mentioned in Section 4.2.2 can be seen as corollaries of the item (c) of Theorem 5. Moreover, the basic result stating that, if either one of the partial traces ρ_1 or ρ_2 of a bipartite state ρ is pure, then $\rho = \rho_1\otimes\rho_2$, follows directly from this result.

The more novel insights provided by Theorem 5 include the following: If the observables M and N are compatible and either one is extreme, then they have a unique joint observable, and if both of them are extreme, this joint observable is extreme as well. Similarly, if an observable M and a channel $\mathscr{E}$ are compatible and either one is extreme, then their joint instrument is unique, and if both devices are extreme, so is their joint instrument.

We may also give a corollary concerning a particular class of channels. Let us fix the Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{K}_1$, and $\mathscr{K}_2$ and study channels $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}_1\otimes\mathscr{H}_2,\mathscr{K}_1\otimes\mathscr{K}_2)$ between the joint systems described by $\mathscr{H}_1\otimes\mathscr{H}_2$ and $\mathscr{K}_1\otimes\mathscr{K}_2$. Such a channel $\mathscr{E}$ is called *causal* or *non-signalling*, if there are $\mathscr{E}_1 \in \mathbf{Ch}(\mathscr{H}_1,\mathscr{K}_1)$ and $\mathscr{E}_2 \in \mathbf{Ch}(\mathscr{H}_2,\mathscr{K}_2)$ such that

$$\operatorname{tr}_{\mathscr{K}_2}[\mathscr{E}(\rho\otimes\sigma)] = \mathscr{E}_1(\rho), \quad \operatorname{tr}_{\mathscr{K}_1}[\mathscr{E}(\rho\otimes\sigma)] = \mathscr{E}_2(\sigma)$$

for all $\rho \in \mathscr{S}(\mathscr{H}_1)$ and $\sigma \in \mathscr{S}(\mathscr{H}_2)$. The above condition can also be stated in the dual form $\mathscr{E}^*(A \otimes \mathbb{1}_{\mathscr{K}_2}) = \mathscr{E}_1^*(A) \otimes \mathbb{1}_{\mathscr{H}_2}$ for all $A \in \mathscr{L}(\mathscr{K}_1)$ and $\mathscr{E}^*(\mathbb{1}_{\mathscr{K}_1} \otimes B) = \mathbb{1}_{\mathscr{H}_1} \otimes \mathscr{E}_2^*(B)$ for all $B \in \mathscr{L}(\mathscr{K}_2)$. A particular class of causal channels consists of the *local channels* $\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$, i.e. $\mathscr{E}(\rho \otimes \sigma) = \mathscr{E}_1(\rho) \otimes \mathscr{E}_2(\sigma)$ for all $\rho \in \mathscr{S}(\mathscr{H}_1)$ and $\sigma \in \mathscr{S}(\mathscr{H}_2)$, but there are much more causal channels, even outside the convex hull of the set of local channels [2, 20, 24]. However, according to [31], already when either $\mathscr{E}_1$ or $\mathscr{E}_2$ is extreme, then the causal channel $\mathscr{E}$ is local:

Theorem 6. *If either of the subchannels $\mathscr{E}_1 \in \mathbf{Ch}(\mathscr{H}_1, \mathscr{K}_1)$ or $\mathscr{E}_2 \in \mathbf{Ch}(\mathscr{H}_2, \mathscr{K}_2)$ of a causal channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}_1 \otimes \mathscr{H}_2, \mathscr{K}_1 \otimes \mathscr{K}_2)$ is extreme, then $\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$.*

3.2.1 Extremality and coexistence

Extremality also clarifies the relations between coexistent observables. When observables M and N are coexistent with a mother observable G, i.e., $\operatorname{ran} \mathsf{M} \cup \operatorname{ran} \mathsf{N} \subset \operatorname{ran} \mathsf{G}$, there is typically no computable connection, let alone classical connection given by post-processing, between the outcome statistics of M and N or between the statistics of either of these and that of the mother G as shown by a simple counter-example in [73]. When any one of the observables involved is extreme, the statistical connection can be recovered, however, and coexistence implies joint measurability. The following results have been proven in [29].

Theorem 7. *Assume that (Ω, Σ) and (Ω', Σ') are standard Borel spaces and $(\overline{\Omega}, \overline{\Sigma})$ is any measurable space. Let $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ and $\mathsf{N} \in \mathbf{Obs}(\Sigma', \mathscr{H})$ be coexistent with a mother observable $\mathsf{G} \in \mathbf{Obs}(\overline{\Sigma}, \mathscr{H})$.*

(a) *If M (or N) is discrete and extreme, then M and N are jointly measurable.*

(b) *If G is extreme, then M and N are post-processings of G in the way that there are measurable functions $f : \overline{\Omega} \to \Omega$ and $g : \overline{\Omega} \to \Omega'$ such that $\mathsf{M} = \mathsf{G} \circ f^{-1}$ and $\mathsf{N} = \mathsf{G} \circ g^{-1}$ and, consequently, M and N are jointly measurable.*

The preceding theorem generalizes the corresponding well-known results involving sharp observables. The assumption on discreteness in item (a) of the claim is, however, an additional assumption that remains to be lifted. To give a taste how extremality can be used to recover the classical connection between coexistent observables, let us look closer at the situation in item (b) of the above theorem. Let $(\mathscr{M}, \mathsf{P}, J)$ be a minimal Naĭmark dilation of the extreme observable G, i.e., $\mathscr{M}$ is a Hilbert space, $\mathsf{P} \in \mathbf{Obs}(\overline{\Sigma}, \mathscr{M})$ is a PVM and $J : \mathscr{H} \to \mathscr{M}$ is an isometry such that $\mathsf{G}(Z) = J^*\mathsf{P}(Z)J$ for all $Z \in \overline{\Sigma}$ and vectors $\mathsf{P}(Z)J\varphi$, $Z \in \overline{\Sigma}$, $\varphi \in \mathscr{H}$, span a dense subspace of $\mathscr{M}$. Since $\operatorname{ran} \mathsf{M} \subset \operatorname{ran} \mathsf{G}$, it follows that there is a function $\Sigma \ni X \mapsto Z_X \in \overline{\Sigma}$ such that $\mathsf{M}(X) = \mathsf{G}(Z_X)$ for all

$X \in \Sigma$. Pick now a disjoint sequence $(X_j)_j \subset \Sigma$, any $N \in \mathbb{N}$, and define

$$\mathsf{P}(Z_{\cup_{j=1}^N X_j}) - \sum_{j=1}^N \mathsf{P}(Z_{X_j}) =: D_N \in (\mathrm{ran}\,\mathsf{P})'.$$

One has $J^* D_N J = \mathsf{M}(\cup_{j=1}^N X_j) - \sum_{j=1}^N \mathsf{M}(X_j) = 0$ so that $D_N = 0$ according to Theorem 3. Hence, $\sum_{j=1}^N \mathsf{P}(Z_{X_j}) = \mathsf{P}(Z_{\cup_{j=1}^N X_j}) \leq \mathbb{1}_{\mathscr{M}}$ implying that the sequence of finite sums $\sum_{j=1}^N \mathsf{P}(Z_{X_j})$, $N = 1, 2, \ldots$, is increasing (since the summands are positive) and bounded from above by the identity, and thus the weak limit $w-\lim_{N\to\infty} \sum_{j=1}^N \mathsf{P}(Z_{X_j}) := \sum_j \mathsf{P}(Z_{X_j})$ bounded from above by the identity exists. Now, writing $D = \mathsf{P}(Z_{\cup_j X_j}) - \sum_j \mathsf{P}(Z_{X_j})$, a similar calculation as above shows that $D = 0$ implying the σ-additivity of $X \mapsto \mathsf{P}(Z_X)$. Similarly, $\mathsf{P}(Z_\Omega) = \mathbb{1}_{\mathscr{M}}$ and, hence, the map $X \mapsto \mathsf{P}(Z_X)$ is a PVM that has P as its mother. If $\Sigma' \ni Y \mapsto W_Y \in \overline{\Sigma}$ is such that $\mathsf{N}(Y) = \mathsf{G}(W_Y)$ for all Y, it, likewise, follows that $Y \mapsto \mathsf{P}(W_Y)$ is a PVM with the same mother P. It is immediate that these two new PVMs are jointly measurable from which we deduce that M and N are jointly measurable. Since the PVMs $X \mapsto \mathsf{P}(Z_X)$ and $Y \mapsto \mathsf{P}(W_Y)$ have the sharp mother P, there are functions $f : \overline{\Omega} \to \Omega$ and $g : \overline{\Omega} \to \Omega'$ such that $X \mapsto Z_X$ is essentially the pre-image map $f^{-1} : \Sigma \to \overline{\Sigma}$ and $Y \mapsto W_Y$ coincides essentially with g^{-1} [50, Theorem 3.5] and the claim of the item (c) follows.

Chapter 4

Extreme covariant quantum devices

Until now, we have concentrated on global properties of the essential quantum structures. The reality is, however, often that we have access only to a restricted class of devices or it is physically feasible only to study devices with particular interesting properties. Symmetry is one such property, and in this chapter we study the structure and extreme points of covariance structures, i.e., classes of covariant quantum devices. The theorems of this chapter have been proven in the articles accompanying this thesis which is why the proofs are not included in this chapter. Note, however, that the results of Section 4.2.3 have not appeared in any publication and thus they are accompanied by proofs. For basic definitions for representations, see the short introduction given in Section 2.0.3.

In order to define covariance in quantum framework, let us fix a group G and a Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. We also pick a projective unitary representation $U : G \to \mathscr{U}(\mathscr{H})$ with the associated multiplier $m : G \times G \to \mathbb{T}$. Similarly, we pick another projective representation $V : G \to \mathscr{U}(\mathscr{K})$. Let us assume that G acts measurably on a measurable space (Ω, Σ), i.e., there is a map $G \times \Omega \ni (g, \omega) \mapsto g \cdot \omega \in \Omega$ such that the map $\omega \mapsto g \cdot \omega$ is Σ-measurable for all $g \in G$ and $(gh) \cdot \omega = g \cdot (h \cdot \omega)$ for all g, $h \in G$ and $\omega \in \Omega$.

We concentrate on covariant instruments and observables. We define $\mathbf{Ins}_U^V(\Sigma)$ as the set of those $\Gamma \in \mathbf{Ins}(\Sigma, \mathscr{H}, \mathscr{K})$ such that

$$\Gamma\big(g \cdot X, U(g)\rho U(g)^*\big) = V(g)\Gamma(X, \rho)V(g)^*$$

for all $g \in G$, $X \in \Sigma$, and $\rho \in \mathscr{S}(\mathscr{H})$. Often the σ-algebra Σ is evident from the context (e.g., the Borel σ-algebra of a topological space), in which case we often simplify our notations and write $\mathbf{Ins}_U^V(\Sigma) = \mathbf{Ins}_U^V(\Omega)$. Similarly, for observables, we denote by $\mathbf{Obs}_U(\Sigma)$ or by $\mathbf{Obs}_U(\Omega)$, if the σ-algebra is evident from the context, the set of those $\mathsf{M} \in \mathbf{Obs}(\Sigma, \mathscr{H})$ such that

$$\mathsf{M}(g \cdot X) = U(g)\mathsf{M}(X)U(g)^*, \qquad g \in G, \quad X \in \Sigma.$$

A typical example of covariant observables are the covariant phase space observables, where $\mathscr{H} = L^2(\mathbb{R})$, $\Omega = \mathbb{R}^2$ (equipped with the Borel σ-algebra associated with the natural topology of $\mathbb{R}^2$), $G = \mathbb{R}^2$, and the representation is the *Weyl representation* W of $\mathbb{R}^2$, i.e.,

$$\big(W(q,p)\varphi\big)(x) = e^{iqp/2}e^{ipx}\varphi(x+q)$$

for all $q,\,p \in \mathbb{R}$, $\varphi \in L^2(\mathbb{R})$, and a.a. $x \in \mathbb{R}$. Any covariant phase space observable $\mathsf{M} \in \mathbf{Obs}_W(\mathbb{R}^2)$ has the structure

$$\mathsf{M}(X) = \frac{1}{2\pi}\int_X W(q,p)TW(q,p)^*\,dq\,dp, \qquad X \in \mathscr{B}(\mathbb{R}^2), \tag{4.1}$$

where T is a positive trace-class operator on $L^2(\mathbb{R})$ of trace 1. For other examples on covariance in quantum measurements and preliminaries for the techniques involved in the study of covariance structures we refer to [41, 42, 43, 44, 59, 61]. Covariant observables and the extreme points of covariance structures in the case of finite-dimensional Hilbert spaces are studied particularly in [14, 15]. For theory related with (covariant) sesquilinear-form-valued maps on modules, see, e.g., [28, 65, 70].

Systems of imprimitivity and the imprimitivity theorem are of paramount importance in the study of covariance structures. Let us assume that G is a locally compact group with a closed subgroup $H \leq G$ and π is an *ordinary* strongly continuous representation of H on a Hilbert space $\mathscr{K}_\pi$. We equip G/H (the space of left cosets) with the topology induced by that of G and the associated Borel σ-algebra. The group G acts naturally on G/H as $g\cdot(g'H) = gg'H$ for all $g,\,g' \in G$ and we assume that $\mu_{G/H}$ is a strongly quasi-invariant measure on the Borel σ-algebra of G/H, i.e., the translated measures $\mu_g : X \mapsto \mu_{G/H}(g\cdot X)$ are mutually absolutely continuous with respect to $\mu_{G/H}$ with continuous Radon-Nikodým derivatives $d\mu_g/d\mu_{G/H} =: \rho_g$. We denote $gH = \overline{g}$ for all $g \in G$. Let $\mathfrak{K}_0$ be the vector space of continuous functions $f : G \to \mathscr{K}_\pi$ such that the projected support $\{\overline{g}\,|\,f(g) \neq 0\} \subset G/H$ is relatively compact and $f(gh) = \pi(h)^* f(g)$ for all $g \in G$ and $h \in H$ and let us define the inner product $\langle\cdot|\cdot\rangle$ on $\mathfrak{K}_0$:

$$\langle f|f'\rangle = \int_{G/H}\langle f(g)|f'(g)\rangle\,d\mu_{G/H}(\overline{g}), \qquad f,\,f' \in \mathfrak{K}_0.$$

Note that the above integrand is constant as a function of g on all cosets of G/H. We define the unitary operators $V_\pi(g)$, $g \in G$, on $\mathfrak{K}_0$ through

$$\big(V_\pi(g)f\big)(g') = \sqrt{\rho_g(g'H)}\,f(g^{-1}g'), \qquad g,\,g' \in G.$$

Denote by $\mathfrak{K}_\pi$ the Hilbert space completion of $\mathfrak{K}_0$ with respect to the above inner product. The operators $V_\pi(g)$ can be extended into unitary operators on $\mathfrak{K}_\pi$ giving rise to a strongly continuous unitary representation $V_\pi : G \to \mathscr{U}(\mathfrak{K}_\pi)$ which is called as the representation *induced* by π and the induction $H \to G$ and it is often denoted $V_\pi = \mathrm{ind}_H^G \pi$. One can also define the spectral measure $\mathsf{P}_\pi : \mathscr{B}(G/H) \to \mathscr{L}(\mathfrak{K}_\pi)$,

$$\big(\mathsf{P}_\pi(X)f\big)(g) = \chi_X(\overline{g})f(g), \qquad X \in \mathscr{B}(G/H), \quad g \in G.$$

The triple $(\mathfrak{K}_\pi, \mathsf{P}_\pi, V_\pi)$ is called as the *canonical system of imprimitivity associated with π and the induction $H \to G$*. Note that $\mathsf{P}_\pi \in \mathbf{Obs}_{V_\pi}(G/H)$. For more on the inducing construction and its uses, we refer to [59] and for a contemporary view [25, Chapter 6]. The inducing construction can be generalized also for projective unitary representations [60] but, for simplicity, we concentrate on ordinary representations. Moreover, in the more detailed examples we study in sections 4.2.1 and 4.2.2, the symmetry groups G involved are locally compact second countable unimodular type-I Hausdorff groups, in which case the existence of truly G-invariant measures on transitive G-spaces is guaranteed which further simplifies the inducing construction (the density functions ρ_g coincide a.e. with 1).

Let us assume that G is a lcsc (locally compact and second-countable) Hausdorff group and Ω is a lcsc transitive G-space which is Hausdorff; this means essentially that there is a closed subgroup $H \leq G$ such that Ω is homeomorphic with G/H. Suppose that $U : G \to \mathscr{U}(\mathscr{H})$ is an ordinary strongly continuous unitary representation. According to [13, 25] for every $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$, there is a Hilbert space $\mathscr{K}_0$, a strongly continuous unitary representation $\pi : H \to \mathscr{L}(\mathscr{K}_0)$, and an isometry $J_\pi : \mathscr{H} \to \mathfrak{K}_\pi$ such that the triple $(\mathfrak{K}_\pi, \mathsf{P}_\pi, J_\pi)$ is a minimal Naĭmark dilation for M and $J_\pi U(g) = V_\pi(g)J_\pi$ for all $g \in G$.

4.1 Structure of covariant quantum devices

The discussion of this section can be generalized to the context of C^*-modules and especially Hilbert C^*-modules or von Neumann modules as discussed in [28, 69, 70], but we suppress generality for clarity. We fix a von Neumann algebra $\mathscr{A}$ and a Hilbert space $\mathscr{H}$. Moreover, G is a group and $U : G \to \mathscr{U}(\mathscr{H})$ is a projective unitary representation. We also fix a G-action (a homomorphism) $G \ni g \mapsto \beta_g \in \mathrm{Aut}(\mathscr{A})$. We say that the action β is *inner* if there are unitaries $u(g) \in \mathscr{A}$, $g \in G$, such that $\beta_g(a) = u(g)au(g)^*$ for all $g \in G$; in this case, the map $g \mapsto u(g)$ is, essentially, a projective representation. Let us define $\mathbf{NCP}_U^\beta$ as the set of those $\Phi \in \mathbf{NCP}(\mathscr{A}; \mathscr{H})$ such that

$$(\Phi \circ \beta_g)(a) = U(g)\Phi(a)U(g)^*, \qquad g \in G, \quad a \in \mathscr{A}.$$

One can prove the following [28]:

Theorem 8. *Assume that* $\Phi \in \mathbf{NCP}_U^\beta$*. When* $(\mathscr{M}, \pi, J)$ *is a minimal Stinespring dilation for* Φ*, there is a projective unitary representation* $\tilde{U} : G \to \mathscr{U}(\mathscr{M})$ *such that* $JU(g) = \tilde{U}(g)J$ *and* $\tilde{U}(g)\pi(a) = (\pi \circ \beta_g)(a)\tilde{U}(g)$ *for all* $g \in G$ *and* $a \in \mathscr{A}$*. If the action* β *is inner,* $\beta_g(a) = u(g)au(g)^*$*, there is also a projective unitary representation* $\overline{U} : G \to \mathscr{U}(\mathscr{M})$*,* $\overline{U}(g) = \pi(u_g^*)\tilde{U}(g)$ *for all* $g \in G$*, such that* $(\pi \circ u)(g)JU(g) = \overline{U}(g)J$ *and* $\overline{U}(g)\pi(a) = \pi(a)\overline{U}(g)$ *for all* $g \in G$ *and* $a \in \mathscr{A}$*.*

We call quadruples $(\mathscr{M}, \pi, \tilde{U}, J)$ and $(\mathscr{M}, \pi, \overline{U}, J)$ of Theorem 8 as (β, U)-*covariant dilations of* $\Phi \in \mathbf{NCP}_U^\beta$, where the latter one we call as the covariant dilation associated with an inner action. This latter one is particularly important in the study of covariant instruments and observables. Note that Theorem 8 in the form presented above has been proven in different forms and levels of abstraction in multiple publications; see, e.g., [44, 74]. Especially this result for covariant POVMs has been presented in [13] and the case of covariant observables and instruments has been dealt with in [16]. However, the original form of this result presented in [28] is considerably more abstract. The set $\mathbf{NCP}_U^\beta$ is convex and the following result characterizes its extreme points [28]:

Theorem 9. *Let* $(\mathscr{M}, \pi, \tilde{U}, J)$ *be a* (β, U)*-covariant dilation for* $\Phi \in \mathbf{NCP}_U^\beta$*. One has* $\Phi \in \mathrm{ext}\,\mathbf{NCP}_U^\beta$ *if and only if the map* $\tilde{U}(G)' \cap \pi(\mathscr{A})' \ni D \mapsto J^*DJ \in \mathscr{L}(\mathscr{H})$ *is an injection. Again,* $\mathscr{D}'$ *is the commutant of the set* $\mathscr{D}$ *of operators.*

The representation $\tilde{U}$ in the above theorem can be replaced by the representation $\overline{U}$ of Theorem 8 in the case of an inner action. Indeed, in this case the commutants $\tilde{U}(G)' \cap \pi(\mathscr{A})'$ and $\overline{U}(G)' \cap \pi(\mathscr{A})'$ coincide.

Let now G be a lcsc Hausdorff group and Ω a lcsc Hausdorff Borel G-space so that $\Omega = G/H$ with a closed subgroup $H \leq G$. Also fix the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ and the ordinary strongly continuous unitary representations $U : G \to \mathscr{U}(\mathscr{H})$ and $V : G \to \mathscr{U}(\mathscr{K})$. One has the following corollary for covariant instruments [9, 28]. Note that $\tilde{\Gamma}$ stands for the dual instrument for an instrument Γ as defined in the beginning of Section 3.1.1.

Corollary 1. *Pick* $\Gamma \in \mathbf{Ins}_U^V(\Omega)$*. There is a Hilbert space* $\mathscr{K}_0$*, a strongly continuous unitary representation* $\pi : H \to \mathscr{U}(\mathscr{K}_0)$*, and hence the canonical system* $(\mathfrak{K}_\pi, \mathsf{P}_\pi, V_\pi)$ *of imprimitivity, and an isometry* $Y : \mathscr{H} \to \mathscr{K} \otimes \mathfrak{K}_\pi$ *such that*

$$\tilde{\Gamma}(X, B) = Y^*\big(B \otimes \mathsf{P}_\pi(X)\big)Y, \qquad X \in \mathscr{B}(\Omega), \quad B \in \mathscr{L}(\mathscr{K}),$$

and $YU(g) = \big(V(g) \otimes V_\pi(g)\big)Y$ *for all* $g \in G$ *and the linear span of the vectors* $\big(B \otimes \mathsf{P}_\pi(X)\big)Y\varphi$*,* $B \in \mathscr{L}(\mathscr{K})$*,* $X \in \mathscr{B}(\Omega)$*,* $\varphi \in \mathscr{H}$*, is a dense subspace of* $\mathscr{K} \otimes \mathfrak{K}_\pi$*. One has* $\Gamma \in \mathrm{ext}\,\mathbf{Ins}_U^V(\Omega)$ *if and only if for any* $A \in \mathscr{L}(\mathfrak{K}_\pi)$*, defined by a* $A_0 \in \mathscr{L}(\mathscr{K}_0)$ *such that* $(Af)(g) = A_0 f(g)$ *for all* $f \in \mathfrak{K}_\pi$ *and a.a.* $g \in G$ *and* $\pi(h)A_0 = A_0\pi(h)$ *for all* $h \in H$*, the condition* $Y^*(\mathbb{1}_{\mathscr{K}} \otimes A)Y = 0$ *implies*

$A = 0$. *Furthermore,* $\Gamma \in \mathrm{ext}\,\mathbf{Ins}\big(\mathscr{B}(\Omega), \mathscr{H}, \mathscr{K}\big)$ *if and only if the map*

$$(\mathrm{ran}\,\mathsf{P}_\pi)' \ni D \mapsto Y^*(\mathbb{1}_{\mathscr{K}} \otimes D)Y \in \mathscr{L}(\mathscr{H})$$

is an injection.

4.2 Structure of covariant observables

Next, we take a look at the properties and general structure of covariant observables and the characterization of the extreme points of the corresponding covariance structures. Two example cases are studied: observables whose symmetry group is a lcsc Abelian Hausdorff group and whose value space is an arbitrary transitive space for the group and observables whose symmetry group is a lcsc Hausdorff group of type I and whose value space is a transitive space for the group associated with a compact stability subgroup. The decomposition results exhibited briefly in Section 2.0.3 are used extensively especially in the study of the latter case. Also an example concentrating on Euclidean-covariant spin-0 localization observables is studied.

4.2.1 The case of an Abelian symmetry group

In this section, we study the structure and extreme points of the set of quantum observables that are covariant with respect to an Abelian group G the value space of which is a transitive G-space. Thus, we fix a lcsc Abelian Hausdorff group G with a closed subgroup $H \leq G$ and denote $\Omega = G/H$. Again, we denote $gH = \overline{g}$. The representation dual of G consisting of the characters $G \to \mathbb{T}$ is denoted by $\hat{G}$ that has the closed subgroup $H^\perp$, the annihilator of H, i.e., $\langle h, \eta\rangle = 1$ for all $\eta \in H^\perp$ whenever $h \in H$. Note that we treat $\hat{G}$ as an additive group in our notation, i.e., $\langle g, \gamma + \gamma'\rangle = \langle g, \gamma\rangle\langle g, \gamma'\rangle$ for all $g \in G$ and γ, $\gamma' \in \hat{G}$. We fix the Haar measures μ_G, μ_Ω, and $\mu_{H^\perp}$, where the last two are chosen so that the Fourier-Plancherel operator $\mathscr{F} : L^2(\mu_\Omega) \to L^2(\mu_{H^\perp})$,

$$(\mathscr{F}\varphi)(\eta) = \int_\Omega \langle g, \eta\rangle \varphi(\overline{g})\, d\mu_\Omega(\overline{g}), \qquad \varphi \in (L^1 \cap L^2)(\mu_\Omega), \quad \eta \in H^\perp,$$

is unitary. Also the coset space $\hat{G}/H^\perp$ is of importance, and we denote $\gamma + H^\perp =: \overline{\gamma}$ for all $\gamma \in \hat{G}$.

We fix a strongly continuous unitary representation $U : G \to \mathscr{U}(\mathscr{H})$ with $\mathscr{H}$ being our system Hilbert space. Because of the SNAG-theory, there is a measure $\nu_U : \mathscr{B}(\hat{G}) \to \mathbb{R}$ and a measurable field $\gamma \mapsto \mathscr{H}(\gamma)$ of Hilbert spaces such that $\mathscr{H} = \int_{\hat{G}}^\oplus \mathscr{H}(\gamma)\, d\nu_U(\gamma)$ and $\big(U(g)\varphi\big)(\gamma) = \langle g, \gamma\rangle\varphi(\gamma)$ for all $g \in G$, $\varphi \in \mathscr{H}$, and a.a. $\gamma \in \hat{G}$. The representations appearing in the covariant Naĭmark dilations of U-covariant observables are induced from representations $\pi : H \to \mathscr{U}(\mathscr{K}_0)$

with some Hilbert space $\mathscr{K}_0$. Such a representation is, likewise, associated with a Borel measure ν_π on $\hat{G}/H^\perp \simeq \hat{H}$ and a measurable field $\overline{\gamma} \mapsto \mathscr{K}(\overline{\gamma})$ such that $\mathscr{K}_0 = \int_{\hat{G}/H^\perp} \mathscr{K}(\overline{\gamma})\, d\nu_\pi(\overline{\gamma})$ and $\big(\pi(h)\xi\big)(\overline{\gamma}) = \langle h, \gamma\rangle \xi(\overline{\gamma})$ for all $h \in H$, $\xi \in \mathscr{K}_0$, and a.a. $\overline{\gamma} \in \hat{G}/H^\perp$.

Any measure $\nu : \mathscr{B}(\hat{G}/H^\perp) \to \mathbb{R}$ that is finite on compact sets gives rise to a measure $\tilde{\nu} : \mathscr{B}(\hat{G}) \to \mathbb{R}$ such that

$$\int_{\hat{G}} f\, d\tilde{\nu} = \int_{\hat{G}/H^\perp} \int_{H^\perp} f(\gamma + \eta)\, d\mu_{H^\perp}(\eta)\, d\nu(\overline{\gamma})$$

for all compactly supported continuous functions $f : \hat{G} \to \mathbb{C}$.

One can show [12] that the covariance structure $\mathbf{Obs}_U(\Omega)$ is non-empty if and only if there is a strongly continuous representation $\pi : H \to \mathscr{U}(\mathscr{K}_0)$, where $\mathscr{K}_0$ is a Hilbert space, associated with the canonical system of imprimitivity $(\mathfrak{K}_\pi, \mathsf{P}_\pi, V_\pi)$, and an isometry $J_\pi : \mathscr{H} \to \mathfrak{K}_\pi$ such that $J_\pi U(g) = V_\pi(g) J_\pi$ for all $g \in G$. When this is the case and ν_π is the Borel measure on $\hat{G}/H^\perp$ associated with π, one has $\nu_U \ll \tilde{\nu}_\pi$. From now on, let us assume that $\mathbf{Obs}_U(\Omega) \neq \emptyset$. Using the preceding observation, we may henceforth assume that there is a fixed Borel measure ν on $\hat{G}/H^\perp$ such that $\nu_U \ll \tilde{\nu}$, and, for simplicity, we may assume that $\nu_U = \tilde{\nu}$.[1]

Whenever $\varphi \in \mathscr{H}$, we write

$$\|\varphi_{\overline{\gamma}}\|_1 = \int_{H^\perp} \|\varphi(\gamma + \eta)\|\, d\nu_{H^\perp}(\eta)$$

for ν-a.a. $\overline{\gamma} \in \hat{G}/H^\perp$ whenever the integral exists. Let us denote by $\mathscr{D}$ the subset of vectors $\varphi \in \mathscr{H}$ such that

$$\int_{\hat{G}/H^\perp} \|\varphi_{\overline{\gamma}}\|_1^2\, d\nu(\overline{\gamma}).$$

This is a dense subspace of $\mathscr{H}$ [27]. Also note that $\mathscr{D}$ is invariant under U in the sense that $U(g)\mathscr{D} \subset \mathscr{D}$ for all $g \in G$. One can prove the following [12, 27]:

Theorem 10. *Fix an infinite-dimensional separable Hilbert space $\mathscr{M}$. For any $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$ there is a decomposable isometry $W = \int_{\hat{G}}^{\oplus} W(\gamma)\, d\tilde{\nu}(\gamma) : \mathscr{H} \to L^2_{\mathscr{M}}(\tilde{\nu})$, where $W(\gamma) : \mathscr{H}(\gamma) \to \mathscr{M}$ are isometries, such that defining $\Phi : \mathscr{D} \to L^2_{\mathscr{M}}(\nu)$ through*

$$(\Phi\varphi)(\overline{\gamma}) = \int_{H^\perp} W(\gamma + \eta)\varphi(\gamma + \eta)\, d\nu_{H^\perp}(\eta), \quad \varphi \in \mathscr{D}, \quad \overline{\gamma} \in \hat{G}/H^\perp,$$

[1] For otherwise, we could include the density function of ν_U with respect to $\tilde{\nu}$ in the component spaces $\mathscr{H}(\gamma)$.

one can write

$$
\begin{aligned}
&\langle\varphi|\mathsf{M}(X)\psi\rangle = \int_X \langle \Phi U(g)^*\varphi|\Phi U(g)^*\psi\rangle\, d\mu_\Omega(\overline{g}) \\
= \quad & \int_X \int_{\hat{G}/H^\perp} \int_{H^\perp} \int_{H^\perp} \langle g, \zeta-\xi\rangle \langle W(\gamma+\zeta)\varphi(\gamma+\zeta)|W(\gamma+\xi)\psi(\gamma+\xi)\rangle \times \\
\times \quad & d\mu_{H^\perp}(\zeta)\, d\mu_{H^\perp}(\xi)\, d\nu(\overline{\gamma})\, d\mu_\Omega(\overline{g}) \qquad (4.2)
\end{aligned}
$$

for all φ, $\psi \in \mathscr{D}$ and $X \in \mathscr{B}(\Omega)$; note that the integrands are independent of the choice of representatives within cosets in $\Omega = G/H$ and $\hat{G}/H^\perp$. Moreover, any M with this structure is an element in $\mathbf{Obs}_U(\Omega)$.

The covariance structure $\mathbf{Obs}_U(\Omega)$ contains sharp observables if and only if the function $\gamma \mapsto \dim \mathscr{H}(\gamma)$ is $\mu_{H^\perp}$-essentially constant on each coset $\overline{\gamma} \in \hat{G}/H^\perp$, and the $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$ like that above is sharp if and only if, for a.a. $\overline{\gamma} \in \hat{G}/H^\perp$ and a.a. $\gamma_1, \gamma_2 \in \overline{\gamma}$, the operator $W(\gamma_2)^*W(\gamma_1) : \mathscr{H}(\gamma_1) \to \mathscr{H}(\gamma_2)$ is unitary.

Note that the map Φ introduced in Theorem 10 is well defined on $\mathscr{D}$. One can show [27] that, given the decomposable isometry $W : \mathscr{H} \to L^2_{\mathscr{M}}(\tilde{\nu})$ and the associated $\Phi : \mathscr{D} \to L^2_{\mathscr{M}}(\nu)$ of Theorem 10, the closure $\mathbf{H} \subset L^2_{\mathscr{M}}(\nu)$ of the image space $\Phi(\mathscr{D})$ is of the direct-integral form $\mathbf{H} = \int^\oplus_{\hat{G}/H^\perp} \mathbf{H}(\overline{\gamma})\, d\nu(\overline{\gamma})$ with a measurable field $\hat{G}/H^\perp \ni \overline{\gamma} \mapsto \mathbf{H}(\overline{\gamma}) \subset \mathscr{M}$ of Hilbert spaces. Using this fact, one can give the following extremality characterizations [27].

Theorem 11. *Let $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$ be associated with the decomposable isometry W and the linear map Φ as in Theorem 10. Also assume that the direct-integral space $\mathbf{H}$ is defined as above. One has $\mathsf{M} \in \operatorname{ext} \mathbf{Obs}_U(\Omega)$ if and only if there is no non-zero decomposable $A \in \mathscr{L}(\mathbf{H})$,*

$$
A = \int^\oplus_{\hat{G}/H^\perp} A(\overline{\gamma})\, d\nu(\overline{\gamma}), \quad A(\overline{\gamma}) \in \mathscr{L}(\mathbf{H}(\overline{\gamma})), \quad \overline{\gamma} \in \hat{G}/H^\perp,
$$

such that for ν-a.a. $\overline{\gamma} \in \hat{G}/H^\perp$ and $\nu_{H^\perp}$-a.a. $\gamma' \in \overline{\gamma}$

$$
W(\gamma')^* A(\overline{\gamma}) W(\gamma') = 0.
$$

Moreover, one has $\mathsf{M} \in \operatorname{ext} \mathbf{Obs}\big(\mathscr{B}(\Omega), \mathscr{H}\big)$ if and only if there is no non-zero decomposable $D \in \mathscr{L}\big(L^2_{\mathbf{H}}(\mu_\Omega)\big)$, $(D\eta)(\omega) = D(\omega)\eta(\omega)$, $\eta \in L^2_{\mathbf{H}}(\mu_\Omega)$, $\omega \in \Omega$, *such that for some measurable section* $s:\Omega\to G$ *(i.e.,* $s(\omega)\in\omega$ *for all* $\omega\in\Omega$*)*

$$\begin{aligned} 0 &= \int_\Omega \langle \Phi(U\circ s)(\omega)^*\varphi | D(\omega)\Phi(U\circ s)(\omega)^*\psi\rangle\, d\mu_\Omega(\omega) \\ &= \int_\Omega\int_{\hat{G}/H^\perp}\int_{H^\perp}\int_{H^\perp} \langle \zeta-\xi, s(\omega)\rangle \times \\ &\times\ \langle W(\gamma+\zeta)\varphi(\gamma+\zeta)|D(\omega)W(\gamma+\xi)\psi(\gamma+\xi)\rangle \times \\ &\times\ d\mu_{H^\perp}(\xi)\, d\mu_{H^\perp}(\zeta)\, d\nu(\overline{\gamma})\, d\mu_\Omega(\omega) \end{aligned}$$

for all φ, $\psi\in\mathscr{D}$.

Using Theorems 10 and 11, one can construct extreme observables that are not sharp. Consider, e.g., the time evolution $U:\mathbb{R}\to\mathscr{U}\big(L^2(\mathbb{R})\big)$ of a free non-relativistic particle of mass m confined to a line given by $U(t)=e^{itH_0}$, where $H_0 = P^2/(2m)$ is the free Hamiltonian and P is the canonical momentum operator defined through $(P\varphi)(x)=-i\varphi'(x)$ for all absolutely continuous $\varphi\in L^2(\mathbb{R})$ and a.a. $x\in\mathbb{R}$. Because of the spectral structure of this representation, U does not support any covariant sharp observables with value space $\mathbb{R}$. Indeed, one can define the unitary map $L^2(\mathbb{R})\ni\varphi\mapsto\tilde{\varphi}\in L^2_{\mathbb{C}^2}[0,\infty)$ through

$$\tilde{\varphi}(\varepsilon)=\left(\frac{m}{2\varepsilon}\right)^{1/4}\big(\hat{\varphi}(\sqrt{2m\varepsilon}),\hat{\varphi}(-\sqrt{2m\varepsilon})\big),\quad \varphi\in L^2(\mathbb{R}),\quad 0\le\varepsilon<\infty \tag{4.3}$$

such that $\big(\widetilde{U(t)\varphi}\big)(\varepsilon)=e^{it\varepsilon}\tilde{\varphi}(\varepsilon)$ for all $t\in\mathbb{R}$, $\varphi\in L^2(\mathbb{R})$, and a.a. $\varepsilon\in[0,\infty)$. Thus we can see that, according to Theorem 10, the spectral structure of the time evolution does not allow covariant sharp observables, however, the covariance structure $\mathbf{Obs}_U(\mathbb{R})$ is non-empty.

The observables in $\mathbf{Obs}_U(\mathbb{R})$ are called as *time observables*, and any such observable is obtained by fixing a measurable field $[0,\infty)\ni\varepsilon\mapsto W(\varepsilon)$ of isometries $W(\varepsilon):\mathbb{C}^2\to\mathscr{M}$, where $\mathscr{M}$ is a fixed infinite-dimensional Hilbert space and constructing the operator Φ as in Theorem 10. Note that, in this case, the stability subgroup H is trivial. Picking two orthonormal vectors ζ_0, $\zeta_1\in\mathscr{M}$ and fixing $W(\varepsilon)|n\rangle=\zeta_n$ for $n=0$, 1 and all $\varepsilon\in[0,\infty)$, where $\{|0\rangle, |1\rangle\}$ is the natural basis of $\mathbb{C}^2$ in which the vector $\varphi(\varepsilon)$ is given in (4.3), one arrives at the canonical time observable τ given by

$$\begin{aligned} \langle\varphi|\tau(X)\psi\rangle &= \frac{1}{2\pi m}\int_X\int_0^\infty\int_0^\infty e^{\frac{it}{2m}(p_2^2-p_1^2)}\big(\hat{\varphi}(p_1)\hat{\psi}(p_2)+\hat{\varphi}(-p_1)\hat{\psi}(-p_2)\big)\times \\ &\times\ \sqrt{p_1p_2}\, dp_2\, dp_1\, dt \end{aligned} \tag{4.4}$$

for all $X\in\mathscr{B}(\mathbb{R})$ and all φ and ψ in the Schwartz space of rapidly decreasing functions. It is immediate that $\tau\in\operatorname{ext}\mathbf{Obs}_U(\mathbb{R})$ but τ is an extreme point of the whole set $\mathbf{Obs}\big(\mathscr{B}(\mathbb{R}),L^2(\mathbb{R})\big)$ of $\mathbb{R}$-valued observables as well [27]. Compare this result with the earlier finding stating that the canonical phase observable is extreme [36].

4.2.2 The case of a compact stability subgroup

This section deals with instruments and observables whose value space Ω is a transitive G-space for a symmetry group G such that the stability subgroup of G for each point in Ω is compact. It is yet unclear whether this technical assumption on compactness of the stability subgroup can be removed.

For the duration of this section, we fix a separable Hilbert space $\mathscr{H}$ and a unimodular lcsc type-I group G which is Hausdorff. Let $H \leq G$ be a compact subgroup and denote the space G/H of left H-cosets by Ω which we endow with the Borel σ-algebra $\mathscr{B}(\Omega)$ generated by the topology induced by G. We also pick Haar measures μ_G and μ_H for G and H, respectively, and assume, for simplicity, that μ_H is normalized, i.e., $\mu_H(H) = 1$. We also choose the G-invariant Borel measure μ_Ω on Ω such that for each $f \in L^1(\mu_G)$,

$$\int_G f\, d\mu_G = \int_\Omega \int_H f(gh)\, d\mu_H(h)\, d\mu_\Omega(\overline{g}).$$

We denote the Plancherel measure on $\mathscr{B}(\hat{G})$ defined by μ_G by $\mu_{\hat{G}}$. Let us pick a representative π_γ for each $\gamma \in \hat{G}$ with the representation space $\mathscr{K}(\gamma)$. Denote $\mathbb{1}_{\mathscr{K}(\gamma)} =: \mathbb{1}(\gamma)$. We denote the antilinear dual space of $\mathscr{K}(\gamma)$ by $\mathscr{K}(\gamma)^*$ and set up a bijection $\mathscr{K}(\gamma) \ni \zeta \mapsto \zeta^* \in \mathscr{K}(\gamma)^*$ where $\zeta^*(\xi) = \langle \zeta | \xi \rangle$ for all $\gamma \in \hat{G}$ and $\zeta, \xi \in \mathscr{K}(\gamma)$. Note that the space $\mathscr{K}(\gamma)^* \otimes \mathscr{K}(\gamma)$ can be identified with the Hilbert space of Hilbert-Schmidt operators on $\mathscr{K}(\gamma)$. Let us define measurable vector fields $\gamma \mapsto e_j(\gamma)$, $j = 1, 2, \ldots$, where $\{e_j(\gamma)\}_{j=1}^{\dim \mathscr{K}(\gamma)}$ is an orthonormal basis for $\mathscr{K}(\gamma)$ whenever $n(\gamma) := \dim \mathscr{K}(\gamma) > 0$ and $e_j(\gamma) = 0$ whenever $j > n(\gamma)$.

Let us fix a strongly continuous unitary representation $U : G \to \mathscr{U}(\mathscr{H})$. The type-I property of G insures that there is a Borel measure μ_U on $\hat{G}$ and a measurable field $\gamma \mapsto \mathscr{L}(\gamma)$ of Hilbert spaces such that $\mathscr{H} = \int_{\hat{G}}^{\oplus} \mathscr{K}(\gamma) \otimes \mathscr{L}(\gamma)\, d\mu_U(\gamma)$ and for all $g \in G$, $\varphi \in \mathscr{H}$, and μ_U-a.a. $\gamma \in \hat{G}$

$$\big(U(g)\varphi\big)(\gamma) = \big(\pi_\gamma(g) \otimes \mathbb{1}_{\mathscr{L}(\gamma)}\big)\varphi(\gamma).$$

It turns out that if the covariance structure $\mathbf{Obs}_U(\Omega)$ is non-empty, one has $\mu_U \ll \mu_{\hat{G}}$. This is exactly our assumption and, from now on, we simply assume that $\mu_U = \mu_{\hat{G}}$. The following structure result for U-covariant observables with values in Ω giving a generalization for the results obtained in [8, 45] is proved in [28].

Theorem 12. *There is a dense subspace $\mathscr{D} \subset \mathscr{H}$ that is U-invariant in the sense that $U(g)\mathscr{D} \subset \mathscr{D}$ for all $g \in G$ such that for each $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$ there is a* *separable Hilbert space* $\mathbf{H}$*, a linear map* $\Phi : \mathscr{D} \to \mathbf{H}$ *whose image* $\Phi(\mathscr{D})$ *is total in* $\mathbf{H}$*, and a strongly continuous unitary representation* $\rho : H \to \mathscr{U}(\mathbf{H})$ *such that* $\Phi U(h) = \rho(h)\Phi$ *for all* $h \in H$ *and*

$$\langle\varphi|\mathsf{M}(X)\psi\rangle = \int_X \langle\Phi U(g)^*\varphi|\Phi U(g)^*\psi\rangle \, d\mu_\Omega(\overline{g}) \tag{4.5}$$

for all φ*,* $\psi \in \mathscr{D}$ *and* $X \in \mathscr{B}(\Omega)$*. The map* Φ *has the following structure: there are measurable fields* $\gamma \mapsto \Phi_j(\gamma)$*,* $j = 1, 2, \ldots$*, of operators* $\Phi_j(\gamma) : \mathscr{L}(\gamma) \to \mathbf{H}$ *such that*

$$\sum_j \Phi_j(\gamma)^*\Phi_j(\gamma) = \mathbb{1}_{\mathscr{L}(\gamma)} \tag{4.6}$$

for a.a. $\gamma \in \hat{G}$ *and, defining for a.a.* $\gamma \in \hat{G}$ *the operator* $\Phi(\gamma) = \sum_j e_j(\gamma)^* \otimes \Phi_j(\gamma)$*,*

$$\Phi\varphi = \int_{\hat{G}} \Phi(\gamma)\varphi(\gamma) \, d\mu_{\hat{G}}(\gamma), \qquad \varphi \in \mathscr{D}.$$

For a covariant observable M *like that above, one has* $\mathsf{M} \in \operatorname{ext} \mathbf{Obs}_U(\Omega)$ *if and only if for an operator* $A \in \mathscr{L}(\mathbf{H})$ *the conditions* $\rho(h)A = A\rho(h)$ *for all* $h \in H$ *and*

$$\sum_j \Phi_j(\gamma)^* A \Phi_j(\gamma) = 0$$

for a.a. $\gamma \in \hat{G}$ *imply* $A = 0$*.*

Let the observable $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$ and the representation $\rho : H \to \mathscr{U}(\mathbf{H})$ be as in the theorem above. The decomposition obtained for M above implies that the canonical system of imprimitivity $(\mathfrak{K}_\rho, \mathsf{P}_\rho, V_\rho)$ together with an isometry $J : \mathscr{H} \to \mathfrak{K}_\rho$ constitutes a minimal Naĭmark dilation for M. The isometry J is defined through $(J\varphi)(g) = \Phi U(g)^*\varphi$ for all $\varphi \in \mathscr{D}$.

4.2.3 Example: Euclidean-covariant observables of a spin-0 object

Let us concentrate on a more detailed example on the results exhibited in Theorem 12. Consider an elementary spin-0 object whose relevant Hilbert space is $L^2(\mathbb{R}^3)$. Denote by $\mathscr{E}$ the semidirect product group $\mathbb{R}^3 Ⓢ SO(3)$, where the normal subgroup $\{0\} \times SO(3) \simeq SO(3)$ operates on $\mathbb{R}^3$ in the natural way, i.e., the group law is given by

$$(\mathbf{x}, R)(\mathbf{y}, S) = (\mathbf{x} + R\mathbf{y}, RS), \qquad (\mathbf{x}, R),\ (\mathbf{y}, S) \in \mathscr{E}.$$

The group $\mathscr{E}$ is the Euclidean group of rigid motions in a three-dimensional Euclidean space.

We study the Euclidean covariant observables of our spin-0 object with the value space $\mathbb{R}^3$; such observables could be considered as approximate localization observables. In the case of half-integer spin, one should consider the covering group of $\mathscr{E}$, instead. However, in our simple spin-0 case, the relevant symmetry action for our object is provided by the representation $U:\mathscr{E}\to\mathscr{U}\big(L^2(\mathbb{R}^3)\big)$,

$$\big(U(\mathbf{a},R)\varphi\big)(\mathbf{x})=\varphi\big(R^T(\mathbf{x}-\mathbf{a})\big)$$

for all $(\mathbf{a},R)\in\mathscr{E}$, $\varphi\in L^2(\mathbb{R}^3)$, and a.a. $\mathbf{x}\in\mathbb{R}^3$, and the covariance structure under study is $\mathbf{Obs}_U(\mathbb{R}^3)$, i.e., the value space is the coset space $\mathbb{R}^3\simeq\mathscr{E}/\big(\{0\}\times SO(3)\big)$.

The character group of the subgroup $\mathbb{R}^3$ is denoted by $\mathbb{P}^3$ and, as usual, we identify $\mathbb{P}^3\simeq\mathbb{R}^3$. We denote the unit ball in $\mathbb{P}^3$ by $\mathbb{S}^2$ where the group $SO(3)$ operates transitively. The $SO(3)$-space $\mathbb{S}^2$ possesses the invariant measure that in spherical coordinates (φ,ϑ) reads $\sin\vartheta\,d\varphi\,d\vartheta$; the associated L^2-space on $\mathbb{S}^2$ we denote simply as $L^2(\mathbb{S}^2)$. We denote elements in $\mathbb{S}^2$ (unit vectors) as $\hat{\mathbf{p}}$. We may define the irreducible unitary representations π_r, $r>0$, of $\mathscr{E}$ in $L^2(\mathbb{S}^2)$, e.g., using the Mackey machine [25] through

$$\big(\pi_r(\mathbf{a},R)\eta\big)(\hat{\mathbf{p}})=e^{-ir\mathbf{a}\cdot\hat{\mathbf{p}}}\eta(R^T\hat{\mathbf{p}})$$

for all $(\mathbf{a},R)\in\mathscr{E}$, $\eta\in L^2(\mathbb{S}^2)$, and a.a. $\hat{\mathbf{p}}\in\mathbb{S}^2$. Define the unitary map $L^2(\mathbb{R}^3)\ni\varphi\mapsto\tilde{\varphi}\in L^2\big([0,\infty),r^2\,dr;L^2(\mathbb{S}^2)\big)$, where for a.a. $r>0$ the vector $\tilde{\varphi}(r)$ is the function $\hat{\mathbf{p}}\mapsto\hat{\varphi}(r\hat{\mathbf{p}})$ where, in turn, $\hat{\varphi}$ is the Fourier-Plancherel transform for $\varphi\in L^2(\mathbb{R}^3)$, i.e., whenever $\varphi\in(L^1\cap L^2)(\mathbb{R}^3)$,

$$\hat{\varphi}(\mathbf{p})=(2\pi)^{-3/2}\int_{\mathbb{R}^3}e^{-i\mathbf{p}\cdot\mathbf{x}}\varphi(\mathbf{x})\,d\mathbf{x}$$

for a.a. $\mathbf{p}\in\mathbb{P}^3$. It is simple to check that

$$\big(\widetilde{U(\mathbf{a},R)\varphi}\big)(r)=\pi_r(\mathbf{a},R)\tilde{\varphi}(r)$$

for all $(\mathbf{a},R)\in\mathscr{E}$, $\varphi\in L^2(\mathbb{R}^3)$, and a.a. $r>0$, i.e., we have decomposed the representation U into its irreducible components. Note that the multiplicity spaces $\mathscr{L}(r)$ are now one-dimensional.

Fix an infinite-dimensional Hilbert space $\mathscr{M}$ and the orthonormal basis for $L^2(\mathbb{S}^2)$ consisting of the spherical harmonic functions Y_{lm}, $l=0,\,1,\,2,\ldots$, $m=-l,\ldots,\,l$,

$$Y_{lm}(\varphi,\vartheta)=(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(1-|m|)!}{(1+|m|)!}}P_l^{|m|}(\cos\vartheta)e^{im\varphi},$$

given in the spherical coordinates, $0\le\varphi<2\pi$, $0\le\vartheta\le\pi$, where $P_l^n:[-1,1]\to$

$\mathbb{R}$, $l = 0, 1, 2, \ldots$, $n = -l, \ldots, l$, are the associated Legendre polynomials given by

$$P_l^n(t) = (-1)^{l+n}\frac{(l+n)!}{(l-m)!}(1-t^2)^{-n/2}2^l l!\frac{d^{l-m}}{dt^{l-m}}\big((1-t^2)^l\big).$$

Define the projections

$$P_l = \sum_{m=-l}^{l} |Y_{lm}\rangle\langle Y_{lm}|$$

for all $l = 0, 1, 2, \ldots$. Also define for each $r > 0$ the representation $D_r : \mathbb{R}^3 \to \mathscr{L}\big(L^2(\mathbb{S}^2)\big)$, $\big(D_r(\mathbf{a})\eta\big)(\hat{p}) = e^{-ir\mathbf{a}\cdot\hat{\mathbf{p}}}\eta(\hat{\mathbf{p}})$ and denote by $C_c(\mathbb{R}^3)$ the subspace of continuous and compactly supported members of $L^2(\mathbb{R}^3)$. We are now ready to characterize all covariant observables $\mathsf{M} \in \mathbf{Obs}_U(\mathbb{R}^3)$ using the template given in Theorem 12.

Theorem 13. *For each $\mathsf{M} \in \mathbf{Obs}_U(\mathbb{R}^3)$ there are measurable fields $\xi_l : (0,\infty) \to \mathscr{M}$, $l = 0, 1, 2, \ldots$, such that for a.a. $r > 0$*

$$\sum_{l=0}^{\infty}(2l+1)\|\xi_l(r)\|^2 = 1 \tag{4.7}$$

and

$$\begin{aligned}\langle\varphi|\mathsf{M}(X)\psi\rangle &= \frac{1}{2\pi^2}\int_X\int_0^\infty\int_0^\infty\sum_{l=0}^{\infty}\langle\xi_l(r_1)|\xi_l(r_2)\rangle \times \\ &\times \langle D_{r_1}(\mathbf{a})^*\tilde{\varphi}(r_1)|P_l D_{r_2}(\mathbf{a})^*\tilde{\psi}(r_2)\rangle r_1^2 r_2^2\, dr_1\, dr_2\, d\mathbf{a}\end{aligned} \tag{4.8}$$

for all φ, $\psi \in C_c(\mathbb{R}^3)$ and $X \in \mathscr{B}(\mathbb{R}^3)$.

Proof. Since the multiplicities of the irreducible components of U are all 1, we can give the maps $\Phi(r) : L^2(\mathbb{S}^2) \to \mathscr{M}$, $r > 0$, of Theorem 12 in the form

$$\Phi(r)\eta = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\langle Y_{lm}|\eta\rangle\zeta_{lm}(r) \tag{4.9}$$

for all $r > 0$ and $\eta \in L^2(\mathbb{S}^2)$, where $\zeta_{lm} : [0,\infty) \to \mathscr{M}$ are measurable vector fields. Let us define $K(r_1, r_2) = \Phi(r_1)^*\Phi(r_2)$ for r_1, $r_2 > 0$, so that the invariance condition of Theorem 12 translates as

$$K(r_1, r_2)\pi_{r_2}(0, R) = \pi_{r_1}(0, R)K(r_1, r_2), \qquad r_1,\, r_2 > 0, \quad R \in SO(3). \tag{4.10}$$

Define the representation $L : SO(3) \to \mathscr{U}\big(L^2(\mathbb{S}^2)\big)$ through $\big(L(R)\eta\big)(\hat{\mathbf{p}}) = \eta(R^T\hat{\mathbf{p}})$ for all $R \in SO(3)$, $\eta \in L^2(\mathbb{S}^2)$ and a.a. $\hat{\mathbf{p}} \in \mathbb{S}^2$. It follows immediately that the condition (4.10) is equivalent to $K(r_1, r_2)$ being L-invariant. Since

$L = \bigoplus_{l=0}^{\infty} \rho_l$ where $\rho_l : SO(3) \to \mathscr{L}\big(P_l L^2(\mathbb{S}^2)\big)$ are the (odd-dimensional) irreducible representations of $SO(3)$, it follows that

$$K(r_1, r_2) = \sum_{l=0}^{\infty} \alpha_l(r_1, r_2) P_l, \qquad r_1,\, r_2 > 0, \tag{4.11}$$

where $\alpha_l(r_1, r_2)$ are complex numbers.

Comparing (4.9) with (4.11), one finds that $\zeta_{jk}(r_1) \perp \zeta_{lm}(r_2)$ whenever $j \neq l$ or $k \neq m$ and $\langle \zeta_{lm}(r_1) | \zeta_{lm}(r_2) \rangle$ does not depend on $m = -l, \dots, l$. Defining $\xi_l(r) := \zeta_{ll}(r)$, it is easy to check that the condition (4.6) equals (4.7), since the operators $\Phi_{lm}(r)$ on the trivial multiplicity spaces are of the form $1 \mapsto \zeta_{lm}(r) = \xi_l(r)$. The equation

$$\begin{aligned} \langle \varphi | \mathsf{M}(X) \psi \rangle \;&=\; \int_X \int_0^\infty \int_0^\infty \langle \pi_{r_1}(\mathbf{a}, \mathbb{1}_3)^* \tilde{\varphi}(r_1) | K(r_1, r_2) \pi_{r_2}(\mathbf{a}, \mathbb{1}_3)^* \tilde{\psi}(r_2) \rangle \times \\ &\times\; r_1^2 r_2^2 \, dr_1 \, dr_2 \, d\mathbf{a} \end{aligned}$$

for all $\varphi,\, \psi \in C_c(\mathbb{R}^3)$ and $X \in \mathscr{B}(\mathbb{R}^3)$ now implies (4.8). □

We did not deal with extremality within $\mathbf{Obs}_U(\mathbb{R}^3)$ in the preceding theorem since determining the representation ρ of the subgroup $\{0\} \times SO(3) \simeq SO(3)$ such that $\Phi U(0, R) = \rho(R) \Phi$ is a bit tricky. However, let us briefly study a family of Euclidean covariant observables that are extreme points of this covariance structure. For each $l = 0,\, 1,\, 2, \dots$ define $\mathsf{M}_l \in \mathbf{Obs}_U(\mathbb{R}^3)$ determined in (4.8) with a single non-zero constant vector field $\xi_l(r) = \xi$ such that $\|\xi\| = (2l{+}1)^{-1/2}$, i.e.,

$$\begin{aligned} \langle \varphi | \mathsf{M}_l(X) \psi \rangle \;&=\; \frac{1}{2\pi^2 (2l+1)} \int_X \int_0^\infty \int_0^\infty \langle D_{r_1}(\mathbf{a})^* \tilde{\varphi}(r_1) | P_l D_{r_2}(\mathbf{a}) \tilde{\psi}(r_2) \rangle \times \\ &\times\; r_1^2 r_2^2 \, dr_1 \, dr_2 \, d\mathbf{a} \end{aligned}$$

for all $\varphi,\, \psi \in C_c(\mathbb{R}^3)$ and $X \in \mathscr{B}(\mathbb{R}^3)$.

Let ρ_l, $l = 0,\, 1,\, 2, \dots$, be the irreducible representations of $SO(3)$ and $W_l : \mathbb{C}^{2l+1} \to L^2(\mathbb{S}^2)$ isometries such that $W_l W_l^* = P_l$ and $W_l \rho_l(R) = L(R) W_l$ for all $R \in SO(3)$ where $L : SO(3) \to \mathscr{L}\big(L^2(\mathbb{S}^2)\big)$ is the representation introduced in the proof of Theorem 13. For each ρ_l, the triplet $(\mathfrak{K}_l, \mathsf{P}_l, V_l)$ is the canonical imprimitivity system where $V_l = \mathrm{ind}_{SO(3)}^{\mathscr{E}} \rho_l$. We express the space $\mathfrak{K}_l$ as the L^2-space of functions $f : \mathbb{R}^3 \to \mathbb{C}^{2l+1}$ where

$$\big(V_l(\mathbf{a}, R) f\big)(\mathbf{x}) = \rho_l(R) f\big(R^T(x - a)\big)$$

for all $(\mathbf{a}, R) \in \mathscr{E}$, $f \in \mathfrak{K}_l$, and a.a. $\mathbf{x} \in \mathbb{R}^3$ and $\mathsf{P}_l(X) f = \chi_X f$ for all $X \in \mathscr{B}(\mathbb{R}^3)$

and $f \in \mathfrak{K}_l$. Moreover, let us define the isometries $J_l : L^2(\mathbb{R}^3) \to \mathfrak{K}_l$ through

$$(J_l\varphi)(\mathbf{x}) = \big(2\pi^2(2l+1)\big)^{-1/2} \int_0^\infty W_l \pi_r(\mathbf{x}, \mathbb{1}_3)\tilde{\varphi}(r) r^2\, dr$$

for all $\varphi \in C_c(\mathbb{R}^3)$ and a.a. $\mathbf{x} \in \mathbb{R}^3$.

It is immediate that $(\mathfrak{K}_l, \mathsf{P}_l, V_l, J_l)$ is a minimal covariant dilation for M_l. Any $A \in \mathscr{L}(\mathfrak{K}_l)$ that commutes with both P_l and V_l is of the form $a\mathbb{1}_{\mathfrak{K}_l}$ with $a \in \mathbb{C}$ (the condition on commuting with V_l enables the use of Schur's lemma), and thus it follows that for such an operator A one has $J_l^* A J_l = 0$ if and only if $A = 0$ implying, according to Theorem 9, that $\mathsf{M}_l \in \operatorname{ext} \mathbf{Obs}_U(\mathbb{R}^3)$ for all l. Direct calculation shows that M_0 is the sharp observable given by $\mathsf{M}_0(X)\varphi = \chi_X \varphi$ for all $X \in \mathscr{B}(\mathbb{R}^3)$ and $\varphi \in L^2(\mathbb{R}^3)$.

4.3 Structure of covariant instruments

We assume that G is a lcsc Hausdorff group, $H \leq G$ is a closed subgroup, and $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces. Furthermore, let $U : G \to \mathscr{U}(\mathscr{H})$ and $V : G \to \mathscr{U}(\mathscr{K})$ be strongly continuous unitary representations (not projective, for simplicity). Denote $\Omega = G/H$. We make the following extra assumptions for the remainder of this chapter.

Requirements. We require that there be a dense subspace $\mathscr{D} \subset \mathscr{H}$ which is invariant under U such that each covariant observable $\mathsf{M} \in \mathbf{Obs}_U(\Omega)$ allows a Kolmogorov decomposition like that in Equations (4.2) and (4.5), i.e., fixing a quasi-G-invariant Borel measure μ_Ω on Ω, there be a linear map $\Phi : \mathscr{D} \to \mathbf{H}$ with a Hilbert space $\mathbf{H}$ intertwining $U|_H$ with a unitary representation $\pi : H \to \mathscr{U}(\mathbf{H})$ such that

$$\langle \varphi | \mathsf{M}(X)\psi \rangle = \int_X \langle \Phi U(g)^* \varphi | \Phi U(g)^* \psi \rangle \, d\mu_\Omega(\overline{g})$$

for all φ, $\psi \in \mathscr{D}$ and $X \in \mathscr{B}(\Omega)$.

These requirements are, naturally, met in the case where G is Abelian or G is of type I and unimodular and H is compact. Similar conditions also hold for the case of non-unimodular symmetry groups with square-integrable U studied, e.g., in [52].

We can prove the following structure theorem for covariant instruments $\Gamma \in \mathbf{Ins}_U^V(\Omega)$ [28]:

Theorem 14. *Let us retain the assumptions and definitions made above. For each $\Gamma \in \mathbf{Ins}_U^V(\Omega)$, there are linear operators $K_j : \mathscr{D} \to \mathscr{K}$, $j = 1, 2, 3, \ldots$ such that $\sum_j \langle K_j \varphi | V(h)^* B V(h) K_j \psi \rangle = \sum_j \langle K_j U(h) \varphi | B K_j U(h) \psi \rangle$ for all φ, $\psi \in \mathscr{D}$,*

$h \in H$, and $B \in \mathscr{L}(\mathscr{K})$,

$$\int_{\Omega} \sum_j \langle K_j U(g)^* \varphi | K_j U(g)^* \varphi \rangle \, d\mu_\Omega(\overline{g}) = \|\varphi\|^2 \tag{4.12}$$

for all $\varphi \in \mathscr{D}$ and

$$\langle \varphi | \tilde{\Gamma}(X,B)\psi \rangle = \int_X \sum_j \langle V(g) K_j U(g)^* \varphi | B V(g) K_j U(g)^* \psi \rangle \, d\mu_\Omega(\overline{g}) \tag{4.13}$$

for all φ, $\psi \in \mathscr{D}$, $X \in \mathscr{B}(\Omega)$, and $B \in \mathscr{L}(\mathscr{K})$.

A particular exemplary case, where the theorem above can be simplified is an instance where the representation U is *square integrable* in the sense that there is a finite constant $d > 0$ such that for all unit vectors φ, $\psi \in \mathscr{H}$

$$\int_G |\langle \varphi | U(g)\psi \rangle|^2 \, d\mu_G(g) = d.$$

In this case, the operators K_j associated to $\Gamma \in \mathbf{Ins}_U^V(\Omega)$ according to Theorem 14 are, in fact, Hilbert-Schmidt operators and the normalization condition (4.12) can be written in the form $\sum_j \mathrm{tr}[K_j^* K_j] = 1/d$ [53], [28]. Denoting the positive trace-1 operator $d \sum_j K_j^* K_j$ on $\mathscr{H}$ by T, one can write for the observable margin M associated with Γ

$$\langle \varphi | \mathsf{M}(X)\psi \rangle = \frac{1}{d} \int_X \langle U(g)^* \varphi | T U(g)^* \psi \rangle \, d\mu_\Omega(\overline{g})$$

for all φ, $\psi \in \mathscr{H}$ and $X \in \mathscr{B}(\Omega)$.

Note that carrying out the standard extension procedure $\mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{T} =: G$ for the Abelian group $\mathbb{R}^2$ resulting in the Schrödinger group and lifting the projective Weyl-representation W introduced as an example in the beginning of this chapter into an ordinary square-integrable ($d = 2\pi$) unitary representation $\overline{W} : G \to \mathscr{U}\big(L^2(\mathbb{R})\big)$ we are now able to give an exhaustive description for the $\mathbb{R}^2 = G/(\{(0,0)\} \times \mathbb{T})$-valued observables and instruments in $\mathbf{Obs}_W(\mathbb{R}^2)$ and $\mathbf{Ins}_W^W(\mathbb{R}^2)$, respectively. Indeed, for any $\mathsf{M} \in \mathbf{Obs}_W(\mathbb{R}^2)$, there is a trace-1 positive operator T on $L^2(\mathbb{R})$ such that Equation (4.1) holds, and for any instrument $\Gamma \in \mathbf{Ins}_W^W(\mathbb{R}^2)$ having M as its margin, there is a decomposition $T = 2\pi \sum_j K_j^* K_j$ with Hilbert-Schmidt operators K_j on $L^2(\mathbb{R})$ such that

$$\tilde{\Gamma}(X,B) = \int_X \sum_j W(q,p) K_j^* W(q,p)^* B W(q,p) K_j W(q,p)^* \, dq \, dp$$

for all $X \in \mathscr{B}(\mathbb{R}^2)$ and $B \in \mathscr{L}\big(L^2(\mathbb{R})\big)$. In this particular case we do not have to require any covariance property for the operators K_j with respect to the subgroup $H = \{(0,0)\} \times \mathbb{T}$ since the restriction $\overline{W}|_H$ only has values in the centre of $\mathscr{L}\big(L^2(\mathbb{R})\big)$.

Chapter 5

Convex-geometric measures

The convex geometry of the quantum structures can be directly used to define measures for discriminating quantum devices from others. In this chapter, we introduce two such classes of measures: the boundariness and robustness measures. The first one is used to quantify how close a device is to the boundary of the structure and we show that this measure is associated with quantum discrimination related tasks. The latter one measures the distance of a device to the boundary of a subset of devices within the same structure, and they can be used to discriminate devices from the given subset. We apply these robustness measures to quantifying how incompatible a given pair of quantum devices is.

5.1 Boundariness and robustness measures

Let us fix a real (or complex) vector space V where K and let F to be the minimal affine subspace containing K. Let us define a preorder $\leq_C$ in K by defining $x \leq_C y$ for $x, y \in K$ if there is $t \in (0, 1]$ and $z \in K$ such that $y = tx + (1 - t)z$, i.e., x appears in a convex decomposition of y with non-zero weight. The singletons consisting of the extreme points of K (if such exist) are the minimal elements according to this order. The maximal equivalence class of K is the (algebraic) interior $\operatorname{int} K$ of K and all the rest constitutes the (algebraic) boundary ∂K of K. Hence, for $y \in K$, one has $y \in \partial K$ if and only if there is $x \in K$ such that $x \not\leq_C y$.

The number $t_y(x) \in [0, 1]$ for all $x, y \in K$ was defined originally in [32] as the supremum

$$t_y(x) = \sup\{t \in [0, 1) \,|\, (1 - t)^{-1}(y - tx) \in K\}.$$

Thus $t_y(x)$ is the supremum of the possible weights of x in a convex decomposition of y. It is elementary to show that this supremum is obtained at a boundary point $\big(1 - t_y(x)\big)\big(y - t_y(x)x\big) \in \partial K$ whenever $x \neq y$ and K is a convex and compact subset of a locally convex vector space V. We call the function $t_y : K \to [0, 1]$ as the *weight function associated with* y. One finds out

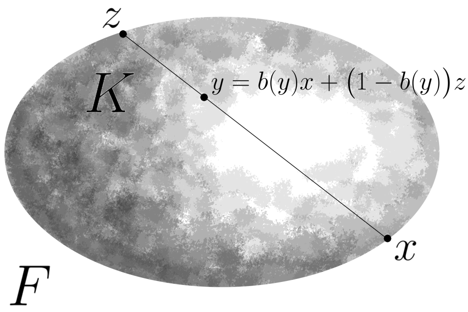

Figure 5.1: The element $y \in K$ is expressed as an optimal decomposition where $x \in \operatorname{ext} K$ is the point fractionally furthest away from y and z is the boundary point of K that is the closest of all boundary points to y.

that the inverse $x \mapsto 1/t_y(x)$ is a convex function and, additionally, whenever V is finite-dimensional, the weight function is continuous on K [32].

Let $y \in Z$ and denote

$$b(y) = \inf_{x \in K} t_y(x).$$

We call the function $b : K \to [0, 1/2]$ as the *boundariness* and from the properties of the weight function mentioned above it follows that, whenever V is finite-dimensional and K is compact and convex, for any $y \in K$, there is an extreme point $x \in \operatorname{ext} K$ such that $b(y) = t_y(x)$, and thus further, there exists a boundary point $z \in \partial K$ such that $y = b(y)x + \big(1 - b(y)\big)z$ [32]. The boundariness $b(y)$ is a measure of the distance of y to the boundary ∂K and, in the case of a compact K, $b(y) = 0$ if and only if $y \in \partial K$.

Let $p : V \to \mathbb{R}$ be a seminorm with the property that there is a finite constant $a \geq 0$ such that $p(x) \leq a$ for all $x \in K$. It follows that for any x, $y \in K$ one has [32]

$$p(x - y) \leq 2a\big(1 - b(y)\big). \tag{5.1}$$

Let now K be a base for a pointed generating cone C for V.[1] In these settings,

[1] Recall that $C \subset V$ is a cone when it is convex and $\alpha v \in C$ for all $\alpha \geq 0$ and $v \in C$. The cone is pointed when $C \cap (-C) = \{0\}$ and generating if $C - C = V$. A convex set $K \subset C$ is a base for the cone C if for any $v \in C$ there are unique $\alpha \geq 0$ and $x \in K$ such that $v = \alpha x$.

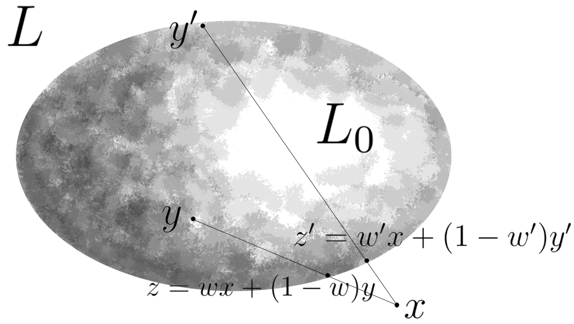

Figure 5.2: Consider, as is illustrated, points $x \in L$ and $y,\, y' \in L_0$ situated so that y' is fractionally further away from x than y is in the sense that, when we consider the boundary points z and z' found where the line segment connecting x to y and, respectively, to y' meets the boundary of L_0, x has a higher weight in z' than in z, i.e., $w(x|y) = w' > w = w(x|y')$. It follows, as can be seen in the illustration, that these further-away points are located towards the boundary of the 'other end' of L_0 seen from x and, hence, it is here where the relative robustness function approaches $w_L(x)$.

one can define the *base norm* $\|\cdot\|_K : V \to \mathbb{R}$ through

$$\|v\|_K = \inf_{\lambda,\, \mu \geq 0} \{\lambda + \mu \,|\, v = \lambda x - \mu y,\ x,\, y \in K\}.$$

It follows that $\|x\|_K \leq 1$ for all $x \in K$ so that from Equation (5.1) it follows that $b(y) \leq 1 - \frac{1}{2}\|x - y\|_K$ for all $x,\, y \in K$. It can, in fact, be shown [71] that this bound is tight, so that

$$b(y) = 1 - \frac{1}{2} \sup_{x \in K} \|x - y\|_K, \qquad y \in K. \tag{5.2}$$

Let us move on to study another type of measures arising from convex structures. Let now $L_0 \subset V$ be a convex set with the minimal affine subspace F. We denote the relative complement $F \setminus L_0$ by L. We want to define a measure of how distant a point $x \in F$ is from L_0. To this end, let us denote

$$w_L(x|y) = \sup\{w \in [0,1] \,|\, wx + (1-w)y \in L_0\}$$

for all $x,\, y \in F$ where $\sup \emptyset$ is defined to be 0. We call $w_L(x|y)$ as the *relative L-robustness of x relative to y*. It can be shown that, for fixed y and x, the

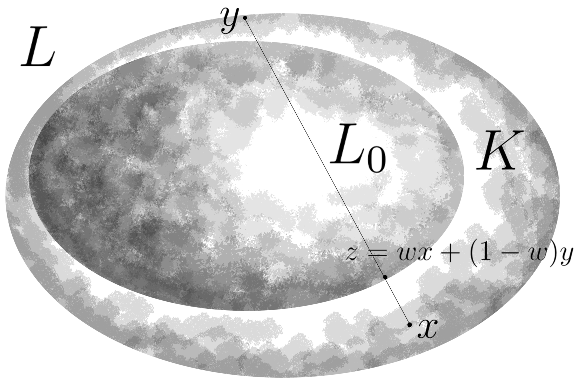

Figure 5.3: The absolute (K, L)-robustness of a point $x \in K \setminus L_0$ (i.e., x is in the outer layer of the highlighted area in the illustration) is obtained by considering the points y of K being fractionally as far away from x as possible in sense clarified in Figure 5.2. However, now we do not require y to be in L_0. Clearly, the optimizing y is on the boundary of K.

function $1/w_L(\cdot|y)$ is convex and $1/\big(1 - w_L(x|\cdot)\big)$ is concave [26].

Furthermore, one may set up the function $w_L : F \to [0, 1]$,

$$w_L(x) = \sup_{y \in L_0} w_L(x|y), \qquad x \in F.$$

If $K \subset F$ is a convex set such that $L_0 \subset K$, one can also define the function $w_L^K : F \to [0, 1]$,

$$w_L^K(x) = \sup_{y \in K} w_L(x|y), \qquad x \in F.$$

We call the measure w_L (respectively w_L^K) as the *(absolute) L-robustness* (respectively *(absolute)* (K, L)*-robustness*). They both measure the distance of points $x \in F$ to L_0 in the sense that, whenever V is a locally convex vector space, these measures acquire the value 1 if and only if x is in the closure of L_0. One can think of $w_L(x)$ as the least amount of noise from L_0 that has to be added to x in order to make it indiscernible from L_0 or as the greatest amount of noise from L_0 that x tolerates without being immersed in L_0. The same can be said about $w_L^K(x)$ with the exception that we allow for noise from the whole of the larger set K. The L-robustness is geometrically (and often physically) better motivated but the (K, L)-robustness is typically easier to calculate. Both $1/w_L(\cdot)$ and $1/w_L^K(\cdot)$ are convex functions [26].

One can, additionally, define the associated convex robustness measures

$R_L = 1/w_L(\cdot) - 1$ and $R_L^K = 1/w_L^K(\cdot) - 1$ that acquire the value 0 if and only if their argument is in the closure of L_0. This is the form in which the robustness measures (robustness of entanglement) have typically been introduced in literature.

5.2 Boundariness and quantum discrimination

For the duration of this section, we fix the separable Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ and an integer $N > 0$. We study the boundary and boundariness measure associated with the convex quantum structures of states $\mathscr{S}(\mathscr{H})$, N-outcome observables $\mathbf{Obs}_N(\mathscr{H})$, and channels $\mathbf{Ch}(\mathscr{H}, \mathscr{K})$. The value space of the observables in $\mathbf{Obs}_N(\mathscr{H})$ is the finite set $\{1, \ldots, N\}$ equipped with its power set for the σ-algebra. We denote elements $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$ as ordered N-tuples $(\mathsf{M}_j)_{j=1}^N$ such that $\mathsf{M}_j = \mathsf{M}(\{j\})$.

It is easy to show in the finite-dimensional case that the boundary $\partial\mathscr{S}(\mathscr{H})$ of the set of states consists of those state operators that have 0 as their eigenvalue. The direct generalization of this holds also in the infinite-dimensional case [32]: if $\dim\mathscr{H} = \infty$, then $\mathscr{S}(\mathscr{H}) = \partial\mathscr{S}(\mathscr{H})$, which is quite remarkable. Moreover, the boundariness $b(\rho)$ for a state ρ coincides with the minimum of the spectrum $\mathrm{sp}\,\rho$ of ρ [32]. Similar results were also proven in [32] to hold for the set $\mathbf{Obs}_N(\mathscr{H})$ of observables: an observable $\mathsf{M} \in \mathbf{Obs}_N(\mathscr{H})$ is on the boundary $\partial\mathbf{Obs}_N(\mathscr{H})$ if and only if $0 \in \mathrm{sp}\,\mathsf{M}_j$ for some j and $b(\mathsf{M}) = \min_j \min \mathrm{sp}\,\mathsf{M}_j$.

The structure of the set of channels is more complicated. Again, if $\dim\mathscr{K} = \infty$, one has $\mathbf{Ch}(\mathscr{H}, \mathscr{K}) = \partial\mathbf{Ch}(\mathscr{H}, \mathscr{K})$ [32]. Let us assume that $\dim\mathscr{H}, \dim\mathscr{K} < \infty$, $\dim\mathscr{H} = d$, and fix an orthonormal basis $\{|n\rangle\}_{n=1}^d$ for $\mathscr{H}$. For any channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H}, \mathscr{K})$, define the Choi-Jamiołkowski operator

$$J_{\mathscr{E}} = d^{-1} \sum_{j,k=1}^{d} \mathscr{E}(|j\rangle\langle k|) \otimes |j\rangle\langle k| \in \mathscr{L}(\mathscr{K} \otimes \mathscr{H}).$$

It turns out that $\mathscr{E} \in \partial\mathbf{Ch}(\mathscr{H}, \mathscr{K})$ if and only if $0 \in \mathrm{sp}\,J_{\mathscr{E}}$ and $b(\mathscr{E}) \geq d \min \mathrm{sp}\,J_{\mathscr{E}}$ [32].

The boundariness measure has a natural connection to quantum discrimination related questions. In a discrimination task, we are given a quantum device which is either one of the known devices Φ and Ψ (which are states, observables, or channels) and we have to determine which one it is by a measurement. Such a task cannot typically be carried out without a risk of failure and the minimum probability $p_{\mathrm{error}}(\Phi, \Psi)$ of failure is [40]

$$p_{\mathrm{error}}(\Phi, \Psi) = \frac{1}{2}\Big(1 - \frac{1}{2}\|\Phi - \Psi\|\Big)$$

where the norm is the base norm defined by the set of devices considered as a basis of a generating cone for the ambient vector space [48]. Because of the result given in Equation (5.2), we may say the following:

Theorem 15. *For any quantum device Φ of any of the sets $\mathscr{S}(\mathscr{H})$, $\mathbf{Obs}_N(\mathscr{H})$, and $\mathbf{Ch}(\mathscr{H},\mathscr{K})$ with finite-dimensional $\mathscr{H}$ and $\mathscr{K}$, there is another device Ψ (of the same type) that is an extreme point of the set of devices such that Φ is best discriminable from Ψ in the sense*

$$p_{\mathrm{error}}(\Phi,\Psi) = \min_{\Psi'} p_{\mathrm{error}}(\Phi,\Psi') = \frac{1}{2}b(\Phi).$$

This means, especially, that the devices on the boundary are best discriminable from all the rest of the devices of the same type. This result was proven to hold for the sets of states and observables in [32]. The result in all its generality has been proven in [71]. Given a device Φ, the extreme device Ψ that is the best discriminable from Φ with respect to the error probability can be restricted to a particular subset: In the case of observables, the best discriminable observable can be chosen to be sharp [32]. Moreover, in the case of channels with identical input and output spaces, the best discriminable channel can be chosen amongst the unitary channels [71]. In [71], a method for calculating the boundariness of a channel $\mathscr{E} \in \mathbf{Ch}(\mathscr{H},\mathscr{H})$ with a finite-dimensional $\mathscr{H}$, $\dim\mathscr{H} = d$, was given: If $\mathscr{E}$ is not a boundary point of $\mathbf{Ch}(\mathscr{H},\mathscr{H})$, then the Choi-Jamiołkowski operator $J_{\mathscr{E}}$ is invertible, and one can show that the boundariness for such a channel is given by

$$b(\mathscr{E}) = \frac{d}{\max_{U\in\mathscr{U}(\mathscr{H})}\langle\psi_U|J_{\mathscr{E}}^{-1}\psi_U\rangle},$$

where the maximization runs over all unitary operators U on $\mathscr{H}$ and $\psi_U = d^{-1/2}\sum_{j=1}^{d}(U\otimes\mathbb{1}_{\mathscr{H}})|jj\rangle$. When U is the unitary operator giving the maximum in the above formula, the unitary channel $\mathscr{U} : \rho \mapsto U\rho U^*$ is a channel that is the best discriminable from $\mathscr{E}$ in the sense that $\mathscr{E} = b(\mathscr{E})\mathscr{U} + \big(1-b(\mathscr{E})\big)\mathscr{F}$ with some (boundary) channel $\mathscr{F}$.

5.3 Robustness of incompatibility

We turn our attention again to compatibility questions. We use the robustness measures introduced in the beginning of this chapter to set up a measure of incompatibility of a pair of quantum devices generalizing the ideas presented in [6, 10, 38]; earlier, similar measures have been used to measure quantum entanglement for states [79]. The results of this section are from [26].

Let us pick the sets $\mathbf{Q}_1$ and $\mathbf{Q}_2$ of quantum devices, namely, $\mathbf{Q}_1$ is either $\mathbf{Obs}(\Sigma,\mathscr{H})$ or $\mathbf{Ch}(\mathscr{H},\mathscr{K}_1)$ and $\mathbf{Q}_2$ is either $\mathbf{Obs}(\Sigma',\mathscr{H})$ or $\mathbf{Ch}(\mathscr{H},\mathscr{K}_2)$ with separable Hilbert spaces $\mathscr{H}$, $\mathscr{K}_1$, and $\mathscr{K}_2$ and standard Borel spaces (Ω,Σ) and

(Ω', Σ'). Let us define the naturally convex set $K = \mathbf{Q}_1 \times \mathbf{Q}_2$ containing the convex subset L_0 of pairs $(\Phi, \Psi) \in K$ such that Φ and Ψ are compatible. Denote by L the relative complement $F \setminus L_0$ where F is the minimal affine subspace of the ambient space containing K. This affine subspace is, in fact, minimal also for L_0. For simplicity, we denote the robustness measures $w_L =: w$ and $w_L^K =: W$, and we call both of them as *robustness of incompatibility*. Hence, $w(\Phi, \Psi)$ (respectively $W(\Phi, \Psi)$) is the maximum amount of noise from the set of compatible pairs (respectively from the entire set of all device pairs) the pair (Φ, Ψ) of devices tolerates before being rendered compatible.

It can be shown that, for any device pair (Φ, Ψ), one has $w(\Phi, \Psi) \geq 1/2$ and similarly for W. When the (input) Hilbert space of the devices is d-dimensional, $d < \infty$, we have an even tighter bound: $w(\Phi, \Psi) \geq (d+2)/\big(2(d+1)\big)$. The same bound applies also for the measure W.

We say that a channel $\mathscr{C}$ is a post-processing of a channel $\mathscr{E}$ if there is a third channel $\mathscr{A}$ such that $\mathscr{C} = \mathscr{A} \circ \mathscr{E}$. This definition for post-processing is analogous with the post-processing order for observables. In general, if a quantum device Φ is a post-processing of another device Φ_0 and a device Ψ is a post-processing of a device Ψ_0, we denote $(\Phi, \Psi) \leq (\Phi_0, \Psi_0)$. The measures W and w behave monotonously with respect to this preorder of device pairs, namely, if $(\Phi, \Psi) \leq (\Phi_0, \Psi_0)$ then $w(\Phi, \Psi) \geq w(\Phi_0, \Psi_0)$ and similarly for W. This means that post-processing makes pairs of devices 'more compatible'. This is natural since post-processing is a zero-resource operation with respect to incompatibility, i.e., when a pair (Φ_0, Ψ_0) is compatible and $(\Phi, \Psi) \leq (\Phi_0, \Psi_0)$, then (Φ, Ψ) is compatible as well.

This observation tells us, especially, that the least compatible observable pairs are found among the post-processing maximal (i.e., rank-1) observables. Let us consider an example of a pair of rank-1 observables: Now $\mathscr{H}$ is a d-dimensional, $d < \infty$, Hilbert space with a fixed orthonormal basis $\{\varphi_n\}_{n=1}^d$. We fix another basis $\{\psi_m\}_{m=1}^d$, where $\psi_m = \mathscr{F}\varphi_m$ with the Fourier-operator $\mathscr{F} \in \mathscr{L}(\mathscr{H})$,

$$\mathscr{F}\varphi_n = \frac{1}{\sqrt{d}} \sum_{m=1}^{d} \langle m, n \rangle \varphi_m, \qquad n = 1, \ldots, d,$$

where $\langle m, n \rangle = e^{i2\pi mn/d}$. We fix the sharp finite-dimensional Weyl pair (Q, P) where Q and P are sharp rank-1 d-outcome observables on $\mathscr{H}$ given by

$$\mathsf{Q}_n = |\varphi_n\rangle\langle\varphi_n|, \quad \mathsf{P}_m = |\psi_m\rangle\langle\psi_m|, \qquad n, m = 1, \ldots, d.$$

One can prove the following:

Theorem 16. *The robustness of incompatibility for the sharp Weyl pair* (Q, P)

is

$$W(\mathsf{Q},\mathsf{P})=\frac{1}{2}\Big(1+\frac{1}{\sqrt{d}}\Big).$$

It is, however, not clear whether this value is actually the minimum value for the robustness measure, i.e., whether the pair (Q,P) is the least compatible pair of observables in $\mathscr{H}$.

On the channel side, the least compatible pairs of channels are those in the post-processing equivalence class of the pair $(\mathrm{id},\mathrm{id})$ defined by the identity channel $\mathrm{id}:\mathscr{S}(\mathscr{H})\to\mathscr{S}(\mathscr{H})$, $\rho\mapsto\rho$. This class of incompatible channel pairs contain, especially, any pair $(\mathscr{U},\mathscr{V})$ of unitary channels $\mathscr{U}$, $\mathscr{V}\in\mathbf{Ch}(\mathscr{H},\mathscr{H})$, i.e., there are unitary operators U, $V\in\mathscr{L}(\mathscr{H})$ such that $\mathscr{U}(\rho)=U\rho U^*$ and $\mathscr{V}(\rho)=V\rho V^*$ for all $\rho\in\mathscr{S}(\mathscr{H})$. Generally, this equivalence class consists of pairs of channels $\mathscr{W}\in\mathbf{Ch}(\mathscr{H},\mathscr{K})$ with varying $\mathscr{K}$ (however, $\dim\mathscr{K}\geq\dim\mathscr{H}$) that are *decodable* in the sense that there is a channel $\mathscr{B}\in\mathbf{Ch}(\mathscr{K},\mathscr{H})$ such that $\mathscr{B}\circ\mathscr{W}=\mathrm{id}$; the exhaustive characterization of this equivalence class (albeit in a considerably more general context) when the input and output spaces are finite dimensional can be found in [49]. The decodable channels are also called as *invertible* or *correctable*. All pairs $(\mathscr{U},\mathscr{V})$ consisting of such channels minimize both w and W among the channel pairs with the input Hilbert space $\mathscr{H}$. Moreover, one can calculate this minimum value:

Theorem 17. *Let $\mathscr{H}$ and $\mathscr{K}$ be finite-dimensional,* $\dim\mathscr{H}=:d$, *and* $\mathscr{U}$, $\mathscr{V}\in\mathbf{Ch}(\mathscr{H},\mathscr{K})$ *be decodable. One has*

$$W(\mathscr{U},\mathscr{V})=\frac{1}{2}\Big(1+\frac{1}{d}\Big).$$

The least compatible observable-channel pairs are amongst the set of pairs consisting of a rank-1 observable and a decodable channel. With a fixed observable $\mathsf{M}\in\mathbf{Obs}(\Sigma,\mathscr{H})$, one has

$$\min_{\mathscr{K}}\ \min_{\mathscr{E}\in\mathbf{Ch}(\mathscr{H},\mathscr{K})} w(\mathsf{M},\mathscr{E})=w(\mathsf{M},\mathrm{id})$$

and similarly for W, implying that the channel that is least compatible with M is the identity channel. The same minimum is reached at any decodable channel with input $\mathscr{H}$. This is yet another manifestation of the fact that any measurement of a physically relevant observable disturbs the system since joining an observable with the identity channel demands the maximum amount of added noise. We have the following special result:

Theorem 18. *Let* $\dim\mathscr{H}=:d<\infty$ *and* $\dim\mathscr{K}<\infty$ *and* $\mathsf{M}\in\mathbf{Obs}_d(\mathscr{H})$ *be a sharp rank-1 observable and* $\mathscr{U}\in\mathbf{Ch}(\mathscr{H},\mathscr{K})$ *a decodable channel. The robustness of incompatibility for the pair* $(\mathsf{M},\mathscr{U})$ *is*

$$W(\mathsf{M},\mathscr{U})=\frac{1}{2}\Big(1+\frac{1}{\sqrt{d}}\Big).$$

Again, it is yet unclear whether the observable-channel pair introduced in the preceding theorem is the most incompatible observable-channel pair, i.e., whether the value obtained for the robustness measure is the minimal one. A possible way to give one of the optimal decompositions

$$\begin{aligned}
\mathsf{A} &= \frac{1}{2}\left(1+\frac{1}{\sqrt{d}}\right)\mathsf{M} + \frac{1}{2}\left(1-\frac{1}{\sqrt{d}}\right)\mathsf{B}, \\
\mathscr{A} &= \frac{1}{2}\left(1+\frac{1}{\sqrt{d}}\right)\mathscr{U} + \frac{1}{2}\left(1-\frac{1}{\sqrt{d}}\right)\mathscr{B}
\end{aligned}$$

in the situation of Theorem 18, where $(\mathsf{A}, \mathscr{A})$ is the compatible observable-channel pair 'closest' to $(\mathsf{M}, \mathscr{U})$ and $(\mathsf{B}, \mathscr{B})$ is the observable-channel pair the 'furthest away' from $(\mathsf{M}, \mathscr{U})$, is such that the joint instrument $\Gamma \in \mathbf{Ins}_d(\mathscr{H}, \mathscr{K})$ for $(\mathsf{A}, \mathscr{A})$ is given by $\Gamma_j = \mathscr{U} \circ \Gamma_j^0$ for all j where $\Gamma^0 \in \mathbf{Ins}_d(\mathscr{H}, \mathscr{H})$ is defined by

$$\Gamma_j^0(\rho) = \frac{\sqrt{d}}{2(\sqrt{d}+1)}\Big(\frac{1}{\sqrt{d}}\mathbb{1}_{\mathscr{H}} + \mathsf{M}_j\Big)\rho\Big(\frac{1}{\sqrt{d}}\mathbb{1}_{\mathscr{H}} + \mathsf{M}_j\Big)$$

for all $j = 1, \ldots, d$ and $\rho \in \mathscr{S}(\mathscr{H})$ so that, defining the trivial observable $\mathsf{T} \in \mathbf{Obs}_d(\mathscr{H})$, $\mathsf{T}_j = d^{-1}\mathbb{1}_{\mathscr{H}}$ for all j, and the Lüders channel $\mathscr{E}_\mathsf{M} \in \mathbf{Ch}(\mathscr{H}, \mathscr{H})$, $\mathscr{E}_\mathsf{M}(\rho) = \sum_j \mathsf{M}_j\rho\mathsf{M}_j$ for all $\rho \in \mathscr{S}(\mathscr{H})$, one can write

$$\mathsf{A} = \frac{\sqrt{d}+2}{2(\sqrt{d}+1)}\mathsf{M} + \frac{\sqrt{d}}{2(\sqrt{d}+1)}\mathsf{T}, \quad \mathscr{A} = \frac{\sqrt{d}+2}{2(\sqrt{d}+1)}\mathscr{U} + \frac{\sqrt{d}}{2(\sqrt{d}+1)}\mathscr{U} \circ \mathscr{E}_\mathsf{M}.$$

Moreover, for the other pair $(\mathsf{B}, \mathscr{B})$ in this optimal decomposition,

$$\mathsf{B} = -\frac{1}{d-1}\mathsf{M} + \frac{d}{d-1}\mathsf{T}, \quad \mathscr{B} = -\frac{1}{d-1}\mathscr{U} + \frac{d}{d-1}\mathscr{U} \circ \mathscr{E}_\mathsf{M}.$$

Chapter 6

Summary

In this thesis, the main results obtained in the research articles written by the PhD student Erkka Haapasalo together with his supervisors and other colleagues are exhibited. These results primarily deal with the convex structures of quantum measurement devices and their convex analysis. In Chapter 2, we have introduced the relevant convex structures of quantum theory and have discussed the incompatibility properties of quantum measurement devices. After determining the extreme points of these structures in Chapter 3, we saw in Theorem 5 that a compatible device pair where one of the devices is extreme can be joined in a single apparatus essentially in a unique way. We also discussed the interplay of coexistence of quantum observables and extremality.

It has now been established that an extreme margin guarantees the uniqueness of a joint quantum measurement device but the question remains what are the necessary conditions for uniqueness of the joint map. We know that extremality of a margin is not necessary as pointed out in [31]. This problem is part of the study of the rich post-processing preorder structure of the set of measurement devices. We have also shown that a pair of coexistent observables where one of the observables is extreme and discrete is, in fact, jointly measurable. The assumption on discreteness seems superfluous and it remains to be seen if this requirement can be removed.

One major research topic in E.H.'s PhD studies has been the convex analysis of covariance structures of quantum theory, particularly the sets of covariant observables and instruments, and the structure of covariant apparati. These issues have been discussed in Chapter 4. In Theorem 9, we identified the extreme points of generalized covariance structures after which we concentrated on a couple of examples on covariance structures of observables. In theorems 10 and 11 we characterized the covariance structure of observables with a lcsc Abelian symmetry group whose value space is a transitive space for the group and the extreme points of this covariance structure. We also introduced another unsharp extreme observable, the canonical time observable. The extreme points of the covariance structure of observables with a lcsc unimodular type-I symmetry group whose value space is associated with a compact stability subgroup of the symmetry group were identified. We also studied the structure of Euclidean-covariant localization observables of elementary spin-0 objects. Finally, Theorem 14 determined the structure of a particular class of covariant instruments. This result applies to instruments whose associated observable margins satisfy certain requirements that are met especially in the cases studied earlier in Chapter 4.

There are many open questions left in the field of quantum covariance structures. We have discussed the case of covariant observables and instruments whose value space is a transitive space for the symmetry group, which was assumed to be lcsc unimodular and of type I, associated with a compact stability subgroup. The assumption on compactness of the stability subgroup makes the study of covariance structures convenient because there is a simple connection to the case where the value space is the symmetry group itself, as made clear in [28]. Removing this assumption is left for future study. We have only discussed the case of transitive value spaces, but generalizing our results to the non-transitive case in the spirit of [14, 15] is an interesting research topic. The structure of covariant channels is not studied in depth in this thesis, but this is of course a well-motivated line of study. The basics for the study of covariant channels is laid in Theorem 8, but in the channel case the problem is that the isometry J of the minimal Stinespring dilation of a covariant channel intertwines the representation on the output Hilbert space to the tensor product of the representation on the input Hilbert space and an *a priori* unknown representation on the ancillary system within the dilation. The structure of such intertwiners is an open problem.

In Chapter 5, two types of measures for quantum devices and device pairs defined purely by convex geometry were introduced: boundariness and robustness of incompatibility. Boundariness is associated with optimal discrimination and the minimum-error probability of quantum discrimination tasks as shown by Theorem 15. Robustness of incompatibility was shown to be a well defined measure of incompatibility for quantum device pairs behaving monotonically under certain compatibility non-decreasing operations. Moreover, the robustness of incompatibility was determined for three types of device pairs: a pair of Fourier-coupled rank-1 sharp observables, a pair of decodable (particularly for unitary) channels, and a pair consisting of a sharp rank-1 observable and a decodable channel. The latter case illuminates the question on how accurately we may measure a von Neumann observable and simultaneously disturb the system as little as possible.

All the examples listed above where the robustness measure was calculated exhibit certain symmetries that make the calculations easier. A general method for calculating the robustness of incompatibility is still missing however. All the examples studied also involved only finite-dimensional Hilbert spaces. One interesting incompatible observable pair on an infinite-dimensional Hilbert space is the canonical position-momentum pair on $L^2(\mathbb{R})$. Now we settle for conjecturing that this pair has the lowest possible value 1/2 for boundariness, meaning that this pair would be an example of a least compatible observable pair with respect to the robustness measure. Such a result would be in line with earlier results obtained, e.g., in [38].

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