Turun yliopisto
University of Turku

MEMORY EFFECTS AND GEOMETRICAL CONSIDERATIONS IN OPEN QUANTUM SYSTEMS

Antti Karlsson

TURUN YLIOPISTON JULKAISUJA – ANNALES UNIVERSITATIS TURKUENSIS
Sarja - ser. AI osa - tom. 541 | Astronomica - Chemica - Physica - Mathematica | Turku 2016

University of Turku

Faculty of Mathematics and Natural Sciences
Department of Physics and Astronomy

Supervised by

Jyrki Piilo
Docent of Theoretical Physics
Department of Physics and Astronomy
University of Turku
Finland

Reviewed by

Andrea Smirne
Researcher
Institute of Theoretical Physics
Ulm University
Germany

Roberta Zambrini
Tenured Researcher
Institute for Cross-Disciplinary Physics
and Complex Systems
Edici Instituts Universitaris de Recerca
Campus Universitat de les Illes Balears
Spain

Opponent

Dariusz Chruściński
Professor of Physics
Institute of Physics, Faculty of Physics,
Astronomy and Informatics
Nicolaus Copernicus University
Poland

ISBN 978-951-29-6503-8 (PRINT)
ISBN 978-951-29-6504-5 (PDF)
ISSN 0082-7002 (Print)
ISSN 2343-3175 (Online)
Painosalama Oy - Turku, Finland 2016

–In loving memory of my father, who now knows the answer to the greatest mystery of life.

Acknowledgments

It all started with stargazing and a popular science book on astrobiology in high school. It then led to another one and another one. Now the circle closes on a book of my own. I am glad that I have not been on this journey alone.

I would like to thank my supervisor Jyrki for sharing his scientific knowledge with me and introducing me to the world of research. Thank you for guiding me through all of this. Thank you to all of my colleagues from the corridor for a great work environment. Especially Kimmo, thank you for the most memorable moments on our trips around Europe. Thank you Otto for all of your advice and support, they really helped a lot. Thank you also to all of my other colleagues and collaborators in Turku and around the world.

I would also like to thank all of my elementary and high school teachers, especially Katja Nivala for laying the first bricks of the foundation of my education. I must also thank my high school mathematics teacher Taina Eskola and physics teacher Risto Laiho (who borrowed the said popular science book to me) for really sparking my interest in mathematics and physics.

Thank you to all of my friends for your support. Especially Roope and Tuomas, thank you for the endless hours doing homework and also other activities during our studies in Turku. Thank you Juho, Jussi and Ville for the countless lost lures to the unforgiving rocks in the waters surrounding Turku. Also everyone from the Pommari gym and TAK: you will probably never read this but thank you for keeping me sane.

Thank you mother, for your unconditional care and love. There are no words to describe all the work you have done which has helped me to get where I am today.

Finally, thank you father for all of your knowledge and encouragement. It breaks my heart not to be able to share this moment with you, but maybe we some day will meet again, there far beyond the stars.

Contents

Abstract

This thesis discusses memory effects in open quantum systems with an emphasis on the Breuer, Laine, Piilo (BLP) measure of non-Markovianity. It is shown how the calculation of the measure can be simplified and how quantum information protocols can benefit from memory effects. The superdense coding protocol is used as an example of this. The quantum Zeno effect will also be studied from the point of view of memory effects. Finally the geometric ideas used in simplifying the calculation of the BLP measure are applied in studying the amount of resources needed for detecting bipartite quantum correlations. It is shown that to decide without prior information if an unknown quantum state is entangled or not, an informationally complete measurement is required.

The first part of the thesis contains an introduction to the theoretical ideas such as quantum states, closed and open quantum systems and necessary mathematical tools. The theory is then applied in the second part of the thesis as the results obtained in the original publications I-VI are presented and discussed.

Tiivistelmä

Tämä väitöskirja käsittelee muisti-ilmiöitä avoimissa kvanttisysteemeissä keskittyen erityisesti Breuer, Laine, Piilo (BLP) ei-markovisuus-mittaan. Työssä osoitetaan, miten mitan laskemista voidaan helpottaa ja miten kvantti-informaatioprotokollat voisivat hyötyä muistiefekteistä. Esimerkkinä tästä käytetään niin sanottua supertiheää koodausta. Myös Zeno-ilmiötä tutkitaan muisti-ilmiöiden näkökulmasta. Lopuksi mitan laskemisen helpottamisessa käytettyjä menetelmiä sovelletaan kvanttikorrelaatioiden olemassaolon osoittamiseen vaadittavien resurssien määrän tutkimiseen. Työssä näytetään, että ilman ennakkotietoja tuntemattoman kvanttitilan kietoutuneisuuden osoittaminen vaatii informatiivisesti täydellisiä mittauksia.

Työn ensimmäisessä osassa esitellään teoreettisia käsitteitä kuten kvanttitiloja, suljettuja ja avoimia kvanttisysteemejä sekä tarvittavia matemaattisia työkaluja. Työn toisessa osassa osajulkaisuissa I-VI saadut tulokset esitellään ja teoriatyökaluja sovelletaan.

List of articles

This thesis consists of an introductory review and the following six articles:

I S.Wißmann, A. Karlsson, E.-M. Laine, J. Piilo, H.-P. Breuer
Optimal state pairs for non-Markovian quantum dynamics
Physical Review A 86, 062108 (2012) (6 pages)

II B.-H. Liu, S. Wißmann, X.-M. Hu, C. Zhang, Y.-F. Huang, C.-F. Li, G.-C. Guo, A. Karlsson, J. Piilo, H.-P. Breuer
Locality and universality of quantum memory effects
Scientific Reports 4, 6327 (2014) (5 pages)

III B.-H. Liu, X.-M. Hu, Y.-F. Huang, C.-F. Li, G.-C. Guo, A. Karlsson, E.-M. Laine, S. Maniscalco, C. Macchiavello, J. Piilo
Experimental demonstration of efficient superdense coding in the presence of non-Markovian noise
EPL 114, 10005 (2016) (5 pages)

IV A. Karlsson, H. Lyyra, E.-M. Laine, S. Maniscalco, J. Piilo
Non-Markovian dynamics in two-qubit dephasing channels with an application to superdense coding
Physical Review A 93, 032135 (2016) (7 pages)

V C. Carmeli, T. Heinosaari, A. Karlsson, J. Schultz, A. Toigo
Verifying the quantumness of bipartite correlations
Accepted to Physical Review Letters, arXiv:1510.03240 [quant-ph] (5 pages)

VI A. Karlsson, F. Francica, J. Piilo, F. Plastina
Quantum Zeno effect and non-Markovianity in a three-level system
Submitted, arXiv:1602.01235 [quant-ph] (8 pages)

Chapter 1

Introduction

Quantum mechanics is at the heart of our understanding of the inner workings of nature on the small scale. The precision of the predictions is unrivalled and some of the implications of the theory shape our understanding about the nature of reality itself. It seems that on the fundamental level, nature is probabilistic instead of deterministic. In a classical world everything works perfectly deterministically and in principle, someone knowing all the positions and velocities of all of the particles in the universe, could predict exactly what happens in the future. All randomness or error would only be because of lack of knowledge or measurement precision or other such technical limitations. However, quantum theory tells us that this is not the case and that nature itself is genuinely probabilistic on the fundamental level; a fact that has sparked countless philosophical debates and confused even Einstein [1].

Quantum phenomena, such as entanglement and superposition of states are incredibly fragile to outside influences and quickly disappear on size and time scales of everyday life [2]. Effects such as entanglement manifest as correlations in measurement statistics unexplainable by classical physics. This makes it possible to complete tasks classically impossible, such as algorithms that quickly factorize large numbers and makes entanglement a vital resource for applications of quantum mechanics. Studying and understanding the correlations themselves and the dynamics of quantum systems interacting with their surroundings is then very important for finding, preserving and protecting the quantum properties.

Initially, quantum theory studied closed quantum systems, the dynamics of which are described by the Schrödinger equation [3]. Closed systems are perfectly isolated from their surroundings and as such do not lose information or energy. However, in practice almost all systems are open as they

cannot be perfectly isolated from their surroundings. The first approaches to this kinds of systems were microscopical models described by Markovian master equations [4, 5]. Markovian quantum channels describe dynamics such that the past states of the system are irrelevant for what happens in the future. The idea is intuitively the same as in a classical, discrete Markov process where the state of the system depends only on the previous time step and not on the possible untold numbers of preceding steps. A general theory of such dynamics was later developed, based on the theory of semigroups and the most general form of their generators [6, 7].

The majority of quantum processes however are non-Markovian, where the history of the system does affect the present. An example could be an atom relaxing to a lower energy state by emitting a photon to a very high quality optical cavity. The photon can return to re-exite the atom at some later time, making the dynamics of the atom non-Markovian. Such dynamics cannot be described by Markovian theory and various analytical and numerical methods have been developed [8–10]. So far no general, all encompassing theory of non-Markovian quantum dynamics exists. Actually the very definition of what non-Markovianity means is under discussion [11, 12].

Defining what non-Markovianity means in the quantum regime is not straightforward and there are many different schools of thought. Some rely on purely mathematical ideas and some prefer a physical interpretation as is seen when we introduce these ideas in later chapters. In this thesis we considered a physically motivated approach called the Breuer, Laine, Piilo (BLP) measure of non-Markovianity [13], which is based on the change of distinguishability of different quantum states. We studied some of its properties in detail and showed how the calculation of it can be simplified greatly. We were also able to clarify the interpretation of it by showing that it treats all parts of the quantum state space equally.

Other main goals of this thesis were finding applications for non-Markovianity and considering cases where it could be considered a resource for accomplishing certain quantum information tasks. Some ideas of taking advantage of non-Markovianity or noise in general have already been introduced for example in metrology [14], excitation transport [15], probing other degrees of freedom [16–18], entanglement distribution [19] and teleportation [20]. For applying non-Markovianity in quantum information we studied the well known superdense coding protocol with different kinds of noise added. We also studied a well known and established peculiarity of quantum mechanics called the quantum Zeno effect from the point of view

of non-Markovianity.

The outline of the thesis is the following. In Chapters 2 and 3 we introduce some of the necessary theoretical tools relevant for our studies. Chapter 2 describes the idea of a quantum state and some of the structures of the quantum state space that turned out to be extremely useful for our work. Chapter 3 discusses a bit more formally what closed and open quantum systems are, discusses non-Markovianity in more detail and introduces the Breuer, Laine, Piilo measure of non-Markovianity mathematically. In Chapter 4 we apply the tools of the previous chapters to show how the measure can be simplified. Chapter 5 contains an introductory overview of our results in applying non-Markovianity in the superdense coding protocol, a study of the quantum Zeno effect and tools from Chapter 2 are applied to studying the resources needed for detecting quantum correlations in a quantum state. Chapter 6 will conclude and summarize the results of the thesis and also discuss some ideas for future work.

The reader of this thesis is assumed to be familiar with basic linear algebra and the very basics of quantum mechanics. We take the Hilbert space structure of quantum mechanics as an axiom. We will use units such that $\hbar = 1$ unless explicitly stated otherwise. We will also take the notion of quantum measurement for granted. Thus in this thesis, observables are described by positive operator valued measures A that assign an effect operator L_j to each measurement outcome j and the effect operators satisfy $O \leq L_j \leq I$ and $\sum_j L_j = I$. For more information and a detailed discussion we point to [21–23] and the references therein. Of course, none of the following is meant to be a comprehensive, ground up constructive study in the subject. Rather, we will introduce various ideas and tools that have been useful and important in our studies.

Chapter 2

Quantum state space

Classically, everything one needs to know about a free particle to completely solve its dynamics, is the position $x(t_0)$ and velocity $\dot{x}(t_0)$ at some instant of time t_0. Then by using Newton's equations, one can solve for the trajectory of the particle and predict where it will be at a later time t. In other words the pair $(x(t_0), \dot{x}(t_0))$ defines the state of the classical system [24, 25] at time t_0. The state is something that encodes all there is to know about the system and can be used to make predictions about the future of the system. Of course this depends on what one is interested in. In our example we are interested in the dynamics of the system and do not care for example about the color of the particle, which of course cannot be solved by knowing $(x(t_0), \dot{x}(t_0))$.

In quantum mechanics the state is a similar set of numbers encoding all information about the system [21–23]. However, the crucial difference to classical mechanics is that we cannot predict exactly for example the path taken by a particle in an interferometer. In general we are only able to give probabilities of different possible outcomes, for example we can say that the particle takes path A with a probability of 70% and path B with a probability of 30%. This uncertainty is not due to our lack of knowledge or experimental limitations but is inherent to quantum theory itself. On the fundamental level nature seems to be probabilistic and this is described extremely well by quantum theory. Since the fifth Solvay conference in 1928 there have been and still are countless debates on what the meaning of quantum theory is or how it should be interpreted and what it tells us about the nature of physical reality [26]. These questions are far beyond the scope of this thesis and we will not address them here. For our purposes it is enough to take Hilbert space quantum mechanics as a set of rules and computational tools that predict and fit experimental data extremely well.

2.1 Quantum states

Let the d dimensional Hilbert space of our quantum system be $\mathcal{H}$. We denote the set of all linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and the set of all self-adjoint operators by $\mathcal{L}_S(\mathcal{H})$. Then the set $\mathcal{S}(\mathcal{H})$ of all possible states of our system is

$$\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{L}_S(\mathcal{H}) | \rho > 0,\ \mathrm{tr}(\rho) = 1\}. \tag{2.1}$$

This means that the set contains all positive linear operators of unit trace. These properties are needed for the states to make sense physically. Positivity guarantees that all measurement outcome probabilities are real and positive and the trace condition guarantees that the probabilities sum to one. The trace constraint is linear, but the positivity constraint is not, which means that $\mathcal{S}(\mathcal{H})$ cannot have the structure of a vector space. A simple calculation shows that the positivity condition is convex. Let ρ_1, $\rho_2 \in \mathcal{S}(\mathcal{H})$ be arbitrary and $\lambda \in [0, 1]$. Then

$$\langle\psi|\, \lambda\rho_1 + (1-\lambda)\rho_2|\psi\rangle = \lambda\langle\psi|\rho_1|\psi\rangle + (1-\lambda)\langle\psi|\rho_2|\psi\rangle \geq 0, \tag{2.2}$$

by the positivity of ρ_1 and ρ_2. As stated above, trace is linear and then trivially

$$\mathrm{tr}\{\lambda\rho_1 + (1-\lambda)\rho_2\} = 1. \tag{2.3}$$

The above result means that if ρ_1 and ρ_2 are states, then their convex combination is also a state. For clarity we only studied the convex combination of two states, but the above considerations generalize trivially for any finite convex combination. In other words $\mathcal{S}(\mathcal{H})$ is a convex set. One could think that convex sets cannot be very complicated geometrically, since the intuition is that they are more or less like deformed circles. For a two dimensional quantum system this is true as we will see later, but for higher dimensions the geometric structure becomes very rich and complicated.

2.1.1 Bloch sphere

The set of quantum states was defined as a convex subset of the real vector space $\mathcal{L}_S(\mathcal{H})$ of all Hermitian operators on the Hilbert space $\mathcal{H}$ of dimension d. These operators can be written in different decompositions in their matrix representation and some of these decompositions hold a special role in quantum mechanics. We will study them next.

Any matrix $H \in \mathcal{L}_S(\mathcal{H})$ can be written as

$$H = \mathrm{tr}(H)\frac{1}{d}I + M, \tag{2.4}$$

where $M = H - \mathrm{tr}(H)\frac{1}{d}I$. M is now traceless and Hermitian, which means that it is in the Lie algebra $\mathfrak{su}(d^2-1)$ (given that we use the physics convention with the factor i in the exponential map that maps elements of the Lie algebra to elements of the Lie group. Without this the Lie algebra consists of traceless anti-Hermitian matrices).

Lie algebras and their representation theory have a rich, widely studied and solved structure and are used in many areas of physics [27–29]. Because of this, there are many canonical choices for their generators. Being a generator of an algebra $\mathfrak{a}$ means that the elements called generators form the smallest subalgebra of $\mathfrak{a}$ that contains the whole of $\mathfrak{a}$. In the case of Lie algebras the algebra structure is given by the Lie bracket, or more familiarly the commutator. This is the mathematical background from which the familiar commutation relations used in many parts of physics come from. Now by using some generators $\{\sigma_i\}$ as discussed above we can write

$$H = \frac{1}{d}\left(\mathrm{tr}(H)I + \sum_{i=1}^{d^2-1} r_i\sigma_i\right) \equiv \frac{1}{d}\left(\mathrm{tr}(H)I + \vec{r}\cdot\vec{\sigma}\right), \tag{2.5}$$

where r_i are suitably chosen real coefficients. We see that after fixing the generators $\{\sigma_i\}$, a unique set of numbers $\{r_i\}$ is associated to each H. These numbers can be collected to a vector $\vec{r}$ in $\mathbb{R}^{d^2-1}$. In this way we have defined a vector space isomorphism between $\mathcal{L}_S(\mathcal{H})$ and $\mathbb{R}^{d^2-1}$. The restriction of this map to the set of quantum states is called the generalized Bloch representation for $d > 2$ and the Bloch representation for $d = 2$. The vector $\vec{r}$ is called the (generalized) Bloch vector.

The Bloch representation is used very often in quantum physics. So often in fact that most physicist do not make any distinction between the state space and its representation in $\mathbb{R}^3$. The reason is that in this case the representation of the state space is a ball of radius 1. This can be seen by studying how the trace and positivity constraints translate to constraints on the coefficients r_i. We will do this for the two dimensional case by studying the eigenvalues of the quantum states. Let $d = 2$, $\rho \in \mathcal{S}(\mathcal{H})$ and let the generators $\{\sigma_1, \sigma_2, \sigma_3\}$ in the Bloch representation be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

Then a straightforward calculation shows that the eigenvalues of ρ are

$$\lambda_1 = \frac{1}{2}\left(1 - \sqrt{r_1^2 + r_2^2 + r_3^2}\right) \quad \lambda_2 = \frac{1}{2}\left(1 + \sqrt{r_1^2 + r_2^2 + r_3^2}\right). \tag{2.7}$$

The trace condition is satisfied by construction and it is evident that the positivity constraint means that the norm of the vector $\vec{r}$ cannot be higher than 1. On the other hand we see that all vectors $\vec{r}$ with norm less than or equal to one correspond to valid quantum states. This means that for $d = 2$ the image of $\mathcal{S}(\mathcal{H})$ in the Bloch representation is the unit ball in $\mathbb{R}^3$. Unfortunately this property does not generalize. Already for $d = 3$ and using for example the Gell-Mann matrices as generators, the image of $\mathcal{S}(\mathcal{H})$ is not the whole unit ball in $\mathbb{R}^8$ anymore but something more exotic [30]. The pure states still have generalized Bloch vectors of unit length, so they live on the unit sphere, but not all of the points on the sphere or inside the unit ball correspond to valid quantum states anymore. The trace condition is of course always met, but the positivity condition on the coefficients of the generalized Bloch vector is not so simple anymore and is broken for many points inside the unit ball.

2.2 Direction in state space

When we speak about direction in everyday life, the intuition we often have is an arrow pointing somewhere, for example from point A to point B on a map. This notion generalizes quite naturally to a vector space, where we can define addition and subtraction of its elements. This is depicted in Fig. 2.1, where we define the direction vector from point A to point B in some arbitrary vector space.

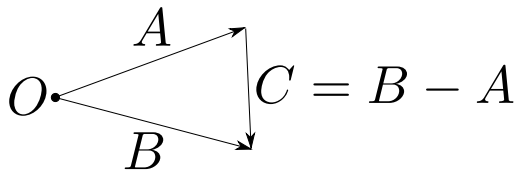

Figure 2.1: Direction from point A to point B. O denotes the origin of the vector space.

As depicted in Fig. 2.2, notice that we can move the vector C around by parallel transport anywhere we want and nothing changes about the

direction it defines. Another way of thinking about this is that there are infinitely many vectors that give the same C as their difference. This is intuitively clear; once we fix a direction vector we are interested in, it does not matter where we draw it.

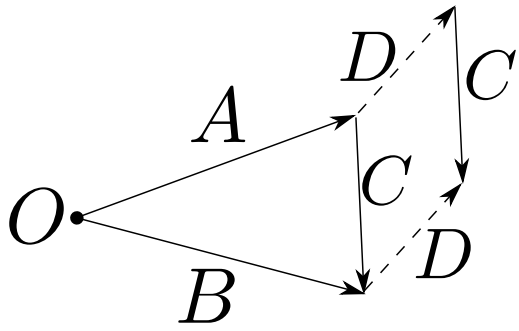

Figure 2.2: Parallel transporting does not change the direction vector C.

On an abstract level, all of the above ideas can be translated to the level of quantum states. The quantum state space itself does not have the structure of a vector space, but it is a subset of the vector space of Hermitian matrices, which means that we can still make sense of addition and subtraction. Using the intuition above, we can think that for some arbitrary quantum states ρ_1 and ρ_2, the vector $\rho_1 - \rho_2$ defines a direction in the state space, or more precisely in the vector space of Hermitian matrices, since the difference matrix is not a valid state anymore.

Returning to our previous example, let ρ_1 and ρ_2 be arbitrary states of a two dimensional system with Bloch vectors $\vec{r}_1$ and $\vec{r}_2$. Then using the Bloch representation we can write

$$\rho_1 - \rho_2 = \frac{1}{2}(\vec{r}_1 - \vec{r}_2) \cdot \vec{\sigma}. \tag{2.8}$$

With this in mind we can identify the direction defined by ρ_1 and ρ_2 with the direction in $\mathbb{R}^3$ defined by the corresponding Bloch vectors. This direction vector can be translated freely like in Fig. 2.2. Thus it is clear that there are many different choices for ρ_1 and ρ_2 that give the same difference. Similar reasoning applies also for higher dimensional examples, which however are difficult to imagine, since already for a three dimensional quantum system the generalized Bloch representation is in $\mathbb{R}^8$ as we saw earlier. On an abstract level the ideas are nevertheless clear.

2.3 Distance in state space

In addition to direction, we want to make sense of the notion of distance in the set of quantum states. For this, we need to define some kind of a metric. Of course any suitable metric would do, but we choose the trace distance [23], which is the metric induced by the trace norm. The reason for our choice is the physical interpretation the trace distance has. We start by defining the trace norm and then studying and listing some of the properties of it and the induced metric.

2.3.1 Trace distance

The trace norm of a matrix A is defined as

$$||A||_{\mathrm{tr}} = \mathrm{tr}|A|, \tag{2.9}$$

where $|A| = \sqrt{A^\dagger A}$. It is clear from the definition that for any matrix A, the trace norm is just the sum of the singular values. For Hermitian matrices in particular, it is then the sum of the absolute values of the eigenvalues

$$||A||_{\mathrm{tr}} = \sum_n |\lambda_n|. \tag{2.10}$$

The induced metric, the trace distance of two quantum states ρ_1 and ρ_2 is

$$D(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2||_{\mathrm{tr}}. \tag{2.11}$$

Using the triangle inequality and the fact that quantum states are positive matrices with trace one, we see that for arbitrary ρ_1 and ρ_2

$$0 \leq D(\rho_1, \rho_2) \leq \frac{1}{2}(||\rho_1||_{\mathrm{tr}} + ||\rho_2||_{\mathrm{tr}}) = 1. \tag{2.12}$$

2.3.2 Jordan-Hahn decomposition

Before continuing further, we will introduce a simple decomposition for Hermitian matrices which turns out to be an extremely useful tool in our calculations. The decomposition is a straightforward consequence of the spectral theorem and is sometimes called the Jordan-Hahn decomposition [31]. The idea is that we can decompose any Hermitian matrix as the difference of two positive matrices with orthogonal supports. To see this,

let A be an arbitrary Hermitian matrix. It follows from the spectral theorem that we can write

$$A = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|, \tag{2.13}$$

which we can further decompose as

$$\sum_n \lambda_n |\phi_n\rangle\langle\phi_n| = \sum_{\substack{k \text{ s.t.} \\ \lambda_k \geq 0}} \lambda_k |\phi_k\rangle\langle\phi_k| - \sum_{\substack{l \text{ s.t.} \\ \lambda_l < 0}} |\lambda_l||\phi_l\rangle\langle\phi_l| \equiv M_+ - M_-, \tag{2.14}$$

where M_+ and M_- are positive and orthogonal as desired.

A few useful properties of the decomposition are now immediate from the definitions. Namely

1. If $\mathrm{tr}(A) = 0$ then $\mathrm{tr}(M_+) = \mathrm{tr}(M_-)$.
2. $|A| = M_+ + M_-$.
3. $||A||_{\mathrm{tr}} = \mathrm{tr}(M_+ + M_-)$ (direct consequence of property 2).

2.3.3 Trace distance as information

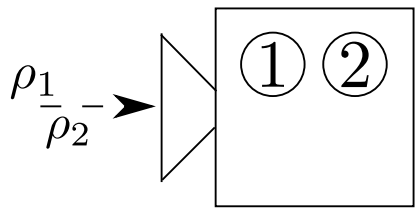

Figure 2.3: Two state discrimination problem.

Trace distance also has a physical interpretation as it is closely related to the optimal probability $P_d(\rho_1, \rho_2)$ of discriminating two unknown quantum states ρ_1 and ρ_2 using a quantum measurement, given a promise that one of them was prepared [23, 32]. This can be seen with the following short calculation following the presentation in [23].

Let us assume that one of two quantum states ρ_1 or ρ_2 are prepared, both with equal probability. As depicted in Fig. 2.3, the task of finding out which of the states was prepared, can be thought of as having a box

taking a state as an input, and flashing light 1 or light 2 as an output if it guesses that the input was ρ_1 or ρ_2. This box can then be described using a POVM with two effects L_1 and $L_2 = I - L_1$ corresponding to the outcomes 1 and 2. The error probability for the discrimination problem is then

$$\begin{aligned} P_{\text{err}} &= P(\rho_1 \text{ was prepared and guess } \rho_2) + P(\rho_2 \text{ was prepared and guess } \rho_1) \\ &= \frac{1}{2}\text{tr}(\rho_1 L_2) + \frac{1}{2}\text{tr}(\rho_2 L_1) = \frac{1}{2}\text{tr}(\rho_1 L_2) + \frac{1}{2}\text{tr}(\rho_2(I - L_2)) \\ &= \frac{1}{2}\left(1 + \text{tr}\left((\rho_1 - \rho_2)L_2\right)\right). \end{aligned} \tag{2.15}$$

We see that to minimize P_{err} we need to make $\text{tr}\left((\rho_1 - \rho_2)L_2\right)$ as small as possible. To do this, we apply the spectral theorem to $\rho_1 - \rho_2$ and write

$$\text{tr}\left((\rho_1 - \rho_2)L_2\right) = \sum_n \lambda_n \langle \phi_n | L_2 | \phi_n \rangle \geq -\sum_{\substack{k \text{ s.t.} \\ \lambda_k < 0}} |\lambda_k|, \tag{2.16}$$

where we have used the fact that $0 \leq \langle \psi | L_2 | \psi \rangle \leq 1$ for all $\psi \in \mathcal{H}$, because L_2 is an element of a POVM. We can achieve the lower bound of equation (2.16) by choosing L_2 to be the projection on the eigenvectors corresponding to the negative eigenvalues of $\rho_1 - \rho_2$, which is

$$L_2 = \sum_{\substack{k \text{ s.t.} \\ \lambda_k < 0}} |\phi_k\rangle\langle\phi_k|. \tag{2.17}$$

It is clear that this is also the optimal lower bound, since it maximizes the contribution of the negative parts to the sum, removes the positive parts completely and we had that $0 \leq \langle \psi | L_2 | \psi \rangle \leq 1$ for all $\psi \in \mathcal{H}$. Thus we have that the minimum error probability is

$$P_{\text{err}} = \frac{1}{2}(1 - \sum_{\substack{k \text{ s.t.} \\ \lambda_k < 0}} |\lambda_k|) = \frac{1}{2}(1 - \frac{1}{2}\text{tr}|\rho_1 - \rho_2|) = \frac{1}{2}(1 - D(\rho_1, \rho_2)), \tag{2.18}$$

where we have used the Jordan-Hahn decomposition and its properties to write

$$\text{tr}(M_-) = \sum_{\substack{k \text{ s.t.} \\ \lambda_k < 0}} |\lambda_k| = \frac{1}{2}\text{tr}(M_- + M_+) = \frac{1}{2}\text{tr}|\rho_1 - \rho_2|. \tag{2.19}$$

We see that if the trace distance goes to one, P_{err} goes to zero and the states can be perfectly distinguished from each other. If the distance is

very small, P_{err} is close to $\frac{1}{2}$ and the discrimination is close to being a coin toss. By the above, the optimal probability P_d is then given as

$$P_d(\rho_1, \rho_2) = 1 - P_{\text{err}} = \frac{1}{2}(1 + D(\rho_1, \rho_2)). \tag{2.20}$$

2.4 Multipartite quantum systems

We have seen that the states of a quantum system S are described by specific kind of linear operators on a specific Hilbert space $\mathcal{H}_S$. But what about scenarios where there are for example two separate systems A and B which could interact, or for some other reason we would like to describe them as a single system? Each system separately is described by a Hilbert space $\mathcal{H}_i$ and it turns out [30] that the correct mathematical description of the system considered as a whole is the tensor product of the subsystem Hilbert spaces:

$$\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.21}$$

The states ρ_{AB} of the combined system are now described by the positive operators of trace one on the product space.

Now, if we have a joint system state ρ_{AB} and want to know the corresponding subsystem states, we use the partial trace operation, which taken over system B is a map $\text{tr}_B : \mathcal{L}(\mathcal{H}_{A+B}) \to \mathcal{L}(\mathcal{H}_A)$, $\rho_{AB} \mapsto \rho_A$ and is defined by the requirement

$$\text{tr}(\rho_A M) = \text{tr}(\rho_{AB} M \otimes I), \tag{2.22}$$

which must hold for all operators M in $\mathcal{L}(\mathcal{H}_A)$ of the system A. If we want to take the partial trace over A, we exchange B and A in the previous definition.

Physically the idea behind the definition of the partial trace is the following. Alice has a particle in her lab in Turku and Bob also has a particle in his lab in Helsinki such that the joint state of the particles is ρ_{AB}. Then, of course Alice can perform measurements on her particle, described by the operators M. Equally well we can think this as a measurement being performed on the combined system such that M is measured on Alice's side and nothing measured on Bob's side. The reduced state ρ_A obtained by the partial trace must be compatible with this, independent of the measurement performed.

2.4.1 Correlations

As a consequence of the mathematical structure of tensor products, ρ_{AB} is in general not determined by knowing ρ_A and ρ_B alone. Thus in a sense, a joint system in quantum mechanics is more than the sum of its parts. As a consequence of the mathematical structure, measurements performed by Alice and Bob on their parts of the joint system can exhibit correlations that are not explainable by classical physics. This phenomenon called entanglement is so much different than what our intuition tells us, that it even made Einstein believe that something about the theory is incomplete [33]. However experiments have shown one after another [34–37], that entanglement is real and the results of the experiments cannot be explained by local hidden variables. The existence of such hidden variables would mean that the observed non-local features of quantum mechanics would only seem non-local because of our lack of knowledge of some features of nature. The experiments say otherwise and it seems that nature really is non-local.

The states exhibiting these non classical correlations can be divided into many different classes and we will now give a few examples. In general any matrix ρ_{AB} in the tensor product space can be written as

$$\rho_{AB} = \sum_{i,j,k,l} c_{ijkl} |\phi_i\rangle\langle\phi_j| \otimes |\eta_k\rangle\langle\eta_l|, \tag{2.23}$$

where $\{|\phi_i\rangle\}$ is a basis of $\mathcal{H}_A$, $\{|\eta_k\rangle\}$ a basis of $\mathcal{H}_B$ and c_{ijkl} some complex numbers. If it happens that ρ_{AB} can be written as

$$\rho_{AB} = \rho_A \otimes \rho_B, \tag{2.24}$$

where ρ_A and ρ_B are quantum states, it is called factorizable. If the sum can be expressed as a convex combination of factorizable states, it is called separable. If it is not separable, it is called entangled. Entanglement has a huge number of applications that are impossible to realize by classical resources. These include superdense coding [38] also studied later in this thesis, teleportation [39] and quantum computing [40–42] with all of its applications just to name a few. The vast amount of applications makes entanglement a desired property to have and a lot of effort has been made in measuring and detecting it [43, 44].

Separable states are not entangled, but can still exhibit correlations not explainable in classical terms [45]. States that cannot be written as a

convex combination of the form

$$\rho_{AB} = \sum_i \lambda_i \rho_i \otimes |i\rangle\langle i|, \tag{2.25}$$

are called (left) discordant. Right discordant states are defined by exchanging the parts of the tensor product above. Discordant states have also been shown to have various applications, in which they outperform purely classical resources [46–50].

As it probably is intuitive, even when given a ρ_{AB} in the general form of equation (2.23), it is a difficult problem to find out with certainty if it can be written in for example one of the forms introduced above [51, 52]. The problem becomes even more difficult if there is no knowledge of the form of ρ_{AB}. One could imagine, that if for example one is not interested in anything else except the fact if a state is entangled or not, one would not necessarily need to know everything about the state itself. In the case of having the prior information that the unknown state is pure, this actually holds and it can be decided if the state is entangled or not without uniquely defining the state itself [53, 54]. However, as we see later in the case of no prior information, it happens that so much needs to be known about the state that this information can be used to identify the state uniquely.

Chapter 3

Open quantum systems

The concept of closed and open quantum systems goes hand in hand with the dynamics of the systems and how that is described. We start by looking at closed quantum systems, after which it is simple to move to the ideas of open systems.

3.1 Closed systems

The time evolution of a closed quantum system starting from a state $\rho(0)$ at time $t = 0$ due to a time independent Hamiltonian operator H is unitary and given by the equation

$$\rho(t) = U(t)\rho(0)U^\dagger(t), \tag{3.1}$$

where $U(t) = e^{-itH}$. By formally taking the derivative, invoking the chain rule and using the fact that any matrix commutes with its own exponential, we can write equation 3.1 in differential form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -iH\rho(t) + i\rho(t)H = -i[H, \rho(t)]. \tag{3.2}$$

This equation is called the Liouville-von Neumann equation and it is the fundamental equation of motion describing the time evolution of a quantum system. It is the density matrix counterpart of and also equivalent to the famous Schrödinger equation, which describes the time evolution of pure states.

Starting from the form of unitary dynamics and then going backwards to see the corresponding differential equation might seem strange. Especially when thinking in terms of canonical quantization, which derives

the Louville-von Neumann equation by starting with classical Hamiltonian mechanics and replaces the Poisson bracket with the Lie bracket and promotes classical variables to operators. However, as it seems that nature is fundamentally quantum, it would be more natural to start with quantum mechanics and obtain classical mechanics as some kind of a limit. For a step to this direction, we observe that from the definition of $U(t)$ it is clear that the dynamics is causal, meaning that the state of the system at time $t > 0$ is completely determined (with fixed $U(t)$) from the state at time $t = 0$. It is also clear that the dynamics is reversible, since

$$U^\dagger(t) = U(-t) = U^{-1}(t). \tag{3.3}$$

The map $t \mapsto U(t)$ is clearly continuous, meaning that if the change in time t is small, then so is the change in $U(t)$. By taking these three properties (causality, reversibility and continuity) as the defining characteristics of the dynamics of a closed system in the context of Hilbert space quantum mechanics, the form of the time evolution can be derived [22]. This means that if we accept the Hilbert space formalism as an axiom, or arrive to it by starting from a set of some other axioms, for example with the quantum logic approach [55, 56], the form of the time evolution of a closed system is fixed by these three requirements. The requirements have a clear physical meaning and seem much more natural than just postulating an abstract equation as the form of the dynamics. This is the whole idea of operational axiomatic approaches to the foundations of quantum mechanics. The goal is to define a set of operationally clear and meaningful axioms from which quantum mechanics can then be constructed. These kinds of studies are beyond the scope of this thesis and we will not go into more detail but turn our attention to open quantum systems.

3.2 Open systems

A closed quantum system can be big, for example it could be a system of 1000 interacting two level systems or it could be the polarization of a photon interacting with the frequency, a continuous degree of freedom. Many times though we are interested in only a small part of the whole system, for example only about the polarization of a photon and not so much about the frequency distribution. The part we are interested in, is called the system of interest or the open system and denoted by S. The rest is called the environment denoted by E as in Fig. 3.1. We assume that the open system can be described with some Hilbert space $\mathcal{H}_S$ and the

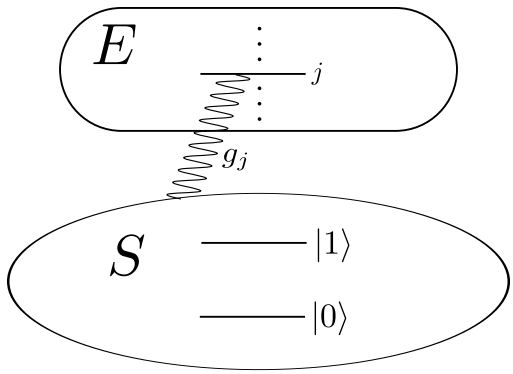

Figure 3.1: An environment E consisting of many different levels indexed by j is coupled to a two level system S such that every level of the environment has a coupling constant of its own.

environment with some Hilbert space $\mathcal{H}_E$. The combined bipartite system formed by S and E is assumed closed and described by the tensor product space $\mathcal{H}_{S+E}$. It is also assumed that initially the open system S and the environment E are uncorrelated, meaning that the state of the bipartite system at time 0 is

$$\rho_{S+E}(0) = \rho_S(0) \otimes \rho_E(0). \tag{3.4}$$

Because the bipartite system is closed, the time evolution is unitary, given by some $U(t)$. We discard the uninteresting environmental degrees of freedom by taking the partial trace and arrive at the state of the open system at time t

$$\rho_S(t) = \mathrm{tr}_E\{U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger(t)\}. \tag{3.5}$$

As can be seen form the form of the equation, different initial states $\rho_E(0)$ of the environment can result in different reduced dynamics. We could interpret the above equation as three transformations we do to the initial open system state $\rho_S(0)$ in sequence:

1. Calculate the tensor product with the state of the environment.

2. Evolve the product unitarily.

3. Take the partial trace over the environment.

All of the above are examples of completely positive (CP) trace preserving (TP) transformations [23]. A completely positive trace preserving map Φ is defined as

(CP) $\Phi \otimes \mathbb{I}^n$, is a positive map for all $n \in \mathbb{N}$. $\mathbb{I}^n$ is the identity map on the operators of the Hilbert space of dimension n

(TP) $\mathrm{tr}\{\Phi(\rho)\} = \mathrm{tr}\{\rho\}$ for all $\rho \in \mathcal{L}(\mathcal{H})$.

Positive maps map positive matrices to positive matrices, which combined with the TP condition seems sufficient to guarantee that states are mapped to states. This is not true however, because one needs to be able to consider the system of interest as a part of an arbitrary larger system. Theory must allow for considering for example the polarization of a photon in a lab in Turku together with a helium particle on the dark side of the moon. Then the combined system state of these two must evolve to a physically sensible state if nothing happens on the moon and the polarization is manipulated in Turku. The complete positivity criterion guarantees this for all possible scenarios such as that. This would not be guaranteed by only positive maps, since for example the partial transposition is a map that is positive, but not completely positive [23]. Completely positive trace preserving maps are usually called quantum channels, channels, CPT maps or CPTP maps in the context of quantum mechanics. These names will also be used in the rest of this thesis. Using the above notions, we rewrite the dynamics of an open system as

$$\rho_S(t) = \Phi_{0,t}\rho_S(0), \tag{3.6}$$

where $\Phi_{0,t}$ is a CPTP map or in other words a quantum channel and the subscripts emphasize that it evolves the state from initial time 0 to a later time $t \geq 0$. If $t = 0$ then $\Phi_{0,0} = \mathbb{I}$, which is the identity operation.

Notice that according to the above discussion, the map $\Phi_{0,t}$ is a channel for each fixed value of t. Of course there are infinitely many different values of t between for example $t = 0$ and $t = 1$, so a formula for $\Phi_{0,t}$ actually defines a continuous family of channels parametrized by t. Usually, especially in the open quantum systems community the words channel, family of channels and dynamical map are used interchangeably and the meaning is clear from the context.

Now, if we forgot the previous discussion and would only state that the most general form of sensible dynamics for an open quantum system is a channel, we could actually derive equation (3.5). The Stinespring

dilation theorem [57] states that all possible CPTP maps Φ can be written in the form of equation (3.5). See also [23, 58] for a more approachable explanation. Thus any quantum channel can be at least simulated by some environment and unitary evolution followed by a partial trace.

In practice, solving the form of the dynamics of the open system can be daunting, for example because of the large number of degrees of freedom of the environment, which we are not interested in anyway. Different kinds of analytical and numerical approaches to this have been developed [13,59–69]. In this thesis we will not go into detail in how the different methods work or how the dynamics can be solved, but will introduce some of the very basic mathematical structures and ideas. These are always in the background, no matter how the dynamics is solved.

3.3 Markovian open systems

As with closed systems, where we had the generic form for the unitary

$$U(t) = e^{-itH}, \tag{3.7}$$

we could hope to find something generic for the dynamical map $\Phi_{0,t}$. Equation (3.5) is a start, but it depends on the environment, which we want to avoid. In general it can be shown [70] that every CP map Φ can be written in what is called the Kraus form as

$$\Phi(\rho) = \sum_{i}^{n} K_i \rho K_i^\dagger \,. \tag{3.8}$$

Further requiring that $\sum_i^n K_i^\dagger K_i = I$ also guarantees the TP property. For d-dimensional Hilbert spaces n can always be chosen to be less than or equal to d^2. The operators K_i encode all the information of the channel and describe the effect of the environment on the open system. However, this is just an equivalent form for the channel and we would like to have something analogous to the Schrödinger equation, which can be used to solve for the form of the channel. A standard approach is to impose some general structure on the dynamical map besides the CPTP property and work from there. One fundamental structure is the semigroup property

$$\Phi_{0,t_1+t_2} = \Phi_{0,t_2}\Phi_{0,t_1}. \tag{3.9}$$

For a semigroup of CPTP maps, there exists a generator $\mathcal{L}$ meaning that the semigroup is a solution to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \mathcal{L}\rho_S(t), \tag{3.10}$$

where the generator can always be put to a general form [6, 7]

$$\mathcal{L}\rho_S = -i[H, \rho_S] + \sum_i \gamma_i (A_i \rho_S A_i^\dagger - \frac{1}{2}\{A_i^\dagger A_i, \rho_S\}). \tag{3.11}$$

Combining equations (3.10) and (3.11) we get the famous Gorini-Kossakowski-Sudarshan-Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i[H, \rho_S(t)] + \sum_i \gamma_i (A_i \rho_S(t) A_i^\dagger - \frac{1}{2}\{A_i^\dagger A_i, \rho_S(t)\}), \tag{3.12}$$

where H is the Hamiltonian describing the unitary part of the evolution due to the system of interest alone with some possible effects caused by the environmnent. The operators A_i describe the open part of the evolution. Notice that they also act only on the system and do not explicitly depend on the environmental degrees of freedom. The non-unitary effect of the environment has been encoded into the operators A_i. This equation is sometimes also called the Markovian master equation and can approximate many different physical scenarios. However, the semigroup requirement is quite stringent and there are many cases where the dynamics is such that it cannot be captured by such equations. For these kinds of cases there are many different approaches, but so far no general, all encompassing form such as equation (3.12). In fact, there is no consensus on what non-Markovian dynamics actually means! This is what we will consider next.

3.4 Non-Markovian open systems

The most obvious generalization moving away from the semigroup structure is to require a property called CP divisibility from the quantum channel. This means that given a channel $\Phi_{0,t}$, for all t_1, t_2 s.t. $t_1 < t_2$ there exist a CPTP map Ψ_{t_1,t_2} such that

$$\Phi_{0,t_1+t_2} = \Psi_{t_1,t_2}\Phi_{0,t_1}. \tag{3.13}$$

This is still quite similar to the semigroup property, but the form of the map connecting the time points t_1 and t_2 is now arbitrary. In fact, for reasons

to follow, maps with this property are still usually considered Markovian and one popular idea is to define everything not divisible as non-Markovian [71, 72].

3.5 Defining non-Markovianity

Equating non-divisibility with non-Markovianity is mathematically very pleasing but lacked a concrete physical interpretation, until some proposals were made very recently [73]. In addition to divisibility, many different physically motivated approaches to defining non-Markovianity have been introduced. Typically, the idea behind them is choosing some more or less physically relevant quantity that tends to decrease under CPTP maps, or in other words, under the action of quantum channels. An increase in the quantity on some time interval is then considered a signature of non-Markovian behaviour. These increases are then added up and the number obtained like this is optimized subject to some fixed conditions to define it uniquely. The magnitude of this number then describes the amount of non-Markovianity in the system, or put differently, the magnitude of the memory effects. Some examples include a measure based on channel capacities [74], change of volumes in state space [75] and entanglement or other correlations with an ancilla [72, 76] just to name a few. An interesting one is also a measure characterizing the degree of non-divisibility not based on the dynamics of some physical quantity but on the form of the channel itself [72]

Still, so far no consensus has been reached on what would be the quantity that best captures the relevant information about the memory effects. Rather, it seems that the different measures are useful in different scenarios and describe different aspects of non-Markovianity. We will now look more closely at a measure based on trace distance and see in detail some of its properties that will be used later in this work.

3.5.1 BLP measure

One of the first and most widely used measures of non-Markovianity is the Breuer, Laine, Piilo (BLP) measure [13]. It is based on trace distance, which we already studied in section 2.3.

In addition to the physical interpretation as kind of information, the trace distance has another desirable property. It is monotonic under positive, trace preserving maps. In other words positive and trace preserving

maps are contractions in terms of the trace distance metric. This can be seen with a straightforward calculation [77]. Let Φ be positive and trace preserving, and let ρ_1 and ρ_2 be arbitrary Hermitian matrices. Using the ideas introduced in section 2.3 we see that

$$\begin{aligned} D(\Phi(\rho_1), \Phi(\rho_2)) &= ||\Phi(\rho_1 - \rho_2)||_{\mathrm{tr}} = ||\Phi(M_+ - M_-)||_{\mathrm{tr}} \\ &\leq ||\Phi(M_+)||_{\mathrm{tr}} + ||\Phi(M_-)||_{\mathrm{tr}} \\ &= \mathrm{tr}|\Phi(M_+)| + \mathrm{tr}|\Phi(M_-)| \\ &= \mathrm{tr}(M_+ + M_-) = \mathrm{tr}|M_+ - M_-| \\ &= ||\rho_1 - \rho_2||_{\mathrm{tr}} = D(\rho_1, \rho_2). \end{aligned} \tag{3.14}$$

This means that positive and trace preserving maps always decrease or don't change the trace distance of quantum states. In particular this then holds for CPTP maps.

We conclude that trace distance has the desired properties to be used as a quantity for measuring non-Markovianity. It is monotonic under CPTP maps and it has a physical interpretation as kind of information that enables us to distinguish different quantum states from each other. A decrease in it is interpreted as information flowing out of the system and a possible increase as information flowing back into the system. The corresponding non-Markovianity measure is defined as

$$\mathcal{N}_{BLP} = \sup_{\rho_1(0), \rho_2(0)} \int_{\sigma(\rho_1(t), \rho_2(t)) > 0} \sigma(\rho_1(t), \rho_2(t))\, \mathrm{d}t, \tag{3.15}$$

where

$$\sigma(\rho_1(t), \rho_2(t)) = \frac{dD(\rho_1(t), \rho_2(t))}{dt}. \tag{3.16}$$

The measure is built by adding up the increases in the trace distance between a pair of states during the evolution. This number is then maximized over all possible choices for the initial states to get a quantity which characterises only the properties of the channel. Specifically, whenever $\mathcal{N}_{BLP} > 0$, the channel is defined as non-Markovian. As we see from the definition, if a channel has the semigroup or CP-divisibility property, the trace distance between any two states is always decreasing under its action and the channel is Markovian in terms of the BLP measure. Notice that this does not hold in the other direction, as there exist channels that are not divisible, but still decrease trace distance [78].

Chapter 4

Optimizing the BLP measure

We will now consider the problem of the maximization included in the BLP measure. The purpose of it is finding a pair of states for which the net increase in the trace distance is largest. This is a very difficult task already for relatively small systems, since it seems to require to search over all possible pairs of states. Fortunately, this can be simplified quite a lot.

The first idea we got is based on linearity of quantum channels. This means that in a sense lines are mapped into lines in the quantum state space. This together with the convex structure of the state space ultimately leads to the result that the maximizing pair must consist of states that lie on the boundary of the state space.

The interior of the state space can be characterized as states that are not singular, in other words they are full rank. The boundary then consists of states with at least one zero in the spectrum. Now, let ρ_1 and ρ_2 be such states that at least one of them, say ρ_2 does not belong to the boundary. Then by continuity of eigenvalues we can form a third state ρ_3 (see Fig. 4.1) such that

$$\rho_3 = (1-\lambda)\rho_1 + \lambda\rho_2, \quad \lambda > 1, \tag{4.1}$$

with λ suitably close to, but larger than one. So we extend the line connecting ρ_1 and ρ_2 a little so that it goes a bit further from ρ_1 than ρ_2. The idea is to do this until we cannot do it anymore, which is when the other end of the line hits the boundary. Now, by linearity of quantum channels, the time evolution given by some channel $\Phi_{0,t}$ is

$$\rho_3(t) = (1-\lambda)\rho_1(t) + \lambda\rho_2(t), \tag{4.2}$$

from which we see that

$$\rho_1(t) - \rho_3(t) = \lambda(\rho_1(t) - \rho_2(t)). \tag{4.3}$$

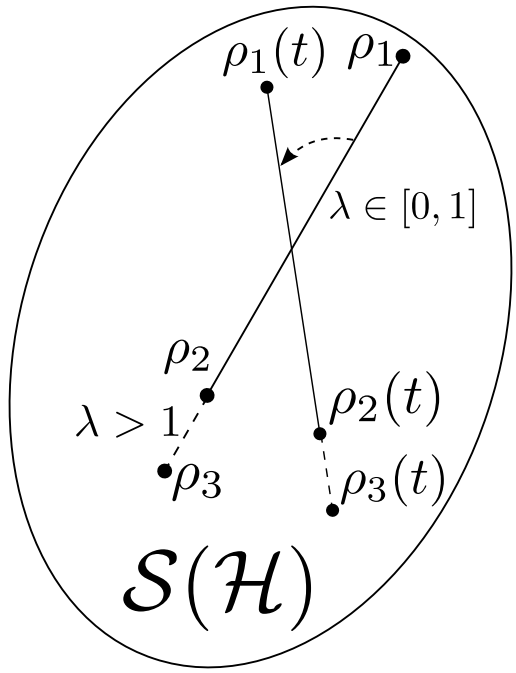

Figure 4.1: Lines are mapped into lines and the extension of the convex combination.

Then it immediately follows that

$$D(\rho_1(t), \rho_3(t)) = \lambda D(\rho_1(t), \rho_2(t)), \tag{4.4}$$

by which it is clear that

$$\int_{\sigma(\rho_1(t),\rho_3(t))>0} \sigma(\rho_1(t), \rho_3(t))\,\mathrm{d}t = \lambda \int_{\sigma(\rho_1(t),\rho_2(t))>0} \sigma(\rho_1(t), \rho_2(t))\,\mathrm{d}t. \tag{4.5}$$

Because $\lambda > 1$ this means that the pair (ρ_1, ρ_3) necessarily gives a higher value for the BLP measure than the pair (ρ_1, ρ_2). We conclude that the maximizing pair must lie on the boundary of the state space, because otherwise one can always do the extension like above and get a higher value for the integral.

It is now clear that the maximization can be restricted to the boundary of the state space, but that is still quite a challenge and we can do better. First we again use the spectral theorem to see that any difference of states ρ_1 and ρ_2 can be written as

$$\rho_1 - \rho_2 = M_+ - M_-, \tag{4.6}$$

for some positive, orthogonal matrices M_+ and M_-. Since ρ_1 and ρ_2 are states and thus of unit trace, the trace of the difference is zero, which implies that

$$\mathrm{tr}(M_+) = \mathrm{tr}(M_-) \equiv c. \tag{4.7}$$

Using the properties of the trace norm, we see that

$$||\rho_1 - \rho_2||_{\mathrm{tr}} = \mathrm{tr}(M_+) + \mathrm{tr}(M_-) = 2c, \tag{4.8}$$

which is equivalent to $D(\rho_1, \rho_2) = c$. Thus we can normalize both M_+ and M_- with the same constant c to make them orthogonal quantum states. The constant c is equal to the trace distance of the states ρ_1 and ρ_2 and thus less than or equal to 1. Defining $\rho_{M_+} = \frac{1}{c}M_+$ and $\rho_{M_-} = \frac{1}{c}M_-$ we find that

$$D(\rho_{M_+}, \rho_{M_-}) = \frac{1}{c}D(\rho_1, \rho_2). \tag{4.9}$$

Because the constant $\frac{1}{c}$ is greater than or equal to 1, it follows as in the line extension argument that the pair (ρ_{M_+}, ρ_{M_-}) gives a larger or at least equal value for the BLP measure as the pair (ρ_1, ρ_2). Since ρ_1 and ρ_2 were arbitrary, we conclude that the maximizing pair must consist of orthogonal states.

We see that the orthogonality condition is the strongest as it implies that the maximizing pair must lie on the boundary of the state space. This is because the orthogonality is defined via the Hilbert-Schmidt inner product, which in turn means that we are interested in the orthogonality of the supports of the matrices. As all states belonging to the interior of $\mathcal{S}(\mathcal{H})$ are positive and have full rank, they cannot have orthogonal supports. So it must hold that the states are not of full rank and thus belong to the boundary. In particular it follows that the maximizing pair consists of pure states for a qubit system as the boundary of the state space consists only of pure states in that case [23].

4.1 BLP measure and direction in state space

We have now confined the maximization needed in calculating the BLP measure to the set of pairs of orthogonal states. Still, we are maximizing over two states and as dimension grows this becomes increasingly difficult. Also, it seems like we need to look "all over" this set in state space for the

maximizing pair. Some more careful consideration shows that this is not needed.

Using the ideas presented in section 2.2 about the concept of direction in state space and the definition of the BLP measure we come to the following conclusion. The state pairs enter the measure only as their difference, from which we immediately recognize that the important concept for the BLP measure are not the pairs themselves but the directions defined by them and as we discussed, it does not matter where this direction is defined. We will now demonstrate how the BLP maximization can be performed in any part of the state space and by fixing a reference state, which is then surrounded by a surface such that all possible directions can be defined by picking the reference state and a point from the surface.

We already saw above that any direction defined by a pair of quantum states can also be defined by a pair of orthogonal quantum states. Actually, the same reasoning applies to any possible direction. In other words, by exactly the same arguments, any nonzero traceless Hermitian matrix A can be written as

$$\lambda A = \rho_{M_+} - \rho_{M_-}, \tag{4.10}$$

where λ is some suitably chosen real constant (we identify all real scalar multiples of A as the same direction). We define the set of all directions in $\mathcal{S}(\mathcal{H})$ as

$$\mathcal{E}_0(\mathcal{H}) = \{A \neq 0,\, A = A^\dagger,\, \mathrm{tr}(A) = 0\}. \tag{4.11}$$

Now, let us fix a state ρ_0 from the interior of the state space. Then a set $\partial U(\rho_0) \subset \mathcal{S}(\mathcal{H})$ is called an enclosing surface of the state ρ_0 if and only if for all $A \in \mathcal{E}_0(\mathcal{H})$ there exists a $\lambda > 0$ such that

$$\rho_0 + \lambda A \in \partial U(\rho_0). \tag{4.12}$$

Notice that by definition ρ_0 does not belong to $\partial U(\rho_0)$ and that every interior point of $\mathcal{S}(\mathcal{H})$ has at least one enclosing surface, an ϵ-ball being an example. Notice also that the shape of an enclosing surface is not restricted in any particular way. It does not need to be a sphere or anything even resembling a sphere nor does ρ_0 need to be in the centre as demonstrated in Fig. 4.2.

To see how the maximization can be simplified, let $\rho \in \partial U(\rho_0)$ be arbitrary. Then by the same argument as used in the previous section,

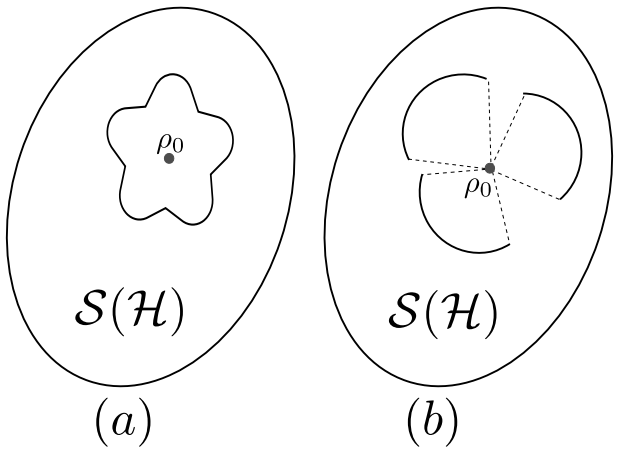

Figure 4.2: Examples of different enclosing surfaces for ρ_0.

there exists orthogonal quantum states ρ_1 and ρ_2 such that

$$\rho_1 - \rho_2 = \frac{1}{D(\rho_0, \rho)}(\rho_0 - \rho). \tag{4.13}$$

Thus for any pair (ρ_0, ρ) there exists a corresponding orthogonal pair (ρ_1, ρ_2). Now on the other hand, for any orthogonal pair (ρ_1, ρ_2) we have by definition that there exist a $\lambda > 0$ such that

$$\rho_0 + \lambda(\rho_1 - \rho_2) \in \partial U(\rho_0), \tag{4.14}$$

which means that there exists a $\rho \in \partial U(\rho_0)$ such that

$$\rho - \rho_0 = \lambda(\rho_1 - \rho_2). \tag{4.15}$$

Since ρ_1 and ρ_2 are orthogonal, we have $D(\rho_1, \rho_2) = 1$, which implies that $\lambda = D(\rho_0, \rho)$. We see that for every pair of orthogonal states (ρ_1, ρ_2) there exists a state $\rho \in \partial U(\rho_0)$ such that

$$\rho_1 - \rho_2 = \frac{1}{D(\rho_0, \rho)}(\rho - \rho_0). \tag{4.16}$$

This shows the other direction and we can conclude that the value of the BLP measure can be found by

1. Choosing a reference state ρ_0 from the interior of the state space

2. Surrounding ρ_0 with an enclosing surface $\partial U(\rho_0)$
3. Calculating the BLP integral for a pair (ρ_0, ρ), where $\rho \in \partial U(\rho_0)$
4. Normalizing the result with $1/D(\rho_0, \rho)$
5. Repeating for all $\rho \in \partial U(\rho_0)$.

This result brings great simplifications to the maximization problem and besides that also has some physical implications. Because ρ_0 is arbitrary, the above shows that the memory effects in the BLP sense are universal in the sense that they can equally well be witnessed in any part of the state space. This can be highly beneficial for example when the dynamics is such that it has a fixed point in the interior of $\mathcal{S}(\mathcal{H})$. Then this point can be chosen as the reference and then only the sampled states evolve in time! Also, the freedom in choosing many different shapes for $\partial U(\rho_0)$ can be useful in experiments. If some states are difficult to prepare, it might be possible to do the maximization over an enclosing surface that does not contain these problematic states at all. The universality also supports the idea that BLP non-Markovianity is an intrinsic property of the channel Φ_t itself even though the measure is defined via the effect on the states alone. In fact, to experimentally evaluate the measure, one does not even need to know the form of the channel. In other words the time evolution of the states could be given by a black box, the inner workings of which are unknown and only the effects of which could be observed.

4.1.1 Experimental demonstration

To verify the implications of the above theorem, a photonic experiment was carried out by our collaborators in the University of Science and Technology of China. In the experiment the polarization degree of freedom of the photon was used as the system of interest and the frequency degree of freedom as the environment.

The experimental setup is depicted in the left panel of Fig. 4.3. An entangled pair of photons is created, one photon put through a setup realizing a non-Markovian channel and the other used as a trigger for detection. Different aspects and details of the experimental realization are discussed in [79–82], but the basic idea is simple. First, any desired pure polarization state and frequency state are prepared. Then the polarization and frequency degrees are coupled causing noise to the polarization state. The type of noise depends on the frequency distribution used. In our case we

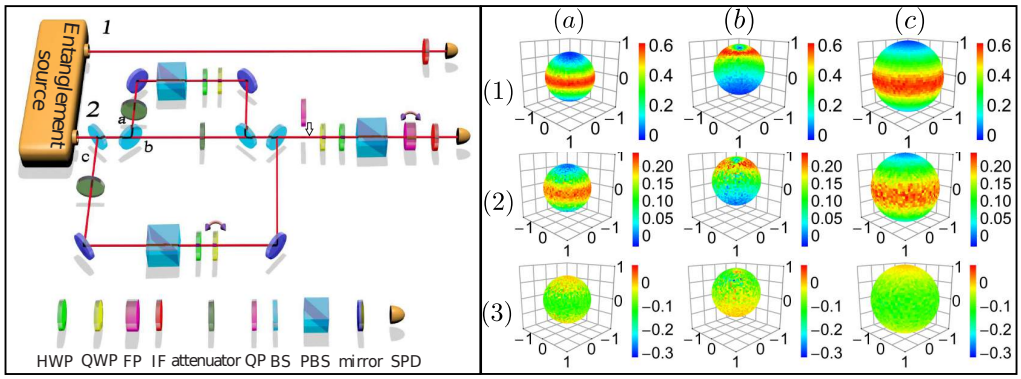

Figure 4.3: The experimental setup and the growth of the trace distance over one period of oscillation of the decay rate for pure states (column a) and two different reference states (columns (b) and (c)). The different rows correspond to different environments correspondig to two different non-Markovian channels (1)-(2) and a Markovian channel (3).

realize a non-Markovian noise channel to demonstrate the maximization procedure. The maximization is done with two different reference states

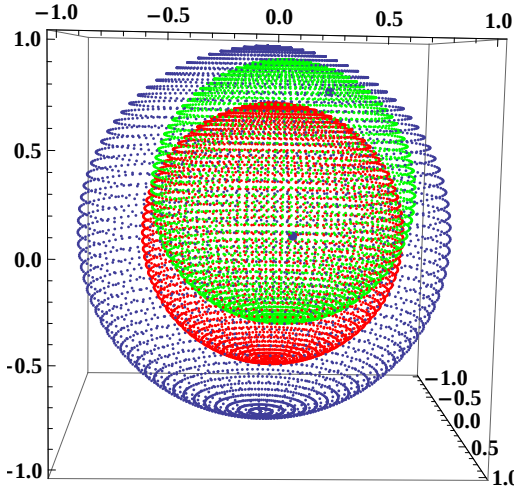

Figure 4.4: The Bloch sphere (blue), the two reference states (isolated blue dots) and the two enclosing surfaces (red and green).

and enclosing surfaces as depicted in Fig. 4.4. One is chosen from the

equatorial plane and the other from the northern hemisphere, close to the surface of the Bloch sphere. The maximization is then done using the method just introduced. Since the measure is not an observable, we must perform state tomography. This means that the desired state is prepared, put through the channel with some fixed time parameter value t and full tomography is performed. Then the parameter value, meaning the length of the noise is increased and the process is repeated. The measure is then calculated from the states obtained by the tomography and the results are shown graphically in the right panel of Fig. 4.3 and numerically in Table 4.1. As expected from theory, the maximizing directions in all of the cases are the ones orthogonal to the line connecting the north and the south poles. We clearly see how this is captured by the reference states and enclosing surfaces as red rings in Fig. 4.3. We also clearly see from the table, that the values for the measure we get from the experiment are very close to the ones calculated theoretically.

	$\mathcal{N}_{theo}$	$\mathcal{N}_{(a)}$	$\mathcal{N}_{(b)}$	$\mathcal{N}_{(c)}$
(1)	0.59	0.59 ± 0.01	0.59 ± 0.02	0.59 ± 0.02
(2)	0.21	0.21 ± 0.01	0.21 ± 0.02	0.21 ± 0.02
(3)	0	0.001 ± 0.013	-0.005 ± 0.008	-0.0002 ± 0.0015

Table 4.1: The numerical values of the BLP measure obtained from the experimental realization.

Chapter 5

Non-Markovianity and quantum information

Entanglement is at the heart of most quantum information tasks. In this section we will introduce one application, namely the superdense coding (SDC) protocol and look at what happens if some or all of the initial entanglement required is lost. We find a scenario where the entanglement on the system level is temporarily lost to the environment or to the correlations between the system and the environment, but can later be retrieved and used for performing the SDC protocol with high fidelity. This serves as a proof of principle example of the possibility of using non-Markovianity as a resource in quantum information.

We will also study the quantum Zeno effect from the point of view of BLP non-Markovianity. The effect is about how frequent measurements or a strong coupling to an external system can greatly effect, even stop the time evolution of a quantum system. We find that especially the strong coupling version of the Zeno effect gives a good test for the information flow interpretation of the BLP measure and also exemplifies how the information can flow into the correlations between the system and the environment even when the state of the environment does not change in time.

5.1 Superdense coding

The purpose of the superdense coding protocol is to send a message from Alice to Bob by exploiting quantum entanglement [38]. The idea is that Alice can encode a message the joint state of her and Bob's qubits by a local operation and send her qubit to Bob who measures the joint state

and decodes the message as described in Fig. 5.1. The trick is that Alice and Bob first share a maximally entangled state of two qubits, which Alice can manipulate locally to four orthogonal, thus perfectly distinguishable quantum states. Of course Alice and Bob need to have decided beforehand what the used encoding is. One example could be

$$\begin{aligned} |\Psi_+\rangle\langle\Psi_+| &= \text{"00"} \\ |\Psi_-\rangle\langle\Psi_-| &= \text{"01"} \\ |\Phi_+\rangle\langle\Phi_+| &= \text{"10"} \\ |\Phi_-\rangle\langle\Phi_-| &= \text{"11"}, \end{aligned} \tag{5.1}$$

where the quantum states are the different Bell states, which expressed in the basis $\{|H\rangle, |V\rangle\}$ are

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|HH\rangle \pm |VV\rangle) \tag{5.2}$$

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|HV\rangle \pm |VH\rangle). \tag{5.3}$$

We see that the information Bob ends up with uniquely determines a state of two classical bits. We conclude that by receiving just one qubit from Alice, Bob has ended up with two bits of classical information, which is where the name of the protocol comes from.

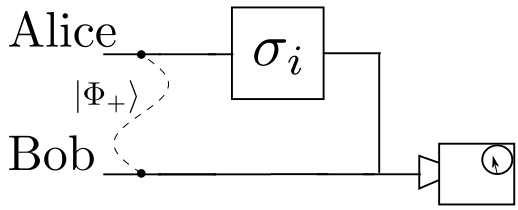

Figure 5.1: The idea of the superdense coding protocol. $|\Phi_+\rangle$ is one of the Bell states and σ_i is one of the Pauli matrices or the identity matrix σ_0.

The only way to perform the superdense coding protocol and successfully send two bits of classical information is to use a set of four orthogonal, maximally entangled states, which implies that entanglement is a crucial resource for such an operation. However, entanglement alone is not enough, since without having Alice's qubit, Bob cannot learn anything about the

joint state, because locally his state is just $\mathrm{tr}_{\mathrm{Alice}}(|\Phi_+\rangle\langle\Phi_+|) = \frac{1}{2}I$, the maximally mixed state! Like in [83], we write a heuristic relation that describes the relationship between these resources and classical information

$$1\,\text{entaglement bit} + 1\,\text{qubit} \succcurlyeq 2\,\text{classical bits}, \tag{5.4}$$

where the $\succcurlyeq$ is used to emphasize that the relation between these quantities is quite special. As stated above, just sharing a bit of entanglement does not allow Alice and Bob to communicate at all; a qubit must be sent for information to travel. Also, just sending a qubit without any entanglement involved does not give any advantage over classical communication. We will now study what happens if some or all of the entanglement is lost.

5.1.1 Noisy superdense coding with non-local memory effects

The intuition from the previous section is that if some or all of the entanglement between Alice and Bob is lost, the amount of information Alice is able to send to Bob should decrease. This idea has been formalized and the protocol generalised for an arbitrary (not necessarily maximally) entangled state ρ_{AB} shared between Alice and Bob [84] and also to many users [85]. The capacity C, in other words the maximal amount of classical information that can be transmitted by Alice to Bob by maximizing over all possible encoding strategies, using a shared state ρ_{AB} and assuming a noiseless channel in the transmission of the qubit, was proven to take the simple form [84]

$$C(\rho_{AB}) = \log_2 d + S(\rho_B) - S(\rho_{AB}), \tag{5.5}$$

where d is the dimension of Alice's system, ρ_B is Bob's reduced density operator, and $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ the von Neumann entropy.

To give a concrete example, we will first theoretically study and then comment on the experimental side of a photonic realization of the SDC protocol schematically depicted in Fig. 5.2. As qubits we will use photon polarization, which can conveniently be prepared to a maximally entangled state using parametric down-conversion. The initial state that Alice and Bob share is

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle). \tag{5.6}$$

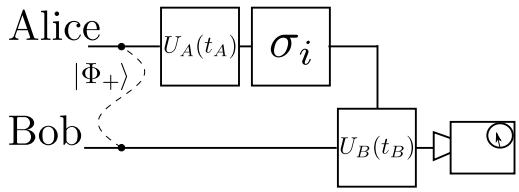

Figure 5.2: SDC with noise on Alice's qubit before encoding and noise on Bob's qubit after encoding.

Alice and Bob could use this state for the SDC protocol as described in the previous section. However, we now introduce noise to the system by coupling it to an environment. For the environment degree of freedom we choose the frequency of the photons. The initial two-photon frequency state, in general, can be written as

$$|\chi\rangle = \int \mathrm{d}\omega_A \mathrm{d}\omega_B g(\omega_A, \omega_B)|\omega_A\rangle|\omega_B\rangle, \tag{5.7}$$

where $g(\omega_A, \omega_B)$ is the joint probability amplitude and the corresponding joint probability distribution is $P(\omega_A, \omega_B) = |g(\omega_A, \omega_B)|^2$. The unitary operator describing the local coupling between the polarization and frequency degrees of freedom of each photon $j = A, B$ (Alice, Bob) is

$$U_j(t) = \int d\omega^j (e^{i\omega^j n_V^j t}|V\rangle\langle V| + e^{i\omega^j n_H^j t}|H\rangle\langle H|) \otimes |\omega^j\rangle\langle\omega^j|. \tag{5.8}$$

where ω_j is the frequency of photon j and n_V^j (n_H^j) the index of refraction of its polarization component V (H). We assume that $n_H^A - n_V^A = n_H^B - n_V^B \equiv \Delta n$. Notice how a non-zero Δn is necessary for anything interesting to happen on the system side.

In our scenario the state on Alice's side is coupled to the environment, which causes the Bell state shared by Alice and Bob to decohere and thus lose entanglement. We will show how this is detrimental for the performance of the SDC protocol. By following the ideas presented in [20, 86, 87] in harnessing the so called non-local memory effects, we will also show how correlations present in the initial state of the environment can allow Bob to repair the broken entanglement on the system side by adding more noise to the system!

Initially, the joint state of the polarization and frequency degrees of freedom is a tensor product of the state defined in equations (5.6) and

(5.7). Then the unitary coupling defined in equation (5.8) is turned on on Alice's side for time t_A. Tracing over the frequency degree of freedom, the joint polarization state shared by Alice and Bob becomes

$$\rho_{AB}(t_A) = \frac{1}{2}(|HH\rangle\langle HH| + \kappa_A(t_A)|HH\rangle\langle VV| \tag{5.9}$$

$$+ \kappa_A^*(t_A)|VV\rangle\langle HH| + |VV\rangle\langle VV|), \tag{5.10}$$

where κ_A is called the decoherence function and is caused by the coupling to the frequency degree of freedom. It is calculated as

$$\kappa_A(t_A) = \int \mathrm{d}\omega_A \mathrm{d}\omega_B e^{it_A\omega_A\Delta n}|g(\omega_A, \omega_B)|^2. \tag{5.11}$$

As it is apparent from the form of $\kappa_A(t_A)$, the distribution $|g(\omega_A, \omega_B)|^2$ of the environment modes completely determines what happens to our system. For this example, we choose that it has a joint Gaussian form where the marginals have equal mean values $\langle\omega_A\rangle = \langle\omega_B\rangle = \omega_0/2$ and variances $C_{AA} = \langle\omega_A^2\rangle - \langle\omega_A\rangle^2 = C_{BB}$. The correlation coefficient between the two frequencies is $K = (\langle\omega_A\omega_B\rangle - \langle\omega_A\rangle\langle\omega_B\rangle)/C_{\mathrm{AA}}$. This represents the case, where the entangled photons are created in a parametric down-conversion process as explained later.

Assuming that there is no noise on Bob's side and no noise in the transmission, the capacity of the protocol is given by equation (5.5) with $\rho_{AB}(t_A)$ given by the expression above, namely

$$C(\rho_{AB}(t_A)) = 2 - H\left(\frac{1 + |\kappa_A(t_A)|}{2}\right), \tag{5.12}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. It is clear that the capacity is 1, when $|\kappa_A(t_A)| = 0$ (worst possible decoherence) and 2 when $|\kappa_A(t_A)| = 1$ (no decoherence at all). This reflects the fact that if there was no noise on Alice's side, the protocol would be just the ordinary SDC, and if there is maximal amount of decohorence, meaning that the coherence terms are completely removed from equation (5.9), there is no entanglement to use, thus nothing extra can be gained with the SDC protocol.

Alice now applies one of the local unitary operations $\{I, \sigma_x, \sigma_y, \sigma_z\}$ to encode her message to the decohered state (5.9). Without loss of generality, let us assume that Alice applies σ_x. Now Bob applies local noise to his qubit for the duration t_B, which he can freely choose. After this step the

two-qubit state is

$$\rho_{AB}(t_A, t_B) = \frac{1}{2}\left[|VH\rangle\langle VH| + h(t_A, t_B)|VH\rangle\langle HV| + h^*(t_A, t_B)|HV\rangle\langle VH| + |HV\rangle\langle HV|\right], \tag{5.13}$$

where the decoherence function, with the specific choice $t_B = t_A$ and by using the fact that the we chose $|g(\omega_A, \omega_B)|^2$ to have the specific Gaussian form, becomes

$$h(t_A, t_A) = e^{i\omega_0 \Delta n t_A} e^{-C_{AA} t_A^2 (1+K)}. \tag{5.14}$$

The absolute value is then just

$$|h(t_A, t_A)| = e^{-C_{AA} t_A^2 (1+K)}, \tag{5.15}$$

from which we see, that in the case of perfect anticorrelations ($K = -1$) in the frequency distribution, the absolute value is equal to one. What this means, is that up to a phase factor, Bob cancels the effect of the noise added on Alice's qubit by adding the exact same amount of noise to the system. The phase factor is irrelevant for the performance of the protocol, since it can be accounted for in the encoding operation. This fact is also reflected by the equation for the capacity, which only depends on the absolute value of the decoherence function.

The reason this approach works is, that locally what the noise can be thought of as giving random phase kicks to the system, the randomness coming from the distribution of the environment modes. So locally the noise seems random both to Alice and Bob, but if there are perfect anticorrelations, the kick on Alice's qubit are always exactly counteracted by kicks on Bob's qubit. Put more precisely, the net effect, the integral over all of the kicks, is that nothing happens.

What makes this interesting is that such anticorrelations in the frequency distributions are easily made in the lab. If the pump laser used in the down-conversion process had an infinitely narrow frequency distributions, the down converted photons would have perfect frequency anticorrelations. Experimentally however the pump frequency distribution is not a delta function at some specific frequency ω_{pump}, but described by a Gaussian centered at ω_{pump}. This means that there is randomness in the sum of the frequencies $\omega_A + \omega_B$ of the created, polarization entangled photons. The amount of randomness depends on the width of the pump frequency distribution. In the case of infinitely narrow distribution we recover the

ideal case of perfect anticorrelations, which get worse and worse as the width is increased.

As it probably is clear, the memory effect does not depend on when the noise is added on Bob's side. The same calculations would hold if Bob added the noise before, during or after what happens on Alice's side. What makes this a bit unintuitive is the following. Assume that Bob adds the noise after he has already received Alice's qubit. Thus at the time of encoding the amount of entanglement in the joint state could be almost arbitrarily small, meaning a small value of $|\kappa_A(t_A)|$. Then Alice performs the encoding and sends her qubit to Bob. Were Bob now to measure the joint state, he couldn't learn much. So where is the information? It is somehow spread into the environment or the correlations between the system and the environment and is not accessible to Bob directly. However, as we saw above, if Bob now adds noise to his qubit he cancels the effect of Alice's noise and brings the information back into the polarization degree of freedom.

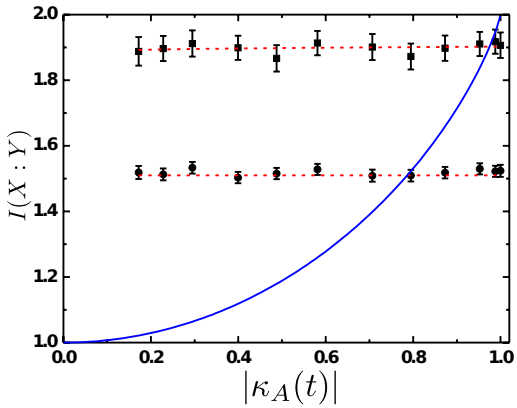

Figure 5.3: Mutual information vs. magnitude of Alice's decoherence function $|\kappa_A|$. The blue solid line corresponds to noise on Alice's side and no noise on Bob's side with four-state encoding. The dots correspond to the measured capacities reached measured in terms of mutual information. The red lines are theoretical fits to the data. The lower line corresponds to the case where we used three-state encoding and the upper line the case of four-state encoding.

The above theoretical consideration was realized experimentally with our collaborators in USTC, China. Fig. 5.3 shows our results. Because of limitations of linear optics, a full Bell state measurement cannot be accom-

plished. This means that there is no way to experimentally distinguish all of the four Bell states from each other. For this reason a modified SDC protocol is used, where only three different Bell states are used for the sending the messages. This of course lowers the maximum amount of classical information that can be sent. In the case of four state encoding, the limitation of linear optics was overcome by sending the same message many times, which made it possible to distinguish between four different ones. This of course does not count as genuine superdense coding. Nevertheless it shows that Bob has been able to cancel the effect of Alice's noise by adding noise of his own, and in principle, near perfect SDC could be performed.

5.1.2 Noisy superdense coding with local noise

We have also studied the case where there are no correlations between the frequencies of Alice's and Bob's photons and the noise is not necessarily identical on both sides anymore. In that case Bob cannot directly cancel the effect of Alice's noise, but it turned out that it is more beneficial to have a non-Markovian channel on Alice's side and a Markovian one on Bob's side than the other way around.

Perhaps the most interesting effect in this study was that Alice could use an effect similar to the spin echo to cancel some noise from her qubit. If there is identical noise on Alice's side before and after encoding, then the capacity does not drop below 1.5. This can be explained by observing what the net effect of the noise-encoding-noise is on the combined system level. Using the equations (5.6), (5.7) and (5.8) we see that the combined system state after the noise interactions and encoding is

$$U_A(t_A)\sigma_i \otimes I\, U_A(t_A)|\Phi_+\rangle|\chi\rangle. \tag{5.16}$$

A straightforward calculation shows that for $i = 0$ and $i = z$ the above can be written as

$$U_A(t_A)\sigma_0 \otimes I\, U_A(t_A) = \sigma_0 \otimes I U_A(2t_A) \tag{5.17}$$

$$U_A(t_A)\sigma_z \otimes I\, U_A(t_A) = \sigma_z \otimes I U_A(2t_A), \tag{5.18}$$

whereas for $i = x$ and $i = y$ it becomes

$$U_A(t_A)\sigma_x \otimes I\, U_A(t_A) = \sigma_x \otimes \int \mathrm{d}\omega_A e^{i\omega_A(n_V^A + n_H^A)t_A}|\omega_A\rangle\langle\omega_A| \tag{5.19}$$

$$U_A(t_A)\sigma_y \otimes I\, U_A(t_A) = \sigma_y \otimes \int \mathrm{d}\omega_A e^{i\omega_A(n_V^A + n_H^A)t_A}|\omega_A\rangle\langle\omega_A|. \tag{5.20}$$

Thus we see that if Alice decides to send the message $|\Phi_+\rangle$ or $|\Phi_-\rangle$, the message states will have suffered from noise for a duration of $2t_A$. If she decides to send $|\Psi_+\rangle$ or $|\Psi_-\rangle$, they will only have a global phase factor. Notice that $|\Phi_\pm\rangle$ decohered by this particular noise remain orthogonal to $|\Psi_\pm\rangle$ but not to each other. This means that three different messages can be sent with this noise scenario by choosing $|\Psi_\pm\rangle$ and one of $|\Phi_\pm\rangle$ as the encoding states. This then explains the lower bound of 1.5 for the capacity.

5.2 Zeno effect

Zeno of Elea, not to be confused with Zeno of Citium [88], the founder of the Stoic school of philosophy, is best known for his many paradoxes or apparent paradoxes, like Achilles chasing a tortoise and the arrow paradox, which is, as put by Aristotle [89]:

> "If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless".

So how can a flying arrow be moving, if at any instant of time when it is observed, it is seen in some place and stationary? Of course we now know how to take limits and observe that flying arrows indeed move, and as Newton realized, keep on moving if no external force is applied. However, about two thousand years after Zeno, von Neumann's reduction postulate [90] laid the foundation for a quantum version of the paradox, which has also been experimentally verified! Namely, if a quantum measurement collapses the state of a quantum system to a state in the measurement basis, this process is repeated N times a second and N is let to tend to infinity, the motion of the system is prevented. This phenomenon was named the quantum Zeno effect in [91] and has also been studied in many earlier works [92–98]

Besides frequent measurements, a strong coupling to an external system can also prevent the original system of interest from evolving in time [99, 100]. Loosely speaking the external system is continuously measuring or gazing at the system of interest and thereby preventing its dynamics. This effect is called dominated evolution or the watchdog effect and is the one that we studied from the point of view of non-Markovianity.

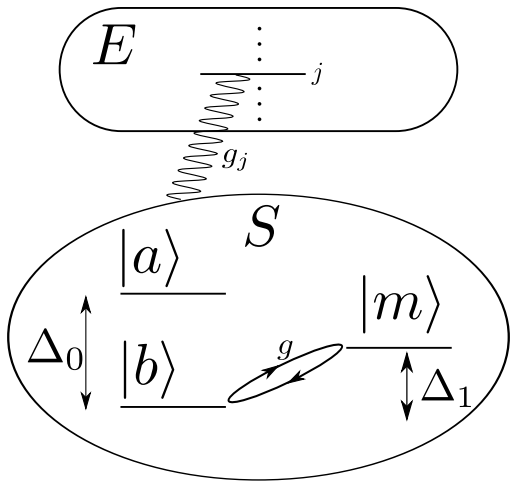

Figure 5.4: We study a two level system interacting with a zero-temperature bosonic environment and an external level. By controlling the coupling strength g, the external level is used to control the dynamics caused by the environment.

5.2.1 Model and dynamics

Our system is schematically described in Fig. 5.4. The idea is to consider the levels a and b as the system of interest, which decoheres due to the coupling to the bosonic environment E. We studied two different cases for E that we call the good and the bad cavity limits. If not interfered with, the good cavity is such that it causes non-Markovian and the bad cavity Markovian dynamics to the system. The name comes from the idea, that experimentally the environment is often realized by an optical cavity, which can for example leak out photons (bad cavity) or efficiently store photons without losing them (good cavity). The third level m then is coherently coupled to level b and works as a watchdog inducing the Zeno effect on levels a and b. The force of this can be controlled by varying the coupling strength g. We then study what happens to the system from the point of view of the information flow in sense of the BLP measure of non-Markovianity. It turns out that the gaze of the watchdog first introduces a mechanism that makes the evolution of the two level system more non-Markovian, but this effect then asymptotically goes to zero as the gaze gets stronger with increasing g, freezing the dynamics due to the Zeno effect.

Put more precisely, the total Hamiltonian describing the whole system

in the rotating wave approximation can be written as

$$H_{tot} = H_S + H_E + H_{int} + H_C, \tag{5.21}$$

where

$$H_S = \omega_a|a\rangle\langle a| + \omega_b|b\rangle\langle b| + \omega_m|m\rangle\langle m|, \tag{5.22}$$

$$H_E = \sum_j \omega_j a_j^\dagger a_j, \tag{5.23}$$

$$H_{int} = \sum_j g_j^*|b\rangle\langle a|a_j^\dagger + g_j|a\rangle\langle b|a_j, \tag{5.24}$$

$$H_C = g(|b\rangle\langle m|e^{i\Delta_1 t} + |m\rangle\langle b|e^{-i\Delta_1 t}), \tag{5.25}$$

where $a_j^\dagger$ and a_j are the bosonic creation and annihilation operators, g_j and g the coupling strength to the jth mode of the environment and the level $|m\rangle$ respectively, ω_a, ω_b, ω_m the frequencies of the levels $|a\rangle, |b\rangle, |m\rangle$, ω_j are the frequencies of the environment modes and

$$\begin{aligned}\Delta_0 &= \omega_a - \omega_b \\ \Delta_1 &= \omega_m - \omega_b.\end{aligned} \tag{5.26}$$

Notice that the level $|m\rangle$ is neither directly coupled with the environment nor with level $|a\rangle$, but still it can be used to freeze the dynamics of the two level system as we will soon see. From here on, we will work in the interaction picture defined by the free Hamiltonian. The interaction Hamiltonian becomes

$$\begin{aligned}H^{(i)} =& \sum_j g_j|a\rangle\langle b|a_j e^{-i(\omega_j-\Delta_0)t} + g_j^*|b\rangle\langle a|a_j^\dagger e^{i(\omega_j-\Delta_0)t} \\ &+ g(|b\rangle\langle m| + |m\rangle\langle b|).\end{aligned} \tag{5.27}$$

To study the dynamics, we consider for simplicity only initial states with one excitation in the system and empty environment modes. This means that the initial state can be written as

$$|\psi(0)\rangle = (\alpha_0|a\rangle + \beta_0|b\rangle + \mu_0|m\rangle) \otimes |\{0\}\rangle. \tag{5.28}$$

Because the excitation number is conserved, the state at any later time is

$$\begin{aligned}|\psi(t)\rangle =& (\alpha(t)|a\rangle + \beta(t)|b\rangle + \mu(t)|m\rangle) \otimes |\{0\}\rangle \\ &+ \sum_j \beta_j(t)|b\rangle \otimes |1_j\rangle + \sum_j \mu_j(t)|m\rangle \otimes |1_j\rangle,\end{aligned} \tag{5.29}$$

where $|1_j\rangle = a_j^\dagger|\{0\}\rangle$ means an excitation in the jth mode in the environment. From this we get the density matrix describing the three level system by tracing out the environment

$$\begin{pmatrix} |\alpha(t)|^2 & \alpha(t)\beta^*(t) & \alpha(t)\mu^*(t) \\ \alpha^*(t)\beta(t) & |\beta(t)|^2 + \sum_j |\beta_j(t)|^2 & \beta(t)\mu^*(t) + \sum_j \beta_j(t)\mu_j^*(t) \\ \alpha^*(t)\mu(t) & \beta^*(t)\mu(t) + \sum_j \beta_j^*(t)\mu_j(t) & |\mu(t)|^2 + \sum_j |\mu_j(t)|^2 \end{pmatrix}. \tag{5.30}$$

The form of the coefficients we obtain by solving the Schrödinger equation for the whole system, which can be done with lengthy but straightforward calculations which we omit here. In the solution, we approximate the environment with a spectral density function, which has a Lorentzian form

$$J(\omega) = \Omega_0^2 \frac{\lambda}{\pi((\omega - \Delta_0)^2 + \lambda^2)}, \tag{5.31}$$

where $\Omega_0^2 = \frac{\lambda\gamma}{2}$. The form of the spectral density function could be almost anything, but this choice makes the calculations fairly simple and the Lorentzian spectral density is also physically relevant [101]. By controlling the parameters γ and λ, which are basically the height and width of the Lorentzian, we can control whether the environment behaves like a bad or a good cavity, causing Markovian or non-Markovian dynamics to the system. Using this, we get the solutions for the coefficients without indices

$$\begin{aligned} \alpha(t) = \alpha_0 \Big(& \frac{(s_1+\lambda)^2 + g^2}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} + \frac{(s_2+\lambda)^2 + g^2}{(s_2 - s_1)(s_2 - s_3)} e^{s_2 t} \\ & + \frac{(s_3+\lambda)^2 + g^2}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t} \Big) \end{aligned} \tag{5.32}$$

$$\beta(t) = \beta_0 \cos(gt) - i\mu_0 \sin(gt) \tag{5.33}$$

$$\mu(t) = \mu_0 \cos(gt) - i\beta_0 \sin(gt), \tag{5.34}$$

where s_1, s_2 and s_3 are some constants. For the other coefficients we get

$$\sum_j |\mu_j(t)|^2 = \Omega_0^2 \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 e^{-\lambda|t_1-t_2|}\alpha(t_1)\alpha^*(t_2) \times \sin(g(t-t_1))\sin(g(t-t_2)) \tag{5.35}$$

$$\sum_j |\beta_j(t)|^2 = \Omega_0^2 \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 e^{-\lambda|t_1-t_2|}\alpha(t_1)\alpha^*(t_2) \times \cos(g(t-t_1))\cos(g(t-t_2)) \tag{5.36}$$

$$\sum_j \mu_j^*(t)\beta_j(t) = \Omega_0^2 \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 e^{-\lambda|t_1-t_2|}\alpha(t_1)\alpha^*(t_2) \times \cos(g(t-t_1))\sin(g(t-t_2)). \tag{5.37}$$

5.2.2 Non-Markovianity

Once we have the solution for the dynamics, we can start to calculate the values of the BLP measure. We do this by sampling random initial states using the direction argument. Because the only purpose of the external level $|m\rangle$ is to control the levels $|a\rangle$ and $|b\rangle$ that we are actually interested in, we restrict the maximization to initial states where $|m\rangle$ is empty. This means that we are interested in initial states with $\mu_0 = 0$, which in turn means that the states are described by the parameters α_0 and β_0 alone. This enables a Bloch sphere representation of the initial sates in the form

$$\rho_0 = \frac{1}{2}(\sigma_0 + \vec{r}\cdot\vec{\sigma}), \tag{5.38}$$

where $||\vec{r}|| \leq 1$ and σ_i are the Pauli matrices. The matrix describing the actual state of our system is of the form ρ_0 above, appropriately padded with zeros making it a 3x3 matrix. With this convention we can do the maximization using the direction argument in the following way. We randomly pick a point from the Bloch sphere and its antipodal point. These then form the pair of initial states and are evolved for a fixed time t. The value of the BLP integral is calculated and this is repeated many times. This gives an approximate idea on what the maximizing pair among the ones studied is. It turned out that for this system the maximizing pair was the north and south pole, in other words the states $|a\rangle\langle a|$ and $|b\rangle\langle b|$ in all but one of the studied cases. In the one exception, which was the good cavity with zero coupling g to the external level, the maximizing pair could have been taken as any pair from the equator.

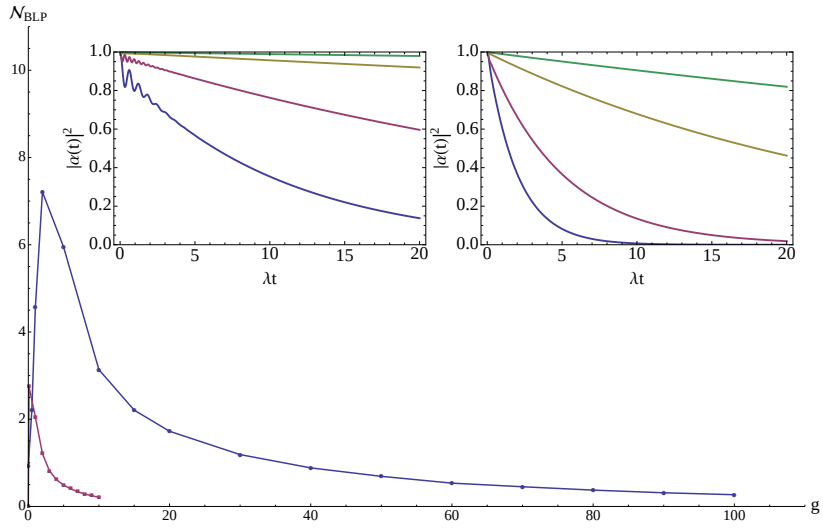

Figure 5.5: The BLP measure as a function of the coupling strength g to the external level. The blue line is for the good cavity regime ($\gamma = 10\lambda$), while purple refers to the bad cavity case ($\gamma = \frac{\lambda}{10}$). The left inset displays the population of the excited state as a function of time with $g = 10\lambda, 20\lambda, 50\lambda, 100\lambda$ (from bottom to top) in the good cavity case. The right inset gives the same in the bad cavity case with $g = \lambda, 2\lambda, 5\lambda, 10\lambda$ (from bottom to top).

As seen from Fig. 5.5, increasing the coupling g first sharply increases the non-Markovianity of the system and after some critical value, the non-Markovianity starts to decrease with increasing g. This can be explained by a mechanism, the effect of which can be seen in Fig. 5.6. In panel (a) we have the good cavity case and the coupling g is relatively small. The population of the excited state $|a\rangle$ quickly reaches zero and shows oscillations typically seen in non-Markovian systems. The populations of the levels $|b\rangle$ and $|m\rangle$ oscillate with a large amplitude and small frequency. Then, moving to panel (b), the coupling g is increased and we see that the Zeno effect starts to influence the population of the excited state, as it decreases much slower. Also, the amplitudes of the oscillations of the levels $|b\rangle$ and $|m\rangle$ are much smaller and the frequencies much higher than in the previous case. The cases (c) and (d) correspond to the bad cavity limit and are quantitatively similar, but with the population of the excited state always monotonically decreasing. Still, as seen form Fig. 5.5, non-zero values of g give a non-zero value for the measure, signalling non-Markovian behaviour.

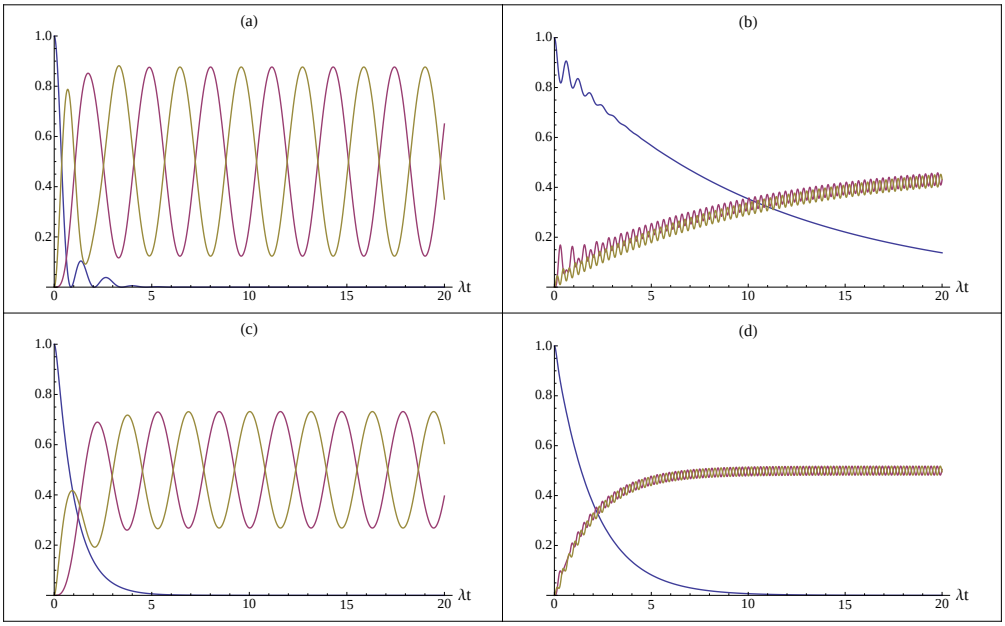

Figure 5.6: Population of the excited state (blue), ground state (yellow) and the external level (purple) in the good (a-b) ($\gamma = 10\lambda$) and bad (c-d) cavity ($\gamma = \frac{\lambda}{10}$) case. In (a) we have used $g = \lambda$ and in (b) $g = 10\lambda$. Similarly in (c) $g = \frac{\lambda}{10}$ and in (d) $g = \lambda$. In all of these cases, we took an initially excited system prepared in $|a\rangle$.

The oscillatory behaviour observed persists even though the excited state population has decreased to zero and is also reflected as oscillations in the trace distance between the initial states $|a\rangle$ and $|b\rangle$. This might seem peculiar, since the excited state is the only thing connecting the system to the environment and it seems that after this coupling is switched off, all that remains is unitary dynamics. This is not true however, since it matters how the oscillations between the lower levels start. An intuitive picture could be that if all of the excitation is initially in the level $|a\rangle$, it has to drop down to $|b\rangle$ before it can start oscillating between $|b\rangle$ and $|m\rangle$. This means that the oscillations get a "slow start" as compared to the case where all of it is initially in the level $|b\rangle$. In this way the system remembers where the oscillating population came from and it matters where it was initially. This is depicted in Fig. 5.7, where we see how the initial state $|a\rangle$ ends up going around an ellipse trajectory in the system formed by the levels $|b\rangle$ and $|m\rangle$. We can also see that the trajectories that the states $|a\rangle$ and $|b\rangle$

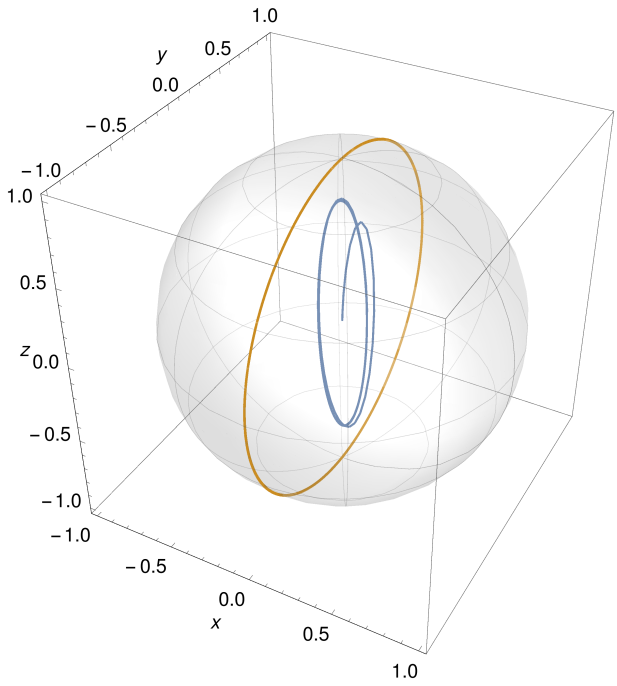

Figure 5.7: The trajectories of the initial state $|b\rangle$ (yellow great circle) and $|a\rangle$ (blue ellipse) in the Bloch sphere representation corresponding to the levels $|b\rangle$ and $|m\rangle$ until time $\lambda t = 20$ with parameter values $\gamma = 10\lambda$ and $g = \lambda$. Notice that since all of the population starting from level $|a\rangle$ is not initially in the system formed by $|b\rangle$ and $|m\rangle$, the blue points near the origin (early times) do not correspond to normalized states. The spiral like behavior describes how population enters the system and eventually starts the oscillations that last forever.

end up in are in perpendicular planes.

The idea of the BLP measure is to quantify the flow of information between the system and the environment. Even though Fig. 5.7 gives an explanation for changing trace distance for the initial states $|a\rangle$ and $|b\rangle$, how can information still flow in the case where the system has been isolated from the influence of the environment by depleting all the population from the level $|a\rangle$? Let us consider the case where the initial state was such that only $|a\rangle$ was initially populated and has now depleted. Thus we set $\mu_0 = \beta_0 = 0$ and time t sufficiently large so that $\alpha(t) = 0$ in equation (5.29)

to get

$$|\psi(t)\rangle = \sum_j \beta_j(t)|b\rangle \otimes |1_j\rangle + \sum_j \mu_j(t)|m\rangle \otimes |1_j\rangle. \tag{5.39}$$

The density matrix $\rho_{\mathrm{tot}}(t)$ describing the total system state can be written using this as

$$\rho_{\mathrm{tot}}(t) = \sum_{k,l} \begin{pmatrix} \beta_k(t)\beta_l^*(t) & \beta_k(t)\mu_l^*(t) \\ \mu_k(t)\beta_l^*(t) & \mu_k(t)\mu_l^*(t) \end{pmatrix} \otimes |1_k\rangle\langle 1_l| \tag{5.40}$$

Tracing out the system, we get the state of the environment

$$\rho_E(t) = \sum_{k,l} (\beta_k(t)\beta_l^*(t) + \mu_k(t)\mu_l^*(t))|1_k\rangle\langle 1_l|, \tag{5.41}$$

where the coefficients $\beta_k(t)\beta_l^*(t) + \mu_k(t)\mu_l^*(t)$ simplify to the form

$$-g_k g_l^* \int_0^t \int_0^t e^{i(\omega_l-\omega_0)t_2 - i(\omega_k-\omega_0)t_1} cos(g(t_2-t_1))\alpha^*(t_1)\alpha(t_2)\mathrm{d}t_1\mathrm{d}t_2, \tag{5.42}$$

which clearly does not change in t after t is such that $\alpha(t) = 0$. So we see that the state of the environment does not change after the time at which the level $|a\rangle$ has depleted. Still, obviously the different terms in the state of the total system (5.40) do change in time. The change in the coherence terms of the total system state show how correlations between the system and the environment can still change in time even though the state of the environment does not change anymore. This now explains the non-Markovianity in the system in terms of the BLP information flow. The information does not go to the change in the state of the environment, but it goes to the correlations between the system and the environment. The change is of course induced by the coupling between $|b\rangle$ and $|m\rangle$, which is switched on forever. Notice that the correlations in the total system state cannot be created without some coupling between the system and the environment at some point, which explains the different behavior for the initial state where there is no initial population in the level $|a\rangle$. This difference in the long time limit causes the trace distance of the initial states $|a\rangle$ and $|b\rangle$ to oscillate forever. To get a finite value for the measure with different coupling strengths as plotted in Fig. 5.5, we use a finite integration time $\lambda t = 20$ in the integral.

5.3 Deciding entanglement of bipartite states

As stated in previous sections, quantum entanglement is a vital resource for quantum information purposes. Because of this, finding out if a given state is entangled or not, is an important problem in quantum mechanics. Many of the existing methods like entanglement witnesses [102, 103] can detect the entanglement of specific kinds of states. In other words, the methods are either probabilistic or they rely on prior information about the state. For example measuring witnesses can be thought of as slicing the state space with hyperplanes. The intuition is then, that if the boundary of separable states is somehow "round" somewhere, an infinite amount of hyperplanes (witnesses) are needed to carve such shape out of the state space. As it turned out in our study, the shape of the set of states with a given property dictates the minimum amount of information needed to decide if a completely unknown state has this property or not.

We have studied this problem using the idea of direction in state space, much like in optimizing the BLP measure. We have shown that with no prior information about the state, an informationally complete measurement is required to decide with certainty if the state is entangled or not. This means that so much information about the state is required that it can be used to identify the state uniquely. This might have been an expected result, but so far lacked a rigorous proof, which we provided based on simple geometrical arguments.

The problem can be formulated as a membership problem, because we can split the state space into the set of entangled states $\mathcal{P}$ and states that are not entangled $\mathcal{P}^c$. The problem is then to decide, purely based on measurement statistics, if the state belongs to $\mathcal{P}$ or $\mathcal{P}^c$. Notice that if some states ρ_1 and ρ_2 give the same statistics for a measurement described by a POVM A, we have that $\mathrm{tr}\left((\rho_1 - \rho_2)L_j\right) = 0$ for all L_j. Thus the direction $\rho_1 - \rho_2$ is orthogonal to the vector space spanned by the POVM elements L_j. In other words $\rho_1 - \rho_2$ belongs to the vector space

$$\mathcal{X}_A = \{\Delta \mid \mathrm{tr}(\Delta L_j) = 0 \ \text{ for all } \ j\}, \tag{5.43}$$

which consists of all the directions orthogonal to all the POVM elements L_j. This forms the main geometric idea of our proof. We see that a measurement of A determines if a state belongs to $\mathcal{P}$, if and only if the subspace $\mathcal{X}_A$ has the property that no direction $\Delta \in \mathcal{X}_A$ can be decomposed as $\Delta = \lambda(\varrho_1 - \varrho_2)$ with $\varrho_1 \in \mathcal{P}$ and $\varrho_2 \in \mathcal{P}^c$ and some scalar λ. This way $\mathcal{X}_A$ characterizes the distinguishing power of a POVM, the extreme case

being $\mathcal{X}_A = \{0\}$, corresponding to an informationally complete POVM, which can distinguish between any two states. Such measurements must always have at least $\dim(\mathcal{H})^2$ elements L_j, meaning that they are the most demanding in terms of required resources. This is intuitive of course, since an informationally complete measurement gives all the information about a state density matrix, meaning that it would be pointless to measure more, since there is nothing new to learn.

The dimension of $\mathcal{X}_A$ depends on the number $\dim(A)$ of linearly independent elements of $\{L_j\}$ by the relation

$$dim(\mathcal{H})^2 = \dim(\mathcal{X}_A) + \dim(A), \tag{5.44}$$

which follows from the fact that the real vector space of all Hermitian matrices acting on $\mathcal{H}$ can be decomposed as a direct sum of the span of $\{L_j\}$ and $\mathcal{X}_A$. Thus to optimize the amount of measurement resources used to decide if a state belongs to $\mathcal{P}$ or $\mathcal{P}^c$ we look for the largest $\mathcal{X}_A$ such that the corresponding POVM can still answer the question with certainty.

5.3.1 Geometric characterization

To summarize the previous discussion, we have that for a given property $\mathcal{P}$ and a POVM A, the relevant question is if any $\Delta \in \mathcal{X}_A$ can be decomposed as

$$\Delta = \lambda(\rho_1 - \rho_2),$$

with $\rho_1 \in \mathcal{P}$ and $\rho_2 \in \mathcal{P}^c$. Solving for

$$\rho_1 = \rho_2 + \lambda^{-1}\Delta,$$

we arrive at the following observation: a POVM A can not be used to verify a property $\mathcal{P}$ if it is possible to draw a line segment from $\mathcal{P}$ to $\mathcal{P}^c$ which is parallel to $\mathcal{X}_A$. Conversely, A can be used for the task if there does not exist a line parallel to $\mathcal{X}_A$ which intersects both $\mathcal{P}$ and $\mathcal{P}^c$.

It is immediate that if $\mathcal{P}$ is contained in the interior of $\mathcal{S}(\mathcal{H})$, it has an enclosing surface (defined in section 4.1), meaning that for any given direction, one can find a parallel line intersecting $\mathcal{P}$ and $\mathcal{P}^c$. Thus to verify the corresponding property with certainty, an informationally complete measurement is required. However, we are interested in entanglement and we know that there are entangled states also on the boundary of $\mathcal{S}(\mathcal{H})$, which makes the problem more intricate. Notice also that if the property is such that it can be verified without informational completeness, there

are implications regarding the shape of the corresponding sets in $\mathbb{R}^{D^2-1}$. Indeed, this means that there exists at least one preferred direction such that no line parallel to it crosses the boundary between $\mathcal{P}$ and $\mathcal{P}^c$, making the boundary flat in that direction. Similarly, if there are several preferred directions which corresponds to one being able to verify the property $\mathcal{P}$ with fewer POVM elements (measurement outcomes), then the boundary is flat in all of these directions.

5.3.2 Results

We did the previous analysis for various choices of the set $\mathcal{P}$. These were non-positive partial transpose (NPT) states, entangled states, left discordant states and non-classical states, which are the complement of the intersection of left discordant and right discordant states. The calculations were lengthy, but basically straightforward linear algebra. For example in the case of entanglement, we showed that for every direction Δ we can take a separable isotropic state

$$\rho_2 = \frac{1}{d+1}|\Psi\rangle\langle\Psi| + \frac{1}{d(d+1)}I, \tag{5.45}$$

where Ψ is some maximally entangled state and find a corresponding ρ_1, which we showed to be entangled using the NPT criterion. The results are tabulated in Table 5.1. As we see, the only set that does not require an informationally complete measurement, is the set of discordant states. This might seem odd, since for example the states that are not discordant and thus can be written in the form $\sum_i \lambda_i \rho_i \otimes |i\rangle\langle i|$ are contained in the set of separable states, but this just neatly demonstrates how the size of the subset is not important for the verification problem. The only thing that matters is the shape of subset.

Property to be verified	Informational completeness	Minimal number of outcomes
NPT	✔	D^2
ENTANGLED	✔	D^2
LEFT DISCORDANT	✘	$D^2 - D + 1$
NON-CLASSICAL	✔	D^2

Table 5.1: Summary of the main results. For the various properties of bipartite states, we indicate whether or not their verification requires an informationally complete measurement, as well as the minimal number of measurement outcomes needed for succeeding in the task. $D = d^2$ is the dimension of the bipartite system's Hilbert space.

Chapter 6

Conclusions

In this thesis we have discussed the mathematical ideas behind quantifying quantum memory effects and some applications of them with a special emphasis on the Breuer, Laine, Piilo (BLP) measure of non-Markovianity and various questions related to it. How can we calculate the measure easier? What is the essential ingredient making the measure what it is? Could there be useful applications of non-Markovianity? Can we find interesting interpretations of well known quantum effects in terms of non-Markovianity? An application for a tool used in studying non-Markovianity was also found in studying the correlations in quantum states and the resources needed for detecting them.

In Chapter 2 the notion of a quantum state was introduced and we explored some of the notions describing the set of all possible states of a quantum system. We also introduced the notion of multipartite quantum states and reviewed some of the mathematical consequences of the tensor product structure, which include the notion of correlations that make a multipartite state more than the sum of its parts. We also introduced the ideas of direction and distance within this set, tools which turned out to be very useful in studying non-Markovianity and also quantum correlations.

Chapter 3 continued our investigations of the theoretical background necessary for this thesis. We started by introducing the notions of closed and open quantum systems and their dynamics. We also came across the fact that many systems behave in a non-Markovian way, and such dynamics lack a general, all encompassing theory. To study the notion of non-Markovianity we discussed on a general level different ideas for quantifying it and then focused on one such idea, the BLP measure, which is based on the notion of distance in state space defined by the trace distance metric.

In Chapter 4 we presented our results on how the calculation of the

BLP measure can be optimized. We showed that the maximization over all quantum states can be restricted to orthogonal pairs of quantum states. We further showed how the state pairs themselves are not important, but the essential idea is the direction defined by them. This makes an enormous simplification to the calculation of the measure and also gives a new insight to the interpretation of memory effects in the BLP sense. These effects manifest in the same way in all parts of the state space, making it possible to learn everything about the measure with access only to a small part of the whole space.

Chapter 5 introduced an idea how non-Markovianity could be useful in quantum information. We studied the superdense coding protocol where the state used for encoding is decohered by noise on Alice's side before the encoding operation. It turned out that the protocol can be performed with almost perfect fidelity even with the decohered state. This was due to the non-local memory effects that Bob can use to recohere the joint state by adding noise to his part of it. This was based on perfect anticorrelations in the joint environment state of the original joint state shared by Alice and Bob. This protocol was also demonstrated experimentally, which acted as a proof of principle test showing that non-Markovianity could be a resource for quantum information tasks. We also studied the quantum Zeno effect from the point of view of non-Markovianity. We found that Zeno dynamics caused by a strong coupling to an external level serves as a good example on how information of the system state can flow to the correlations in the total system plus environment state without the environmental state changing at all. Finally, we showed how the notion of direction in quantum state space can be used to study the amount of resources needed for deciding if an unknown quantum state has some specific property, for example entanglement. So far only partial results have existed and we were able to rigorously show, that deciding with certainty if a completely unknown quantum state is entangled or not, an informationally complete measurement is needed. Our analysis worked on such a general level that it can be applied to any such binary membership problem for quantum states. We also showed that the amount of resources needed depends only on the shape of the set of interest.

Future directions in the study of non-Markovian dynamics could include more work on the interpretation of the different measures and comparison between them. As many of the measures study the net increase of some quantity of interest in time, more emphasis could be put on the interpretation of what the numerical value actually means. Is a system with one

big increase in trace distance as non-Markovian as another system that has many small increases that add up to the same net increase? Another interesting direction would be constructing examples that are more sophisticated. Now many models like the ones studied in this thesis are such that the state itself does a loop in the state space during its evolution. Thus it is quite clear that if one calculates a number based on the state, it must also do some kind of a loop and thus exhibit non-monotonic behaviour. More non-trivial examples could also shed more light on the differences between the different measures of non-Markovianity.

As a possible further application of non-Markovian dynamics could be something in quantum cryptography. With photons the interactions and the initial states are very controllable and adding specially tailored non-Markovian noise to a system to hide its state could be of some use in the future. The possible non-Markovian character of the interactions could be used to recover the hidden state with an interaction unknown to a malicious third party.

For deciding the entanglement content of unknown quantum states, further studies based on the geometry of the set of entangled or separable states could be made. It is known [104] that the set of separable states is a semialgebraic set and as such can be described by a finite number of polynomial inequalities. These inequalities can then be checked with collective measurements (simultaneous measurement on many copies of the state) but there are no known bounds on the number of inequalities needed for general systems. As entanglement is possibly the most valuable resource in quantum information, such studies giving bounds on the number of inequalities and thus the usefulness of collective measurements in entanglement detection, could be extremely valuable.

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