

CONFORMAL MODULE OF THE EXTERIOR OF TWO RECTILINEAR SLITS

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ABSTRACT. We study moduli of planar ring domains whose complements are linear segments and establish formulas for their moduli in terms of the Weierstrass elliptic functions. Numerical tests are carried out to illuminate our results.

Keywords: conformal module, reduced module, capacity, elliptic functions.

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1. INTRODUCTION

The Weierstrass and Jacobian elliptic and theta functions and the Schwarz-Christoffel formula form the foundation for numerous explicit formulas for conformal mappings (N.I. Akhiezer [2], W. Koppenfels, F. Stallmann [23]). During the past thirty years many authors have studied numerical implementation of conformal mappings. We refer the reader to the bibliography of the monograph N. Papamichael and N. Stylianopoulos [29]. In particular, the Schwarz-Christoffel toolbox of T. Driscoll and N. Trefethen [11] has become a standard tool in the field. In a series of papers of T. DeLillo, J. Pfaltzgraff, D. Crowdy and their coauthors have extended the Schwarz-Christoffel method to certain cases of multiply connected domains with polygonal boundary components [8, 10, 9].

In addition to the conformal mapping problem, also the computation of numerical values of conformal invariants is an important issue in geometric function theory. Here one can often use a conformal map onto a canonical domain so as to simplify the computation. Therefore computation of conformal invariants has a natural link to numerical conformal mapping. However, the so called crowding phenomenon can create serious obstacles for computation of conformal maps, e.g. when long rectangles are mapped onto the upper half space [29].

A basic conformal invariant is the module of a ring domain. A ring domain G can be conformally mapped onto an annulus $\{z \in \mathbb{C} : q < |z| < 1\}$ and its conformal module and capacity are defined as

$$\text{mod } G = (\log(q^{-1}))/2\pi, \quad \text{cap } G = 2\pi/\log(q^{-1}).$$

Therefore, $\text{mod } G = 1/\text{cap } G$ and the computation of $\text{mod } G$ can be reduced to the solution of the Dirichlet problem for the Laplace equation and to the computation of the L^2 -norm of its gradient. This method was applied in [4, 18, 19] for the case of bounded ring domains.

Here we shall consider unbounded ring domains whose complementary components are segments. We describe one-parametric families of functions $f(z, t)$ each of which maps conformally an annulus $\{q < |z| < 1\}$ onto the exterior $G = G(t)$ of two disjoint segments A_1A_2 and A_3A_4 .

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Here $A_j = A_j(t)$, $1 \leq j \leq 4$, are some smooth functions and $q = q(t)$; t is a real parameter. Further we will denote such domains by $G(A_1, A_2, A_3, A_4)$. It is also assumed that the straight lines, containing the segments A_1A_2 and A_3A_4 , are fixed.

We note that one-parametric families of conformal mappings were considered earlier. There is the well-known Loewner-Komatu differential equation which is a generalization of the Loewner equation to the doubly-connected case. The approach of Komatu was developed by Goluzin [16] and others (see, e.g. [3, 6, 7]).

We deduce a differential equation for $f(z, t)$ in the considered case (Theorem 3). In contrast to the Loewner-Komatu equation, we do not assume that the family of the images is monotonic as a function of the parameter t . As a corollary, we obtain a system of ODEs to determine the behavior of the accessory parameters, which are the preimages of the points A_j , and the conformal module $m(t) := \text{mod } G(t) = (\log(q(t))^{-1})/(2\pi)$. On the base of the system, we suggest an approximate method for finding the accessory parameters and the conformal module. We note that in our approach we use essentially the Weierstrass elliptic functions.

Further we apply the obtained results to investigate the behavior of the conformal module in the case when one of the segments and the length of the other one are fixed.

Now we briefly describe the structure of the paper. In Section 2 we give some information on the Weierstrass elliptic functions, moduli, and reduced moduli. In Section 3 we describe an integral representation of an annulus onto the exterior of two slits A_1A_2 and A_3A_4 (Theorem 2). In contrast to the known representations [22]; (see also [20], [9], [10]), our representation is based on the Weierstrass σ -functions. The representation contains some unknown constants; they are called accessory parameters. In Section 4 we consider one-parametric families of such functions $f(z, t)$ and deduce a differential equation for them (Theorem 3). As a corollary, we obtain a system of ODEs for accessory parameters (Theorem 4). We note that to deduce the equations we use the approach developed earlier for one-parametric families of rational functions [27] and conformal mappings of complex tori [25, 26]. In Section 5 we give results of some numerical calculation. In Section 6 we study monotonicity of the conformal module of the exterior of two slits when one segment is fixed and the other one slides along a straight line and has a fixed length.

Finally we should note that recently the capacity computation of doubly connected domains with complicated boundary structure has been studied for instance in [19].

2. SOME PRELIMINARY RESULTS

Elliptic functions. First we recall some information about elliptic functions (see, e.g., [2, 28] and also [30, 31]).

A meromorphic in the complex plane function is called elliptic if it has periods ω_1 and ω_2 ^{1*}, linearly independent over \mathbb{R} . In the fundamental parallelogram constructed by the vectors ω_1 and ω_2 , every nonconstant elliptic function takes each value the same number of times; the number r is called the order of the elliptic function.

If a_1, \dots, a_r are zeroes of an elliptic functions of order r and b_1, \dots, b_r are its poles in the fundamental parallelogram, then

$$a_1 + \dots + a_r \equiv b_1 + \dots + b_r \pmod{\Omega}$$

^{1*} In contrast to [2], we denote by ω_1 and ω_2 periods of elliptic functions, not half-periods. The same remark concerns the values η_k defined by (2).

where Ω is the lattice generated by ω_1 and ω_2 . Further we will denote by ω an arbitrary element of the lattice. We note that, by given lattice, the generators ω_1 and ω_2 are not determined by a unique way; we will further assume that $\text{Im}(\omega_2/\omega_1) > 0$.

One of the main elliptic functions is the Weierstrass \wp -function

$$\wp(z) = \frac{1}{z^2} + \sum' \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right];$$

here the summation \sum' is over all nonzero elements of the lattice. The Weierstrass ζ -function

$$(1) \quad \zeta(z) = \frac{1}{z} + \sum' \left[\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right]$$

has the properties: $\zeta'(z) = -\wp(z)$ and

$$(2) \quad \zeta(z + \omega_k) = \zeta(z) + \eta_k, \quad k = 1, 2,$$

where $\eta_k = 2\zeta(\omega_k/2)$. In the fundamental parallelogram it has a unique pole with residue 1. The numbers η_k and ω_k satisfy the equality

$$(3) \quad \omega_2\eta_1 - \omega_1\eta_2 = 2\pi i.$$

At last, we need the Weierstrass σ -function

$$(4) \quad \sigma(z) = z \prod' \left\{ \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right) \right\}.$$

It is an odd entire function with the properties:

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z), \quad \sigma(z + \omega) = \varepsilon \sigma(z) e^{\eta(z + \omega/2)}.$$

Here $\eta = m\eta_1 + n\eta_2$, if $\omega = m\omega_1 + n\omega_2$. Moreover, $\varepsilon = 1$, if $\omega/2$ belongs to the lattice Ω , otherwise, $\varepsilon = -1$.

We recall the Weierstrass invariants g_2 and g_3 :

$$g_2 = 60 \sum' \frac{1}{(m\omega_1 + n\omega_2)^4}, \quad g_3 = 140 \sum' \frac{1}{(m\omega_1 + n\omega_2)^6}.$$

Elliptic functions depend not only on the variable z but also on the lattice. Further we need explicit expressions for the partial derivatives $\zeta(z) = \zeta(z; \omega_1, \omega_2)$ by the periods ω_1 and ω_2 of the lattice. In [25] the following theorem is proved.

Theorem 1. *The partial derivatives of $\zeta(z) = \zeta(z; \omega_1, \omega_2)$ with respect to the periods ω_1 and ω_2 are equal to*

$$\begin{aligned} \frac{\partial \zeta(z)}{\partial \omega_1} &= \frac{1}{2\pi i} \left[\frac{1}{2} \omega_2 \wp'(z) + (\omega_2 \zeta(z) - \eta_2 z) \wp(z) + \eta_2 \zeta(z) - (\omega_2 g_2/12) z \right], \\ \frac{\partial \zeta(z)}{\partial \omega_2} &= -\frac{1}{2\pi i} \left[\frac{1}{2} \omega_1 \wp'(z) + (\omega_1 \zeta(z) - \eta_1 z) \wp(z) + \eta_1 \zeta(z) - (\omega_1 g_2/12) z \right]. \end{aligned}$$

We will also need the Jacobi theta-function $\vartheta_1(z)$. For given lattice Ω , generated by ω_1 and ω_2 , let $\tau = \omega_2/\omega_1$, $\text{Im} \tau > 0$, and $q = e^{\pi i \tau}$. Then, by definition (see, e.g. [2, ch. 1, sect. 3], [30]),

$$(5) \quad \vartheta_1(z) = \vartheta_1(z|\tau) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin((2n+1)z).$$

There is the following connection between σ -function and $\vartheta_1(z)$ (see, e.g. [2, ch. 4, sect. 19, formula (1)]):

$$(6) \quad \sigma(z) = \omega_1 \frac{e^{\frac{\eta_1 z^2}{2\omega_1}} \vartheta_1(z/\omega_1)}{\vartheta_1'(0)}.$$

Conformal moduli, reduced moduli, and capacities of condensers. Let G be a ring domain in the plane, i.e. a doubly-connected domain with non-degenerate boundary components. There is a conformal mapping $\psi : G \rightarrow A$ of G onto an annulus $A = \{q < |z| < 1\}$ (see, e.g., [17]). The value q does not depend on the choice of ψ . We call

$$\text{mod } G = \frac{1}{2\pi} \log(q^{-1})$$

the conformal module of G . It is conformal invariant and plays an important role in the theory of conformal and quasiconformal mappings.

Let G be a ring domain in the plane with complementary components C_1 and C_2 , and let K be the condenser with plates C_1 and C_2 and with field G . We recall that

$$\text{cap } K = \inf_u \iint |\nabla u|^2 dx dy$$

where the infimum is taken over all smooth functions u such that $u = 0$ on C_1 and $u = 1$ on C_2 . We will define $\text{cap } G := \text{cap } K$ and call $\text{cap } G$ the conformal capacity of the ring domain G .

Let D be a simply connected domain with non-degenerate boundary and $z_0 \in D$. For sufficiently small ε consider the condenser defined by $D \setminus B_\varepsilon(z_0)$; here $B_\varepsilon(z_0)$ is the disk of radius ε centered at the point z_0 . Denote by K_ε its capacity. Then there exists the limit

$$r(D, z_0) := \lim_{\varepsilon \rightarrow 0^+} (K_\varepsilon + (1/(2\pi)) \log \varepsilon)$$

which is called the reduced module of D at the point z_0 [13, Section 2.4], [15].

3. INTEGRAL REPRESENTATION

Consider a conformal mapping g of an annulus $\{q < |\zeta| < 1\}$ onto the exterior $G = G(A_1, A_2, A_3, A_4)$ of two disjoint rectilinear slits A_1A_2 and A_3A_4 in the w -plane. With the help of the exponential map $z \mapsto \zeta = \exp(2\pi iz)$ we can consider the map $f := g(2\pi iz)$ from the horizontal strip

$$S := \{-m < \text{Im } z < 0\}, \quad m = \frac{1}{2\pi} \log(q^{-1}),$$

onto G . It maps conformally the rectangle $\Pi = \{0 < \text{Re } z < 1, -m < \text{Im } z < 0\}$ with identified vertical sides onto G (Fig. 1). The value m is the conformal module of G . It is evident that f has a unique pole in Π .

We will find an integral representation for the conformal mapping f of Schwarz–Christoffel type using the Weierstrass σ -function. We should note that analogs of the Schwarz–Christoffel integral for doubly-connected domains were obtained earlier in [22]; it is based on θ -functions (see also [9, 10, 20]).

Using the Riemann–Schwarz reflection principle, we can extend f to \mathbb{C} as a meromorphic function. We see that the function $h(z) = f''(z)/f'(z)$ is doubly periodic in \mathbb{C} with periods $\omega_1 = 1$ and $\omega_2 = 2mi$. Consider h in the double rectangle $\tilde{\Pi} = \{0 < \text{Re } z < 1, -m < \text{Im } z < m\}$; it is its fundamental parallelogram. Here the function h has only polar singularities at points z_k ,

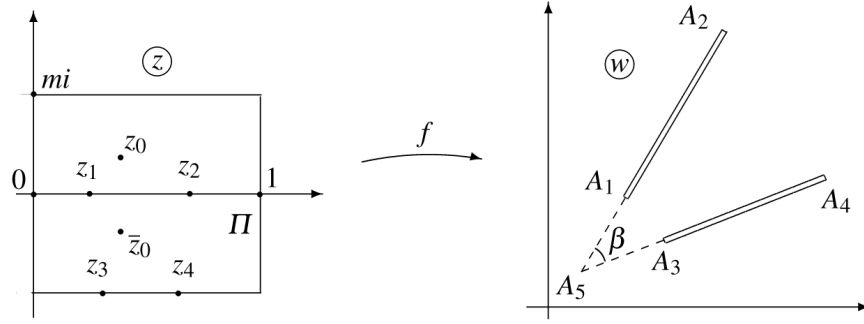


FIGURE 1. Conformal mapping of the rectangle Π with identified vertical sides onto $G(A_1, A_2, A_3, A_4)$.

$1 \leq k \leq 4$, corresponding to the endpoints A_k of the slits, and also at two distinct points, z_0 and \bar{z}_0 , where f has poles. For definiteness, we assume that $y_0 := \text{Im } z_0 > 0$. The residues of h are known, therefore, we can express it with the help of the Weierstrass zeta-function:

$$(7) \quad h(z) = \gamma + \sum_{k=1}^4 \zeta(z - z_k) - 2\zeta(z - z_0) - 2\zeta(z - \bar{z}_0)$$

where γ is a constant. (Here and further, unless otherwise specified, we assume that $\zeta(z)$ and other elliptic functions have periods $\omega_1 = 1$ and $\omega_2 = 2mi$.)

From (7) we have

$$\begin{aligned} \log f'(z) &= \gamma z + \log C + \sum_{k=1}^4 \log \sigma(z - z_k) - 2 \log \sigma(z - z_0) - 2 \log \sigma(z - \bar{z}_0), \\ f'(z) &= C e^{\gamma z} \frac{\prod_{k=1}^4 \sigma(z - z_k)}{\sigma^2(z - z_0) \sigma^2(z - \bar{z}_0)}, \\ (8) \quad f(z) &= C \int_0^z e^{\gamma \xi} \frac{\prod_{k=1}^4 \sigma(\xi - z_k)}{\sigma^2(\xi - z_0) \sigma^2(\xi - \bar{z}_0)} d\xi + C_1 \end{aligned}$$

where $\sigma(z)$ is the Weierstrass sigma-function, $C \neq 0$ and C_1 are complex constants.

The residue of $f'(z)$ at z_0 must vanish, therefore,

$$\gamma + \sum_{k=1}^4 \zeta(z_0 - z_k) - 2\zeta(z_0 - \bar{z}_0) = 0.$$

The σ -function satisfies (2). Because $f'(z)$ must be periodic with period $\omega_1 = 1$, we have

$$\begin{aligned} f'(z+1) &= C e^{\gamma(z+1)} \frac{\prod_{k=1}^4 \sigma(z - z_k + 1)}{\sigma^2(z - z_0 + 1) \sigma^2(z - \bar{z}_0 + 1)} \\ &= \frac{C e^{\gamma} e^{\gamma z} \prod_{k=1}^4 e^{\eta_1(z - z_k + 1/2)} \sigma(z - z_k)}{e^{2\eta_1(z - z_0 + 1/2)} \sigma^2(z - z_0) e^{2\eta_1(z - \bar{z}_0 + 1/2)} \sigma^2(z - \bar{z}_0)} = e^{\gamma + \eta_1(2z_0 + 2\bar{z}_0 - \sum_{k=1}^4 z_k)} f'(z). \end{aligned}$$

Consequently,

$$(9) \quad \gamma + \eta_1 \left(2z_0 + 2\bar{z}_0 - \sum_{k=1}^4 z_k \right) \equiv 0 \pmod{2\pi i}.$$

In a similar way, we have

$$f'(z + \omega_2) = e^{\gamma\omega_2 + \eta_2(2z_0 + 2\bar{z}_0 - \sum_{k=1}^4 z_k)} f'(z).$$

Since $\arg f'(z + \omega_2) - \arg f'(z) = 2\beta$ where β is the angle between the segments A_1A_2 and A_3A_4 , we have

$$(10) \quad \gamma\omega_2 + \eta_2 \left(2z_0 + 2\bar{z}_0 - \sum_{k=1}^4 z_k \right) = 2\beta i \pmod{2\pi i}.$$

Now we will specify the position of the points z_k . We will assume that z_1 and z_2 lie on the real axis, and z_3 and z_4 are on the lower side of Π (Fig. 1). Because z_k can be chosen up to the values $k\omega_1 + n\omega_2$, $k, n \in \mathbb{N}$, for convenience, we shift z_3 by $\omega_2 = 2mi$ and assume that

$$(11) \quad z_1 = x_1, \quad z_2 = x_2, \quad z_3 = x_3 + im, \quad z_4 = x_4 - im,$$

where x_k are real numbers.

Denote

$$(12) \quad a = \sum_{k=1}^4 z_k - 2z_0 - 2\bar{z}_0.$$

Taking into account (11), we see that a is real. We write (9) and (10) in the form

$$(13) \quad \gamma - \eta_1 a = 2\pi ki, \quad \omega_2 \gamma - \eta_2 a = 2\beta i + 2\pi ni, \quad k, n \in \mathbb{N}.$$

Solving (13) as a system of linear equation with respect to γ and a and taking into account that its determinant equals $\omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i$, we obtain

$$(14) \quad \gamma = -k\eta_2 + (n + \beta/\pi)\eta_1, \quad a = -k\omega_2 + (n + \beta/\pi).$$

Since ω_2 is a purely imaginary number, from the second equality in (14) we deduce that $k = 0$. We can change x_3 by entire values, therefore, we can assume that $n = 0$. So (14) has the form

$$(15) \quad \gamma = \beta\eta_1/\pi, \quad a = \beta/\pi.$$

Thus, from (12) and (15) we have

$$\sum_{k=1}^4 x_k = 4x_0 + \beta/\pi, \quad x_0 = \operatorname{Re} z_0.$$

Since every horizontal shift does not change the strip S , we can assume that $x_0 = 0$.

Therefore, we establish the following theorem.

Theorem 2. *The function, mapping the annulus $\{q < |\zeta| < 1\}$ onto $G(A_1, A_2, A_3, A_4)$, is $f(z)$ where $z = (2\pi i)^{-1} \log \zeta$ and f is defined by (8). In (8) $\gamma = \beta\eta_1/\pi$, the points $z_k = x_k + iy_k$ correspond to the endpoints A_k of the slits and satisfy (11) with real x_k and $m = (1/(2\pi)) \log(q^{-1})$, the point $z_0 = iy_0$ matches to the infinity, $C \neq 0$ and C_1 are some complex constants. Moreover, $\sum_{k=1}^4 x_k = \beta/\pi$.*

4. ONE-PARAMETRIC FAMILIES

The parametric method for doubly connected domains was developed by Komatu [21] and Goluzin [16] (in details, see [3], ch. 5). In recent papers [6, 7] some new results were obtained. Here we obtain an equation of Loewner type using ideas of the papers [25, 27].

Taking into account the integral representation (8), obtained in Theorem 2, we consider a smooth one-parametric family of conformal mappings

$$(16) \quad f(z, t) = c(t) \int_0^z e^{\gamma(t)\xi} \frac{\prod_{k=1}^4 \sigma(\xi - z_k(t))}{\sigma^2(\xi - z_0(t))\sigma^2(\xi - \bar{z}_0(t))} d\xi + c_1(t)$$

Here $\sigma(z) = \sigma(z; 1, \omega_2)$ where $\omega_2 = 2mi$, $m = m(t) > 0$. For a fixed t , $f(z, t)$ is periodic with period $\omega_1 \equiv 1$ and maps the half of the fundamental parallelogram (rectangle) $\{0 < \operatorname{Re} x < 1, -m < \operatorname{Im} z < 0\}$ onto the exterior of two rectilinear slits. Without loss of generality we may assume that one slit lies on the positive part of the real axis and the second one is on the ray $\{\arg w = \beta\}$. (The general case can be obtained by multiplying $c(t)$ by $e^{i\theta}$; this means the rotation by the angle θ . Further, in some situations, we will use this remark.)

We note once more that the angle $0 < \beta < 2\pi$ does not depend on t . Moreover,

$$\operatorname{Im} z_1(t) = \operatorname{Im} z_2(t) = 0, \quad \operatorname{Im} z_3(t) = -\operatorname{Im} z_4(t) = m(t), \quad \operatorname{Re} z_0(t) = 0,$$

therefore,

$$z_1(t) = x_1(t), \quad z_2(t) = x_2(t), \quad z_3(t) = x_3(t) + im(t), \quad z_4(t) = x_4(t) - im(t), \quad z_0(t) = iy_0(t),$$

$$x_1(t) < x_2(t) < x_1(t) + 1, \quad x_3(t) < x_4(t) < x_3(t) + 1, \quad 0 \leq y_0(t) \leq m,$$

$$\gamma(t) = (\beta/\pi)\eta_1(t), \quad \sum_{k=1}^4 x_k(t) = \beta/\pi,$$

$$\gamma(t) + \sum_{k=1}^4 \zeta(z_0(t) - z_k(t)) - 2\zeta(z_0(t) - \bar{z}_0(t)) = 0,$$

By the Riemann-Schwarz symmetry principle, we can extend $f(z, t)$ meromorphically to the whole complex plane. It is evident that the extension satisfies

$$(17) \quad f(z + 1, t) = f(z, t), \quad f(z + \omega_2(t), t) = e^{2i\beta} f(z, t),$$

Differentiating (17) with respect to t and z , we obtain

$$\dot{f}(z + 1, t) = \dot{f}(z, t), \quad \dot{\omega}_2(t)f'(z + \omega_2(t), t) + \dot{f}(z + \omega_2(t), t) = e^{2i\beta} \dot{f}(z, t),$$

$$f'(z + 1, t) = f'(z, t), \quad f'(z + \omega_2(t), t) = e^{2i\beta} f'(z, t).$$

Here and further the dot means differentiation with respect to the parameter t and the prime is differentiation with respect to z . Thus, we have

$$\frac{\dot{f}(z + \omega_k(t), t)}{f'(z + \omega_k(t), t)} + \dot{\omega}_k(t) = \frac{\dot{f}(z, t)}{f'(z, t)}.$$

Consequently, the function $h(z, t) := \dot{f}(z, t)/f'(z, t)$ satisfies

$$(18) \quad h(z + \omega_k(t), t) - h(z, t) = -\dot{\omega}_k(t), \quad k = 1, 2,$$

where $\dot{\omega}_1(t) \equiv 0$.

Now we write Taylor's expansion of $f(z, t)$ in a neighborhood of $z_k(t)$:

$$(19) \quad f(z, t) = A_k(t) + \frac{D_k(t)}{2} (z - z_k(t))^2 + \dots,$$

where $D_k(t) = f''(z_k(t), t)$. We have

$$f''(z, t) = c(t)e^{\gamma(t)z} \frac{\prod_{j=1}^4 \sigma(z - z_j(t))}{\sigma^2(z - z_0(t))\sigma^2(z - \overline{z_0(t)})} \times \left[\gamma(t) + \sum_{j=1}^4 \zeta(z - z_j(t)) - 2\zeta(z - z_0(t)) - 2\zeta(z - \overline{z_0(t)}) \right],$$

therefore, as $z \rightarrow z_k(t)$, we obtain

$$(20) \quad D_k(t) = c(t)e^{\gamma(t)z_k(t)} \frac{\prod_{j=1, j \neq k}^4 \sigma(z_k(t) - z_j(t))}{\sigma^2(z_k(t) - z_0(t))\sigma^2(z_k(t) - \overline{z_0(t)})}.$$

From (19) it follows that

$$(21) \quad \begin{aligned} f'(z, t) &= D_k(t)(z - z_k(t)) + \dots, \\ \dot{f}(z, t) &= \dot{A}_k(t) - \dot{z}_k(t)D_k(t)(z - z_k(t)) + \dots, \end{aligned}$$

and, therefore,

$$h(z, t) = \frac{\dot{f}(z, t)}{f'(z, t)} = \frac{\gamma_k(t)}{z - z_k(t)} + O(1), \quad z \rightarrow z_k(t),$$

where

$$(22) \quad \gamma_k(t) := \frac{\dot{A}_k(t)}{D_k(t)}.$$

At the point $z_0(t)$, the function $\dot{f}(z, t)$ has a pole of order at most 2, and $f'(z, t)$ has a pole of order 2. Thus, $h(z, t)$ has a removable singularity at the point. In more details, denoting by $d_{-1}(t)$ the residue of $f(z, t)$ at the point $z_0(t)$, we have

$$(23) \quad \begin{aligned} f(z, t) &= \frac{d_{-1}(t)}{z - z_0(t)} + d_0(t) + O(1), \\ \dot{f}(z, t) &= \dot{z}_0(t) \frac{d_{-1}(t)}{(z - z_0(t))^2} + \frac{\dot{d}_{-1}(t)}{z - z_0(t)} + O(1), \\ f'(z, t) &= -\frac{d_{-1}(t)}{(z - z_0(t))^2} + O(1). \end{aligned}$$

From this we see that in a neighborhood of $z_0(t)$ the function $h(z, t)$ has the expansion

$$(24) \quad h(z, t) = -\dot{z}_0(t) + o(1), \quad z \rightarrow z_0(t).$$

In a similar way, we show that $h(z, t)$ has a removable singularity at the point $\bar{z}_0(t)$.

The function

$$F(z, t) := h(z, t) - \sum_{j=1}^4 \gamma_j(t) \zeta(z - z_j(t))$$

has only removable singularities at the points $z_k(t)$, $1 \leq k \leq 4$, $z_0(t)$, and $\bar{z}_0(t)$, and at points equivalent to them (by mod of the lattice). At other points of the plane it is holomorphic. Consequently, it can be extended holomorphically to the whole plane \mathbb{C} .

From (18) we obtain

$$(25) \quad F(z + \omega_k(t), t) - F(z, t) = -\dot{\omega}_k(t) - \eta_k(t) \sum_{j=1}^4 \gamma_j(t), \quad k = 1, 2.$$

By (25), the function F grows not faster than a linear function, therefore, $F(z, t) = \alpha(t)z + \beta(t)$. So we have

$$(26) \quad h(z, t) = \sum_{j=1}^4 \gamma_j(t) \zeta(z - z_j(t)) + \alpha(t)z + \beta(t).$$

From (24) we find

$$(27) \quad \beta(t) = -\sum_{j=1}^4 \gamma_j(t) \zeta(z_0(t) - z_j(t)) - \alpha(t)z_0(t) - \dot{z}_0(t).$$

From (25) it follows that

$$(28) \quad \alpha(t)\omega_k(t) = -\dot{\omega}_k(t) - \eta_k(t) \sum_{j=1}^4 \gamma_j(t), \quad k = 1, 2.$$

If we put $k = 1$, then, taking into account that $\omega_1(t) \equiv 1$, we obtain

$$(29) \quad \alpha(t) = -\eta_1(t) \sum_{j=1}^4 \gamma_j(t).$$

At last, from (26), (27), and (29) we deduce that

$$(30) \quad h(z, t) = \sum_{j=1}^4 \gamma_j(t) [\zeta(z - z_j(t)) - \zeta(z_0(t) - z_j(t)) - \eta_1(t)(z - z_0(t))] - \dot{z}_0(t).$$

If we put $k = 2$, from (28) we have

$$\dot{\omega}_2(t) = -\alpha(t)\omega_2(t) - \eta_2(t) \sum_{j=1}^4 \gamma_j(t) = (\omega_2(t)\eta_1(t) - \eta_2(t)) \sum_{j=1}^4 \gamma_j(t),$$

and, with the help of the equality (3), we obtain

$$(31) \quad \dot{\omega}_2(t) = 2\pi i \sum_{j=1}^4 \gamma_j(t).$$

Therefore, we proved the following result.

Theorem 3. *The family $f(z, t)$ satisfies the PDE*

$$\frac{\dot{f}(z, t)}{f'(z, t)} = h(z, t)$$

where $h(z, t)$ is defined by (30); here $\gamma_k(t)$ and $D_k(t)$ are specified by (22) and (20). The period $\omega_1(t)$ is equal 1 and the period $\omega_2(t)$ satisfies (31).

Now we will write a system of differential equations to find $z_l(t)$, $1 \leq l \leq 4$. For this, we will write $\dot{f}'(a_l(t), t)$ in two different ways. On the one hand, from (21) it follows that

$$(32) \quad \dot{f}'(z_l(t), t) = -\dot{z}_l(t)D_l(t).$$

On the other hand, by Theorem 3, we have $\dot{f}'(z, t) = h(z, t)f'(z, t)$, therefore,

$$(33) \quad \dot{f}'(z, t) = c(t) \left[\sum_{j=1}^4 \gamma_j(t) [\zeta(z - z_j(t)) - \zeta(z_0(t) - z_j(t)) - \eta_1(t)(z - z_0(t))] - \dot{z}_0(t) \right] \\ \times e^{\gamma(t)z} \frac{\prod_{k=1}^4 \sigma(z - z_k(t))}{\sigma^2(z - z_0(t))\sigma^2(z - \overline{z_0(t)})}$$

and

$$(34) \quad \dot{f}'(z, t) = c(t) \left\{ \left[\sum_{j=1}^4 \gamma_j(t) [\zeta(z - z_j(t)) - \zeta(z_0(t) - z_j(t)) - \eta_1(t)(z - z_0(t))] - \dot{z}_0(t) \right] \right. \\ \times \left(\gamma(t) + \sum_{s=1}^4 \zeta(z - z_s(t)) - 2\zeta(z - z_0(t)) - 2\zeta(z - \overline{z_0(t)}) \right) \\ \left. - \sum_{j=1}^4 \gamma_j(t) [\mathfrak{P}(z - z_j(t)) - \eta_1(t)] \right\} e^{\gamma(t)z} \frac{\prod_{k=1}^4 \sigma(z - z_k(t))}{\sigma^2(z - z_0(t))\sigma^2(z - \overline{z_0(t)})}.$$

From (34) we obtain, as $z \rightarrow z_l(t)$,

$$(35) \quad \dot{f}'(z_l(t), t) = c(t) \left[-\dot{z}_0(t) + \sum_{j=1, j \neq l}^4 \gamma_j(t) [\zeta(z_l(t) - z_j(t)) - \zeta(z_0(t) - z_j(t)) \right. \\ \left. - \eta_1(t)(z_l(t) - z_0(t))] + \gamma_l(t) \left(\sum_{s=1, s \neq l}^4 \zeta(z_l(t) - z_s(t)) + \gamma(t) - \eta_1(z_l(t) - z_0(t)) \right. \right. \\ \left. \left. - \zeta(z_l(t) - z_0(t)) - 2\zeta(z_l(t) - \overline{z_0(t)}) \right) \right] \frac{e^{\gamma(t)z_l(t)} \prod_{k \neq l, k=1}^4 \sigma(z_l(t) - z_k(t))}{\sigma^2(z_l(t) - z_0(t))\sigma^2(z_l(t) - \overline{z_0(t)})}.$$

Comparing (32) and (35), taking into account (20), we see that

$$(36) \quad \dot{z}_l = \dot{z}_0 - \sum_{j=1, j \neq l}^4 \gamma_j [\zeta(z_l - z_j) - \zeta(z_0 - z_j) - \eta_1(z_l - z_0)] \\ - \gamma_l \left(\sum_{s=1, s \neq l}^4 \zeta(z_l - z_s) + \gamma - \eta_1(z_l - z_0) - \zeta(z_l - z_0) - 2\zeta(z_l - \overline{z_0}) \right), \quad 1 \leq l \leq n.$$

Now we will find a differential equation to determine $c(t)$. Comparing (16), (33), and (23), we have

$$(37) \quad d_{-1}(t) = -c(t)e^{\gamma(t)z_0(t)} \frac{\prod_{k=1}^4 \sigma(z_0(t) - z_k(t))}{\sigma^2(z_0(t) - z_0(t))},$$

$$\begin{aligned} \dot{d}_{-1}(t) &= -c(t) \left\{ \sum_{j=1}^4 \gamma_j(t) \mathfrak{P}(z_0(t) - z_j(t)) + \eta_1(t) + \dot{z}_0(t) \right. \\ &\times \left. \left[\gamma(t) + \sum_{k=1}^4 \zeta(z_0(t) - z_k(t)) - 2\zeta(z_0(t) - \bar{z}_0(t)) \right] \right\} e^{\gamma(t)z_0(t)} \frac{\prod_{k=1}^4 \sigma(z_0(t) - z_k(t))}{\sigma^2(z_0(t) - \bar{z}_0(t))}. \end{aligned}$$

Since

$$(38) \quad \gamma + \sum_{k=1}^4 \zeta(z_0 - z_k) - 2\zeta(z_0 - \bar{z}_0) = 0,$$

we have

$$\dot{d}_{-1}(t) = -c(t) \left[\sum_{j=1}^4 \gamma_j(t) \mathfrak{P}(z_0(t) - z_j(t)) + \eta_1(t) \right] e^{\gamma(t)z_0(t)} \frac{\prod_{k=1}^4 \sigma(z_0(t) - z_k(t))}{\sigma^2(z_0(t) - \bar{z}_0(t))}.$$

Therefore,

$$(39) \quad \dot{a}(t) = \sum_{j=1}^4 \gamma_j(t) \mathfrak{P}(z_0(t) - z_j(t)) + \eta_1(t)$$

where $a = \log d_{-1}$.

Differentiating (38), we obtain

$$\begin{aligned} & i \frac{4\beta}{\pi} \frac{\partial \zeta(1/2)}{\partial \omega_2} \dot{m} - \sum_{k=1}^4 \mathfrak{P}(z_0 - z_k) (\dot{z}_0 - \dot{z}_k) + i2 \sum_{k=1}^4 \frac{\partial \zeta(z_0 - z_k)}{\partial \omega_2} \dot{m} \\ & + 2\mathfrak{P}(z_0 - \bar{z}_0) (\dot{z}_0 - \dot{\bar{z}}_0) - i4 \frac{\partial \zeta(z_0 - \bar{z}_0)}{\partial \omega_2} \dot{m} = 0, \\ & \left(4\mathfrak{P}(z_0 - \bar{z}_0) - \sum_{k=1}^4 \mathfrak{P}(z_0 - z_k) \right) \dot{z}_0 = \\ & - \sum_{k=1}^4 \mathfrak{P}(z_0 - z_k) \dot{z}_k + i \left[4 \frac{\partial \zeta(z_0 - \bar{z}_0)}{\partial \omega_2} - \frac{4\beta}{\pi} \frac{\partial \zeta(1/2)}{\partial \omega_2} - 2 \sum_{k=1}^4 \frac{\partial \zeta(z_0 - z_k)}{\partial \omega_2} \right] \dot{m}, \end{aligned}$$

and, therefore,

$$(40) \quad \dot{y}_0 = - \sum_{k=1}^4 \operatorname{Im} \frac{\mathfrak{P}(z_0 - z_k)}{4\mathfrak{P}(z_0 - \bar{z}_0) - \sum_{j=1}^4 \mathfrak{P}(z_0 - z_j)} \dot{x}_k + \operatorname{Re} \left[\frac{4 \frac{\partial \zeta(z_0 - \bar{z}_0)}{\partial \omega_2}}{4\mathfrak{P}(z_0 - \bar{z}_0) - \sum_{k=1}^4 \mathfrak{P}(z_0 - z_k)} \right. \\ \left. - \frac{4\beta}{\pi} \frac{\partial \zeta(1/2)}{\partial \omega_2} - 2 \sum_{k=1}^4 \frac{\partial \zeta(z_0 - z_k)}{\partial \omega_2} - \mathfrak{P}(z_0 - z_3) + \mathfrak{P}(z_0 - z_4) \right] \dot{m}.$$

Theorem 4. *The accessory parameters satisfy the system of ODEs: (36), (39), and (40) where $a = \log d_{-1}$ and d_{-1} is defined by (37).*

Corollary 1. *The conformal module of the domains satisfies the equation*

$$\dot{m}(t) = \pi \sum_{j=1}^4 \gamma_j(t).$$

where $\gamma_k(t) := \dot{A}_k(t)/D_k(t)$, $D_k(t) = f''(z_k)$.

5. SYMMETRIC CASE. NUMERIC RESULTS

Now we will describe an approximate method of finding the accessory parameters in (8). It is based on Theorem 4. If we consider a smooth one-parametric family $f(z, t)$, $0 \leq t \leq 1$, of conformal mappings of the form (16), then, knowing the values of the parameters for $t = 0$, we can solve the Cauchy problem with this initial data and obtain the values of the accessory parameters for all t . We note that it is natural to use the uniform motion of the points $A_k = A_k(t)$, therefore, in our calculations we will take $\dot{A}_k = \text{const}$. Moreover, if we choose the appropriate initial data, then we change only two of A_k , say, A_1 and A_2 ; thus, we can put $\dot{A}_3 = \dot{A}_4 \equiv 0$.

Therefore, to solve the Cauchy problem for the obtained system, we need to know the initial data, i.e. the values of the accessory parameters for some t . For this, it is convenient to use the data for the symmetric case when the segment A_1A_2 and A_3A_4 are symmetric with respect to the real axis and the straight lines, containing these segments, pass through the origin. (This can be achieved by a rotation and a shift.)

Now we describe the conformal mapping for the symmetric case. Because of the Riemann-Schwarz symmetry principle, we can consider the conformal mapping of a strip onto the upper half of the symmetric domain $G = G(A_1, A_2, A_3, A_4)$ and then extend it up to the conformal mapping of the strip, with twice the original width, onto the whole domain G .

a) If $0 < \beta < \pi$, then the conformal mapping has the form (see [23, Part B, Section 8.2, Example 1, p. 354]):

$$f(z) = \tilde{c} \frac{\vartheta_1(z - \alpha)}{\vartheta_1(z + \alpha)}, \quad \alpha = \frac{\beta}{4\pi},$$

where $\vartheta_1(z)$ is the Jacobi theta-function defined by (5) and $\tilde{c} > 0$ is a constant. From (6), taking into account that $\omega_1 = 1$, we easily deduce that

$$(41) \quad f(z) = ce^{2\alpha\eta_1 z} \frac{\sigma(z - \alpha)}{\sigma(z + \alpha)}, \quad c > 0.$$

We should note that, in contrast to (8), here $\sigma(z)$, defined by (4), matches to the periods 1 and im , not to 1 and $i2m$.

The function f , defined by (41), maps the rectangle $R := \{-1/2 < \text{Re } z < 1/2, 0 < \text{Im } z < m/2\}$ with identified vertical sides onto the upper half of $G(A_1, A_2, A_3, A_4)$ and keeps the real axis; it can be extended, by symmetry, to the rectangle $\tilde{R} := \{-1/2 < \text{Re } z < 1/2, -m/2 < \text{Im } z < m/2\}$, and the extended function maps \tilde{R} onto the whole domain $G(A_1, A_2, A_3, A_4)$. The function f has four critical points $\pm z_k$, $1 \leq k \leq 4$, and $z_1 = x_1 + im/2$, $z_2 = x_2 + im/2$, $z_3 = x_1 - im/2$, $z_4 = x_2 - im/2$. Besides, f has a pole at the point $z = -\alpha$ and a zero at $z = \alpha$ (Fig. 2).

The critical points z_k , $1 \leq k \leq 4$, can be found from the equality $f'(z) = 0$, i.e.

$$\zeta(\alpha - z) + \zeta(\alpha + z) = 2\eta_1\alpha.$$

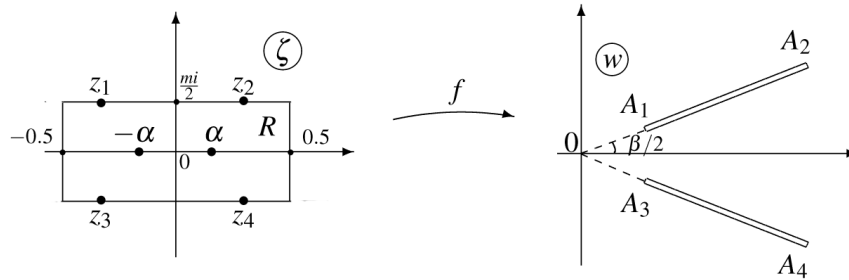


FIGURE 2. Conformal mapping of the rectangle \tilde{R} with identified vertical sides onto symmetric domain $G(A_1, A_2, A_3, A_4)$.

Because of the equality ([2], ch.III, § 15),

$$\zeta(u+v) + \zeta(u-v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)},$$

we have

$$(42) \quad \wp(z) = \wp(\alpha) - \frac{\wp'(\alpha)}{2(\alpha\eta_1 - \zeta(\alpha))},$$

therefore, z_k can be found via the inverse function \wp^{-1} . Because of the evenness of the \wp -function, we see that $z_3 = -z_2$ and $z_4 = -z_1$.

Without loss of generality we can assume that the nearest points of the slits are located at the distance 1 from the origin. Then the farthest points are at the distance $l := if(z_3)/f(z_2) = if(-z_2)/f(z_2)$. Therefore, making use of (41) and oddness of the σ -function, we have

$$(43) \quad l = ie^{-4\alpha\eta_1 z_2} \frac{\sigma^2(z_2 + \alpha)}{\sigma^2(z_2 - \alpha)}.$$

Let z_2 be a root of (42); we note that it depends on m . Then we solve (43) with respect to m , to obtain the initial value of the module. After that, we easily find the initial values of z_k , $1 \leq k \leq 4$. To use them in the non-symmetric case, we need to shift the obtained values of z_k by the vector $\alpha - im/2$.

To find c we use the equalities

$$f(z_3) = f(-z_2) = ce^{-2\alpha\eta_1 z_2} \frac{\sigma(z_2 + \alpha)}{\sigma(z_2 - \alpha)}, \quad f(z_2) = ce^{2\alpha\eta_1 z_2} \frac{\sigma(z_2 - \alpha)}{\sigma(z_2 + \alpha)}.$$

Multiplying them, we have $c^2 = f(z_3)f(z_2) = |f(z_3)f(z_2)|$, therefore,

$$c = \sqrt{|f(z_3)f(z_2)|} = \sqrt{|f(z_3)f(z_4)|}.$$

The residue of $f(z)$, defined by (41), is equal to

$$d_{-1}^0 = -ce^{-2\alpha^2\eta_1} \sigma(2\alpha),$$

therefore, the initial value of a is

$$a^0 = \log d_{-1}^0 = (1/2) \log |f(z_3)f(z_4)| - 2\alpha^2\eta_1 + \log \sigma(2\alpha) + \pi i.$$

We note that $|f(z_3)|$ and $|f(z_4)|$ are the distances l_3 and l_4 from A_3 and A_4 to A_5 ; here A_5 is the point of intersection of the straight lines containing the slits. Finally, we have

$$a^0 = \log d_{-1}^0 = (1/2) \log(l_3 l_4) - 2\alpha^2\eta_1 + \log \sigma(2\alpha) + \pi i.$$

b) Consider the case $\beta = 0$ when the slits lie on the (distinct) parallel lines. Without loss of generality we can assume that the slits are on straight lines parallel to the real axis. Then the conformal mapping has the form (see [23, Part B, Section 8.1, Example 1, p. 339]):

$$f(z) = -\frac{b}{\pi} (\zeta(z) - \eta_1 z).$$

As in the case a), $\zeta(z)$, defined by (1), has the periods 1 and im , not 1 and $i2m$. The parameter b means a half of the vertical distance between the slits. The critical points z_k of the map can be found from the equation $f'(z) = 0$; it is equivalent to the equality $\mathfrak{P}(z) = -\eta_1$. Using the evenness of the \mathfrak{P} -function, we see that $x_1 = -x_2$, $z_3 = -z_2$ and $z_4 = -z_1$. Finding z_2 and using the oddness of $f(z)$, we obtain

$$-\frac{b}{\pi} (\zeta(z_2) - \eta_1 z_2) = l/2$$

where l is the length of each slit. From the last equality we find the initial value of m and z_k . As in the case a), to use the obtained values, we need to shift them; taking into account that here $\alpha = 0$, we see that the shift parameter is the vector $-im/2$. The residue of $f(z)$ at $z = 0$ equals $-b/\pi$, thus, the initial value of a is $\log(b/\pi) + \pi i$ or

$$a^0 = \log(\text{Im}(z_1 - z_3)/(2\pi)) + \pi i.$$

Now we give the Mathematica code, with commentaries, to calculate the values of parameters, the module and the capacity of ring domains with the exterior of two rectilinear slits. For convenience, we divide it into 5 steps.

If $0 < \beta < \pi$, we first find the point A_5 which is the intersection of the straight lines containing the slits. We will assume that A_3A_4 does not contain A_5 ; in the opposite case we renumber the points and use the reflection with respect to the real axis which, in fact, does not change the desired parameters. We also assume that A_4 is farther from A_5 than A_3 . If A_1A_2 also does not contain A_5 , then we number the points so that A_2 is farther from A_5 than A_1 . If A_1A_2 contains A_5 , then we consider that $\arg(A_2 - A_5)/(A_4 - A_5) = \beta$. In the case, either $A_1 = A_5$ or $\arg(A_1 - A_5)/(A_4 - A_5) = \beta \pm \pi$. Dependence on t describes the uniform movement of points $A_1(t)$ and $A_2(t)$ along the corresponding segments; $A_3(t)$ and $A_4(t)$ are herewith constant. Therefore, $\dot{A}_k(t)$ are constant, moreover, $\dot{A}_3(t) = \dot{A}_4(t) = 0$.

Step 1. Input of location of the points A_k , $1 \leq k \leq 4$. (Here we take $A_1 = -2i$, $A_2 = 3i$, $A_3 = 1$, $A_4 = 3$.) Finding A_5 , β , \dot{A}_1 , and \dot{A}_2 .

```
A1=-2.*I; A2=3.*I; A3=1.; A4=3.;
A5=A2+(A1-A2)*Im[(A4-A2)Conjugate[(A3-A4)]]/
Im[(A1-A2)Conjugate[(A3-A4)]];
l1=Sign[Re[(A1-A5)Exp[-I*beta/2]]]*Abs[A1-A5]; l2=Abs[A2-A5];
l3=Abs[A3-A5]; l4=Abs[A4-A5]; beta=Arg[(A2-A1)/(A4-A3)];
alpha=beta/(4*Pi); Adot1=(l1-l3)Exp[I*beta/2];
Adot2=(l2-l4)Exp[I*beta/2];
```

Step 2. Defining Weierstrass elliptic functions ($\mathfrak{P}(z)$, $\mathfrak{P}'(z)$, $\zeta(z)$, $\sigma(z)$, $\partial\zeta(z)/\partial\omega_2$) with periods $\omega_1 = 1$ and $\omega_2 = im$ as functions depending on complex variable z and m . Defining functions $\gamma_k(t)$, $k = 1, 2$.

```
wp1[z_,w1_,w2_] := WeierstrassP[z,WeierstrassInvariants[{w1/2,w2/2}]]; wpp1[z_, w1_,
w2_] := WeierstrassPPrime[z,WeierstrassInvariants[{w1/2,w2/2}]];
wz1[z_,w1_,w2_] := WeierstrassZeta[z,WeierstrassInvariants[{w1/2,w2/2}]];
ws1[z_,w1_,w2_] := WeierstrassSigma[z,WeierstrassInvariants[{w1/2,
```

```

w2/2}]]; wi1[x_,y_]:=WeierstrassInvariants[{x,y}]; g2[w1_,w2_]:=
-4(wp1[w1/2,w1,w2]*wp1[w2/2,w1,w2]+wp1[w1/2,w1,w2]*wp1[(w1+w2)/2,w1,w2]+
wp1[w2/2,w1,w2]*wp1[(w1+w2)/2,w1,w2]);
wz1primeperiod[z_,w1_,w2_]:=-1/(2*Pi*I)((1/2)wpp1[z,w1,w2]+(wz1[z,w1,w2]-
z*2*wz1[1/2,w1,w2])wp1[z,w1,w2]+2*wz1[w1/2,w1,w2]*wz1[z,w1,w2]-g2[w1,w2]*z/12);
wp[z_,t_]:=wp1[z,1,2*I*t]; ws[z_,t_]:=ws1[z,1,2*I*t]; wz[z_,t_]:=wz1[z,1,2*I*t];
wzw2[z_,t_]:=wz1primeperiod[z,1,2*I*t]; gamma[t_]:=2*beta/Pi*wz[0.5,m[t]];
gamma1[t_]:=-ws[z0[t]-z1[t],m[t]]*ws[z0[t]-z2[t],m[t]]*ws[z0[t]-z3[t],m[t]]*
ws[z0[t]-z4[t],m[t]]/(ws[2*z0[t],m[t]])^2*Exp[-(a[t]+gamma[t]*(z1[t]-z0[t]))]*
(ws[z1[t]-z0[t],m[t]])^2*(ws[z1[t]+z0[t],m[t]])^2/(ws[z1[t]-z2[t],m[t]]*
ws[z1[t]-z3[t],m[t]]*ws[z1[t]-z4[t],m[t]]);
gamma2[t_]:=-ws[z0[t]-z1[t],m[t]]*ws[z0[t]-z2[t],m[t]]*ws[z0[t]-z3[t],m[t]]*
ws[z0[t]-z4[t],m[t]]/(ws[2*z0[t],m[t]])^2*Exp[-(a[t]+gamma[t]*(z2[t]-z0[t]))]*
(ws[z2[t]-z0[t],m[t]])^2*(ws[z2[t]+z0[t],m[t]])^2/(ws[z2[t]-z1[t],m[t]]*
ws[z2[t]-z3[t],m[t]]*ws[z2[t]-z4[t],m[t]]);

```

Step 3. Finding initial value of module, critical points, pole, and constant a .

```

f1[t_]:=wp1[alpha,1,I*t]-wpp1[alpha,1,I*t]/(2(alpha*2*wz1[0.5,1,I*t]-
wz1[alpha,1,I*t]));
Z1[t_]=InverseWeierstrassP[f1[t],WeierstrassInvariants[{0.5,0.5*I*t}]];
L[t_]=Abs[Exp[-4*alpha*2*wz1[0.5,1,I*t]*Z1[t]]*(ws1[Z1[t]+alpha,1,I*t]/
ws1[Z1[t]-alpha,1,I*t])^2]-14/13; ar=0.1; bl=3.0; Do[m0=(ar+ bl)/2.;
fc=L[m0]; If[L[bl]*fc>0,bl=m0,ar=m0],{i,70}]; X0=Re[Z1[m0]]; x10=beta/(4*Pi)+X0;
x20=beta/(4*Pi)-X0; x30=beta/(4*Pi)+X0; x40=beta/(4*Pi)-X0;
a20=Pi; y00=m0/2; a10=(1/2)*Log[13*14]-(beta/(2*Pi))^2*Re[wz1[0.5,1,I*m0]]+
Log[Abs[ws1[beta/(2*Pi),1,I*m0]]];

```

Step 4. Solving system of ODEs.

```

sol = NDSolve[{-z1'[t]==Re[Adot1*gamma1[t]*(wz[z1[t]-
z2[t],m[t]]+wz[z1[t]-z3[t],m[t]]+
wz[z1[t]-z4[t],m[t]]+gamma[t]-2*wz[0.5,m[t]]*(z1[t]-z0[t])-wz[z1[t]-z0[t],
m[t]]-2*wz[z1[t]+z0[t],m[t]])+Adot2*gamma2[t](wz[z1[t]-z2[t],m[t]]-
wz[z0[t]-z2[t],m[t]]-2*wz[0.5,m[t]]*(z1[t]-z0[t])),
-z2'[t]==Re[Adot2*gamma2[t]*(wz[z2[t]-z1[t],m[t]]+wz[z2[t]-z3[t],m[t]]+
wz[z2[t]-z4[t],m[t]]+gamma[t]-2*wz[0.5,m[t]]*(z2[t]-z0[t])-wz[z2[t]-z0[t],
m[t]]-2*wz[z2[t]+z0[t],m[t]])+Adot1*gamma1[t](wz[z2[t]-z1[t],m[t]]-
wz[z0[t]-z1[t],m[t]]-2*wz[0.5,m[t]]*(z2[t]-z0[t])),
-z3'[t]==-I*m'[t]+Re[Adot1*gamma1[t](wz[z3[t]-z1[t],m[t]]-wz[z0[t]-z1[t],m[t]]-
2*wz[0.5,m[t]]*(z3[t]-z0[t]))+Adot2*gamma2[t](wz[z3[t]-z2[t],m[t]]-
wz[z0[t]-z2[t],m[t]]-2*wz[0.5,m[t]]*(z3[t]-z0[t])),
-z4'[t]==I*m'[t]+Re[Adot1*gamma1[t](wz[z4[t]-z1[t],m[t]]-wz[z0[t]-z1[t],m[t]]-
2*wz[0.5,m[t]]*(z4[t]-z0[t]))+Adot2*gamma2[t](wz[z4[t]-z2[t],m[t]]-
wz[z0[t]-z2[t],m[t]]-2*wz[0.5,m[t]]*(z4[t]-z0[t])),
m'[t]==Re[Pi*(Adot1*gamma1[t]+Adot2*gamma2[t])],
a'[t]==Adot1*gamma1[t]*wp[z0[t]-z1[t],m[t]]+Adot2*gamma2[t]*wp[z0[t]-z2[t],m[t]]
+2*wz[0.5,m[t]]*(Adot1*gamma1[t]+Adot2*gamma2[t]),
z0'[t]==I*Im[(4*wp[2*z0[t],m[t]]-wp[z0[t]-z1[t],m[t]]-wp[z0[t]-z2[t],m[t]]-
wp[z0[t]-z3[t],m[t]]-wp[z0[t]-z4[t],m[t]])^(-1)(-wp[z0[t]-z1[t],m[t]]z1'[t]-
wp[z0[t]-z2[t],m[t]]z2'[t]-wp[z0[t]-z3[t],m[t]]z3'[t]-wp[z0[t]-z4[t],m[t]]
z4'[t])] + I*Re[(4*wp[2*z0[t],m[t]]-wp[z0[t]-z1[t],m[t]]-wp[z0[t]-z2[t],m[t]]-
wp[z0[t]-z3[t],m[t]]-wp[z0[t]-z4[t],m[t]])^(-1)(4*wzw2[2*z0[t],m[t]]

```

```

-(4*beta/Pi)wzw2[0.5,m[t]]-2(wzw2[z0[t]-z1[t],m[t]]+wzw2[z0[t]-z2[t],m[t]]+
wzw2[z0[t]-z3[t],m[t]]+wzw2[z0[t]-z4[t],m[t]])))*Pi*Re[(Adot1*gamma1[t]+
Adot2*gamma2[t])],
z1[0]==x10,z2[0]==x20,z3[0]==x30+I*m0,z4[0]==x40-I*m0,a[0]==a10+I*a20,
m[0]==m0,z0[0]==-I*y00},{z1,z2,z3,z4,m,a,z0},{t,0,1.}];

```

Step 5. Output of desired values of capacity, module, critical points, pole, and constant a .
`s=1.; {1/m[s], m[s], z1[s], z2[s], z3[s], z4[s], a[s], z0[s]} /.sol`

In the case of slits, parallel to the real axis, we have the same system of ODEs. We recall that we can assume that $\operatorname{Re} A_1 < \operatorname{Re} A_2$, $\operatorname{Re} A_3 < \operatorname{Re} A_4$, and $\operatorname{Im} A_1 = \operatorname{Im} A_2 > \operatorname{Im} A_3 = \operatorname{Im} A_4$. Then we find the values of \dot{A}_1 and \dot{A}_2 by the formulas $\dot{A}_1 = \operatorname{Re}(A_1 - A_3)$, $\dot{A}_2 = \operatorname{Re}(A_2 - A_4)$. We also have other formulas to find the initial values. Thus, Steps 1 and 3 must be changed to the following ones.

Step 1'. Input of location of the points A_k , $1 \leq k \leq 4$. (Here we take $A_1 = i$, $A_2 = 2 + i$, $A_3 = -2 - i$, $A_4 = -1 - i$.) Finding A_5 , β , \dot{A}_1 , and \dot{A}_2 .

```

A1=1.*I; A2=2.+1.*I; A3=-2.-1.*I; A4=-1.-1.*I; Adot1=Re[A1-A3];
Adot2=Re[A2-A4]; beta=0.;

```

Step 3'. Finding initial value of module, critical points, pole, and constant a .

```

g[m_]:= -2*WeierstrassZeta[0.5,WeierstrassInvariants[{0.5,0.5*m*I}]];
h[m_]:= Re[InverseWeierstrassP[g[m],WeierstrassInvariants[{0.5,0.5*m*I}]]];
f[m_]:= Re[(2/Pi)(WeierstrassZeta[h[m]+0.5*m*I,WeierstrassInvariants[{0.5,0.5*m*I}]]-
2*(h[m]+0.5*m*I)WeierstrassZeta[0.5,WeierstrassInvariants[{0.5,0.5*m*I}]])-
2*Abs[A3-A4]/Abs[Im[(A3-A1)]]]; ar=0.1; bl=3.; Do[m0=(ar+bl)/2.;
fc=f[m0]; If[f[bl]*fc>0,bl=m0,ar=m0],{i,70}]; X0=Re[h[m0]]; x10=X0;
x20=-X0; x30=X0; x40=-X0; y00=m0/2; a10=Log[Im[A1-A3]/(2*Pi)]; a20=Pi;

```

Example 1. Consider the case when the endpoints of one of the segments are the points $a - 0.5$, $a + 0.5$ on the real axis and the endpoints of the other one are the points $-i$, $-2i$ of the imaginary axis. Then $\beta = \pi/2$ and $\alpha = 0.125$.

As an initial situation, we take the symmetric case when $a = 1.5$. With the help of (43) and (42) we find the initial module and the real parts of the critical points:

$$m^0 = 0.67578477\dots, \quad \tilde{x}_2^0 = -\tilde{x}_1^0 = 0.22367571\dots$$

Since in the non-symmetric case we have $\operatorname{Im} z_0 = 0$ in (8), we use a shift $z \mapsto z + \alpha$ in the z -plane and take, as an initial data, the values $x_k^0 = \tilde{x}_k^0 + \alpha$. Therefore,

$$x_1^0 = x_3^0 = 0.34867571\dots \quad x_2^0 = x_4^0 = -0.09867571\dots$$

Because of symmetry, we have $y_0^0 = m^0/2$. Consequently,

$$\begin{aligned} z_1^0 &= 0.34867571\dots, & z_2^0 &= -0.09867571\dots, \\ z_3^0 &= 0.34867571\dots + im^0, & z_4^0 &= -0.09867571\dots - im^0. \end{aligned}$$

At last $a^0 = \log d_{-1}^0$ where d_{-1}^0 is the residue of (8) at the point z_0 .

Finding the residue in the symmetric case, we obtain

$$\begin{aligned} a^0 &= [\ln \sqrt{2} - 4\alpha^2 \zeta(0.5; 1, im^0) + \ln \sigma(2\alpha; 1, im^0)] + \pi i \\ &= -1.11526111\dots + i 3.14159265\dots \end{aligned}$$

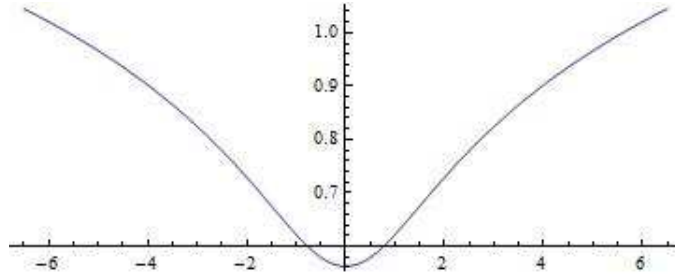


FIGURE 3. The graph of the dependence of the module m on the parameter a (Example 1).

Here the functions $\zeta(z) = \zeta(z; 1, im^0)$ and $\sigma(z) = \sigma(z; 1, im^0)$ correspond to the periods 1 and im^0 . Solving the system of differential equations, we find the dependence of the parameters in (8) on the parameter a (see Fig. 3).

The values of moduli for some a are given on the Table 1.

TABLE 1. The values of moduli and capacities for some a (Example 1).

a	0	1	2	3	4	5	6	7
m	0.56247	0.62207	0.72955	0.82469	0.90239	0.96656	1.02073	1.06743
cap	1.77787	1.60753	1.37070	1.21258	1.10817	1.03459	0.97968	0.93682

Example 2. We also computed the moduli mod G and the corresponding capacities $\text{cap } G$ for some domains $G(A_1, A_2, A_3, A_4)$ when A_k are from the integer lattice in the complex plane. Comparison our results with those obtained by other methods show very good coincidence, up to 10^{-6} . In Table 2 we give some values of capacities obtained by our method and by a MATLAB algorithm written by Prof. M. Nasser [24]; the values are given with 8 digits after the decimal point.

6. MONOTONICITY OF CONFORMAL MODULE

Now we will investigate behavior of the conformal module of $G = G(A_1, A_2, A_3, A_4)$ in the case when the segment A_3A_4 is fixed and the segment A_1A_2 slides along a straight line with a fixed length. This case is equivalent to the situation when the segment A_1A_2 is fixed and A_3A_4 has a fixed length and shifts by vectors with a fixed direction.

We note that some similar problems for quadrangles were investigated by Dubinin and Vuorinen [14].

Without loss of generality we may assume that A_1A_2 lies on the real axis and A_1 is the left endpoint of the segment. Moreover, we can consider the family with $A_1 = t$, $A_2 = t + l$, where l is the length of A_1A_2 . Then $\dot{A}_1(t) = \dot{A}_2(t) = 1$. It is clear that $\dot{A}_3(t) = \dot{A}_4(t) = 0$. From Corollary 1 we obtain that $\dot{m}(t) = \pi(\gamma_1(t) + \gamma_2(t))$ where $\gamma_k(t) = 1/f''(x_k, t)$, $k = 1, 2$. It is easy to see that $f''(x_1, t) > 0$ and $f''(x_2, t) < 0$. Therefore,

$$\dot{m}(t) = \pi (|\gamma_1(t)| - |\gamma_2(t)|) = \pi \left(\frac{1}{|f''(x_1, t)|} - \frac{1}{|f''(x_2, t)|} \right).$$

If $|f''(x_1, t)| > |f''(x_2, t)|$, then, when moving a segment A_1A_2 to the right, the conformal module of $G = G(A_1, A_2, A_3, A_4)$ decreases, otherwise, it increases. At critical points of the module we have $|f''(x_1, t)| = |f''(x_2, t)|$.

TABLE 2. The values of capacities for some domains $G(A_1, A_2, A_3, A_4)$ (Example 2).

	A_1	A_2	A_3	A_4	cap G	
					our results	Nasser's
1	i	$2+i$	$-2-i$	$-1-i$	1.44058466	1.44058486
2	i	$2+i$	$-2-2i$	$-1-2i$	1.30971558	1.30971579
3	i	$2+i$	$3-2i$	$4-3i$	1.35832035	1.35832051
4	i	$2+2i$	$-2-i$	$-1-i$	1.42710109	1.42710150
5	i	$2+2i$	$-2-2i$	$-1-2i$	1.29776864	1.29776889
6	i	$2+2i$	$3-2i$	$4-3i$	1.32814214	1.32814249
7	i	$3+2i$	$-2-i$	$-1-i$	1.49363842	1.49363897
8	i	$3+2i$	$-2-2i$	$-1-2i$	1.36333122	1.36333156
9	i	$3+2i$	$3-2i$	$4-3i$	1.45844055	1.45844094
10	i	$3i$	3	4	1.29126199	1.29126229
11	i	$3i$	0	2	2.18251913	2.18251948
12	i	$3i$	-3	2	2.82846257	2.82846345
13	i	$3+i$	$-i$	$3-i$	2.69941565	2.69941690
14	i	$3+2i$	$-i$	$3-2i$	2.23470313	2.23470399
15	i	$3+3i$	$-i$	$3-3i$	2.11547784	2.11547801

Now we compare $|f''(x, t)|$ at the points x_1 and x_2 using methods of the symmetrization theory. We will temporarily assume that A_1A_2 is symmetric with respect to the imaginary axis, i.e. A_2 lies on the positive part of the real axis symmetrically to A_1 .

In the following lemma we investigate a more general case when the considered doubly connected domain G is the exterior of the segment A_1A_2 and some continuum Q ; if $Q = A_3A_4$, we obtain our case.

Lemma 1. *Let the continuum Q lie in the right half-plane $\operatorname{Re} w > 0$ and let $\psi : \{q < |\zeta| < 1\} \rightarrow G$ be a conformal mapping. If ζ_1 and ζ_2 are the points of the unit circle corresponding to the endpoints A_1 and A_2 of the segment, then $|\psi''(\zeta_1)| > |\psi''(\zeta_2)|$.*

Proof. Without loss of generality we can assume that A_1A_2 coincides with the segment $[-1, 1]$ (Fig. 4, a)). Consider the function φ inverse to the Joukowski function; it maps G onto the exterior of the unit disk with excluded set $Q_1 := \varphi(Q)$. Using the Riemann-Schwarz symmetry principle, we conclude that $\varphi(Q)$ lies in the right half-plane. Now applying the symmetry principle once more, we can extend $\varphi \circ \psi$ to the annulus $A := \{q < |\zeta| < 1/q\}$. The function $\varphi \circ \psi$ maps it onto the doubly connected domain $G_1 := \overline{\mathbb{C}} \setminus (Q_1 \cap Q_2)$; here Q_2 is symmetric to Q_1 with respect to the unit circle, therefore, it also lies in the right half-plane.

Now consider the reduced moduli of A at the points ζ_1 and ζ_2 ; it is obvious that they are equal, i.e. $r(A, \zeta_1) = r(A, \zeta_2)$. On the other side,

$$r(G_1, -1) = r(A, \zeta_1) + \frac{1}{4\pi} \log |\psi''(\zeta_1)|, \quad r(G_1, 1) = r(A, \zeta_2) + \frac{1}{4\pi} \log |\psi''(\zeta_2)|.$$

Therefore, we only need to show that $r(G_1, -1) > r(G_1, 1)$. But this conclusion follows from [12], thm. 1.2, because the configuration $(G_1, -1)$ is obtained from $(G_1, 1)$ by polarization.

Corollary 2. *Let A_3A_4 be a fixed segment in the right half-plane, intersecting the real at the point \tilde{x} , and let one of its endpoints lie on the imaginary axis. Let A_1A_2 be the segment*

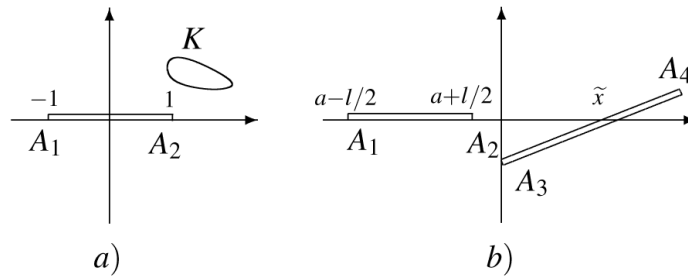


FIGURE 4. The domain G : a) in Lemma 1; b) in Corollary 2.

$[a - l/2, a + l/2]$ on the real axis with a fixed length l (Fig. 4, b)). If $\tilde{x} \leq l/2$, then, when a increases from $-\infty$ to $\tilde{x} - l/2$, the conformal module of $G(A_1, A_2, A_3, A_4)$ decreases from $+\infty$ to 0. If $\tilde{x} > l/2$, then the conformal module decreases from $+\infty$ to some positive value, when a increases from $-\infty$ to 0.

If A_3A_4 does not intersect the real axis then the conformal module decreases for a close to $-\infty$ and increases for a close to $+\infty$. In this connection the problem arises: does the module always have a unique minimum or are there situations when it has more than one (local) minimum?

It is also interesting to investigate the problem for the case when the slits are parallel to each other. Then, using the result that the conformal module decreases after symmetrization with respect to a straight line, we conclude that the minimum of the conformal module is attained for the case of slits symmetric with respect to the orthogonal line.

The same is valid when the slits are perpendicular to each other, one of the slits is fixed and does not intersect the straight line containing the second one. Then the minimal module is attained for the case when the second slit is symmetric with respect to the line containing the first slit.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN, Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series, 55 For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964 xiv+1046 pp.
- [2] N.I. AKHIEZER, Elements of the Theory of Elliptic Functions. Transl. of Mathematical Monographs, vol. 79, American Mathematical Soc., RI, 1990.
- [3] I.A. ALEKSANDROV, Parametric continuations in the theory of univalent functions. Nauka, Moscow, 1976 (Russian).
- [4] D. BETSAKOS, K. SAMUELSSON, AND M. VUORINEN, The computation of capacity of planar condensers. Publ. Inst. Math. (Beograd) (N.S.) 75(89) (2004), 233–252.
- [5] S. BEZRODNYKH, A. BOGATYREV, S. GOREINOV, O. GRIGORIEV, H. HAKULA, AND M. VUORINEN: On capacity computation for symmetric polygonal condensers. J. Comput. Appl. Math. 361 (2019), 271–282, <https://doi.org/10.1016/j.cam.2019.03.030>
- [6] M.D. CONTRERAS, S. DIAZ-MADRIGAL, AND P. GUMENYUK, Loewner theory in annulus I: Evolution families and differential equations. Trans. Amer. Math. Soc. 365 (2013), 2505–2543.
- [7] M.D. CONTRERAS, S. DIAZ-MADRIGAL, AND P. GUMENYUK, Loewner theory in annulus II: Loewner chains. Analysis and Mathematical Physics, 2011, Volume 1, Issue 4, 351–385.
- [8] D. CROWDY, Schwarz-Christoffel mappings to unbounded multiply connected polygonal regions. Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 2, 319–339.
- [9] T.K. DELILLO, A.R. ELCRAT, AND J.A. PFALTZGRAFF, Schwarz-Christoffel Mapping of the Annulus. SIAM Rev., 2001, 43(3), 469–477.

- [10] T.K. DELILLO, T.A. DRISCOLL, A.R. ELCRAT, AND J.A. PFALTZGRAFF, Computation of Multiply Connected Schwarz-Christoffel Maps for Exterior Domains. *Computational Methods and Function Theory*. 2006, Volume 6, Issue 2, pp 301–315.
- [11] T.A. DRISCOLL AND L.N. TREFETHEN, Schwarz-Christoffel mapping. *Cambridge Monographs on Applied and Computational Mathematics*, 8. Cambridge University Press, Cambridge, 2002. xvi+132 pp.
- [12] V.N. DUBININ, Symmetrization in the geometric theory of functions of a complex variable. *Russian Mathematical Surveys*, 1994, 49:1, 1–79.
- [13] V.N. DUBININ, Condenser capacities and symmetrization in geometric function theory. Translated from the Russian by Nikolai G. Kruzhilin, Springer, Basel, 2014, xii+344.
- [14] V.N. DUBININ AND M. VUORINEN, On conformal moduli of polygonal quadrilaterals. *Israel J. Math.* 171, 111–125 (2009).
- [15] J.B. GARNETT, D.E. MARSHALL, Harmonic measure. Reprint of the 2005 original. *New Mathematical Monographs*, 2. Cambridge University Press, Cambridge, 2008. xvi+571 pp.
- [16] G.M. GOLUZIN, On the parametric representation of functions univalent in a ring. *Mat. Sb. (N.S.)*, 29(71):2 (1951), 469–476 (Russian).
- [17] G.M. GOLUZIN, *Geometric Theory of Functions of a Complex Variable*. Translations of Mathematical Monographs, AMS, 1969.
- [18] H. HAKULA, A. RASILA AND M. VUORINEN, On moduli of rings and quadrilaterals: algorithms and experiments. *SIAM J. Sci. Comput.* 33 (2011), no. 1, 279–302.
- [19] H. HAKULA, A. RASILA, AND M. VUORINEN, Conformal modulus and planar domains with strong singularities and cusps. *Electron. Trans. Numer. Anal.* 48 (2018), 462–478.
- [20] P. HENRICI, *Applied and Computational Complex Analysis, Vol. 3: Discrete Fourier Analysis, Cauchy Integrals, Construction of Conformal Maps, Univalent Functions*. Wiley, New York, 1986.
- [21] YU. KOMATU, Untersuchungen über konforme Abbildung zweifach zusammenhängender Bereiche. *Proc. Phys., Math. Soc. Japan* 25(1943), 1–42
- [22] YU. KOMATU, Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung über Polygonalringgebiete. *Jap. J. Math.* 19 (1945), 203–215 (German).
- [23] W. KOPPENFELS AND F. STALLMANN, *Praxis der konformen Abbildung*. Berlin: Springer, 1959 (German).
- [24] M. NASSER AND M. VUORINEN, Computation of conformal invariants. Manuscript, August 2019.
- [25] S.R. NASYROV, Uniformization of One-Parametric Families of Complex Tori. *Russian Mathematics*, 2017, Vol. 61, No. 8, 36–45.
- [26] S.R. NASYROV, Families of elliptic functions and uniformization of complex tori with a unique point over infinity. *Probl. Anal. Issues Anal.*, 2018, Vol. 7(25):2, 98–111.
- [27] S.R. NASYROV, Uniformization of Simply-Connected Ramified Coverings of the Sphere by Rational Functions. *Lobachevskii Journal of Mathematics*. 2018, V.39, No.2 252–258.
- [28] F.W.J. OLVER, D.W. LOZIER, R.F. BOISVERT, AND CH.W. CLARK (eds.) *NIST Handbook of Mathematical Functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. xvi+951 pp.
- [29] N. PAPAMICHAEL AND N. STYLIANOPOULOS, Numerical conformal mapping. Domain decomposition and the mapping of quadrilaterals. *World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ*, 2010. xii+229 pp.
- [30] W.P. REINHARDT, P.L. WALKER *Digital Library of Mathematical functions*. Chapter 20. Theta Functions. <https://dlmf.nist.gov/20>.
- [31] W.P. REINHARDT, P.L. WALKER *Digital Library of Mathematical functions*. Chapter 23. Weierstrass Elliptic and Modular Functions. <https://dlmf.nist.gov/23>.

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