WEIGHTED ORLICZ REGULARITY ESTIMATES FOR FULLY NONLINEAR ELLIPTIC EQUATIONS WITH ASYMPTOTIC CONVEXITY

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ABSTRACT. We prove interior Hessian estimates in the setting of weighted Orlicz spaces for viscosity solutions of fully nonlinear, uniformly elliptic equations

$$F(D^2u, x) = f(x) \text{ in } B_1$$

under asymptotic assumptions on the nonlinear operator F. The results are further extended to fully nonlinear, asymptotically elliptic equations.

1. INTRODUCTION

In this paper, we consider the following fully nonlinear equation

$$F(D^2u, x) = f(x) \text{ in } B_1, \tag{1.1}$$

in which the nonlinearity $F = F(X, x) : S(n) \times B_1 \to \mathbb{R}$ is a Carathéodory function (that is, $F(\cdot, x)$ is continuous for a.e. $x \in B_1$ and $F(X, \cdot)$ is measurable for all $X \in S(n)$), where $S(n)$ denotes the set of $n \times n$ real symmetric matrices. We assume that the operator F is uniformly elliptic with ellipticity constants λ, Λ, that is, there exist constants λ and Λ with $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \|Y\| \leq F(X + Y, x) - F(X, x) \leq \Lambda \|Y\|$$

for all $X, Y \in S(n), Y \geq 0$ and almost all $x \in B_1$, where $\|Y\| := \sup_{|x|=1} |Yx|$.

In the seminal paper [8], Caffarelli established interior $W^{2,p}$ *a priori* estimates for viscosity solutions to (1.1) for all $p > n$, provided that the homogeneous, constant-coefficient equation $F(D^2u) = 0$ has $C^{1,1}$ *a priori* estimates. His works have brought about a significant development in the study of regularity theory for fully nonlinear equations, and in particular, L^p regularity theory for viscosity solutions to (1.1) has been extensively studied, see for instance [3, 4, 16, 38, 43, 45]. We are concerned with interior regularity for viscosity solutions of (1.1) under asymptotic assumptions on the nonlinearity F. In the recent paper [38], Pimentel and Teixeira derived $W^{2,p}$ estimates with $p > n$ for the constant-coefficient equations $F(D^2u) = f(x)$ under the condition of asymptotic convexity imposed on the operator F, and also extended those estimates to (1.1) under Hölder type continuity of the coefficients in the L^n sense.

The purpose of the present paper is to obtain Hessian estimates for (1.1) in the framework of weight Orlicz spaces, extending the results of [38]. The Hessian integrability of solutions to (1.1) is linked closely to the behavior of the nonlinear operator $F(X, x)$ near infinity with respect to $X \in S(n)$. In this regard, in order

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to deal with such an asymptotic property on F, we adopt the idea of [38] by taking the notion of the *recession* operator $F^\star$ associated with F, which is given by

$$F^\star(X,x) := \lim_{\mu\to 0} \mu F(\mu^{-1}X, x).$$

More precisely, under the assumptions that the *recession* operator $F^\star$ satisfies a small oscillation condition in the integral sense with respect to x and the homogeneous, constant-coefficient equation $F^\star(D^2v) = 0$ has $C^{1,1}$ *a priori* estimates, we prove that for $f \in L^\Psi_w(B_1)$, any viscosity solution u to (1.1) belongs to the weighted Orlicz Sobolev space $W^{2,\Psi}_w(B_{\frac{1}{2}})$ satisfying the estimate

$$\|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} \le c\left(\|f\|_{L^\Psi_w(B_1)} + \|u\|^n_{L^\infty(B_1)}\right) \tag{1.2}$$

for *N-function* Ψ and *Muckenhoupt weight* w, where the constant c is independent of f and u. We point out that suitable additional conditions which will be described later should be required on both Ψ and w for ensuring the validity of (1.2). Especially, (1.2) can be reduced to the $W^{2,p}$ estimates in [38] if we take $\Psi(t) = t^{\frac{p}{n}}$ and the trivial weight $w \equiv 1$. Furthermore, we extend the weighted Orlicz estimates (1.2) to a more general equation than (1.1), namely, $F(D^2u, Du, u, x) = f$, provided that the nonlinear operator F satisfies the appropriate structure condition (cf. Theorem 6.2). On the other hand, similar regularity estimates have been obtained in the very recent paper [3] under the convexity of the operator F with respect to D^2u. In view of weakening the convexity condition on F, our works also can be regarded as a sort of these extensions.

Our proof is based on the approach of Caffarelli [8]. We employ the perturbation technique by taking into account regularity available for the solutions to the homogeneous, constant-coefficient equation $F^\star(D^2u, x_0) = 0$ instead of the original operator F. Particularly, the geometric tangential method presented in [38] helps to connect between the regularity theory of F and the one of $F^\star$. Suitable properties of the N-functions and Muckenhoupt weights are needed in order to manage characteristics of weighted Orlicz spaces considered here. Our approach basically relies upon properties of the Hardy-Littlewood maximal function, and in particular, the index characterization of the weights in the reflexive Orlicz spaces for which the maximal function is bounded in the weighted Orlicz spaces. These properties play a central role on the key step in deriving power decay estimates for the upper level set of Hessian of solutions to (1.1).

We further discuss the validity of our result for asymptotically elliptic operator $F(X, x)$ which has a more general behavior near infinity with respect to X. Much research has been done on such asymptotically regular problems, for instance, in [6, 13, 21, 34, 40]. Very recently, Byun, Oh and Wang [7] proved global $W^{2,p}$ estimates for the asymptotically elliptic problem. They made use of a transformation from the given asymptotically elliptic operator, which is neither uniformly elliptic nor convex, into the appropriate uniformly elliptic operator which is uniformly convex. Following their idea, we treat the case of asymptotically elliptic equations, thus extending the regularity results of [7] to the setting of weighted Orlicz spaces and relaxing the convexity condition on the uniformly elliptic operator.

The remainder of this paper is organized as follows. In Section 2, we state our main result (Theorem 2.4) after recalling some basic notions and properties concerning weighted Orlicz spaces. In Section 3, we provide the approximation lemma and introduce the basic tools that are used in the proof of Theorem 2.4.

Section 4 contains the detail proof of the interior $W^{2,p}$ estimates in [38, Theorem 6.1]. The proof of our main theorem (Theorem 2.4) is presented in Section 5. We further extend to a more general equation in Section 6. In Section 7, we state and prove the regularity for the fully nonlinear, asymptotically elliptic equations as an outgrowth of Theorem 2.4.

2. Main result

2.1. Weighted Orlicz space. In order to deal with the weighted Orlicz space, we need to introduce the concept of an *N-function* and its properties. A function $\Phi\colon [0,\infty) \to [0,\infty]$ is said to be an *N-function* if it is convex, continuous and increasing, and satisfies that $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t > 0$,

$$\lim_{t\to\infty} \Phi(t) = +\infty, \text{ and } \lim_{t\to 0^+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0.$$

We say that Φ satisfies Δ_2-condition ($\Phi \in \Delta_2$) if there exists a constant $\kappa_1 > 1$ such that

$$\Phi(2t) \le \kappa_1 \Phi(t) \text{ for all } t > 0,$$

and that Φ satisfies ∇_2-condition ($\Phi \in \nabla_2$) if there exists a constant $\kappa_2 > 1$ such that

$$\Phi(t) \le \frac{1}{2\kappa_2} \Phi(\kappa_2 t) \text{ for all } t > 0.$$

Furthermore, Φ is said to satisfy $\Delta_2 \cap \nabla_2$-condition ($\Phi \in \Delta_2 \cap \nabla_2$) if Φ satisfies both the Δ_2-condition and the ∇_2-condition.

Let U be a bounded domain in $\mathbb{R}^n$ with $n \ge 2$. Consider a weight w, namely, a locally integrable function w on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. Given an N-function Φ satisfying $\Delta_2 \cap \nabla_2$-condition, the *weighted Orlicz space* $L^\Phi_w(U)$ is defined as the set of all Lebesgue measurable functions g on U such that

$$\int_U \Phi\left(|g(x)|\right) w(x)\, dx < +\infty.$$

In particular, since $\Phi \in \Delta_2 \cap \nabla_2$, this *weighted Orlicz space* $L^\Phi_w(U)$ becomes a reflexive Banach space under the following *Luxemburg norm*

$$\|g\|_{L^\Phi_w(U)} = \inf\left\{ s > 0 : \int_U \Phi\left(\frac{|g(x)|}{s}\right) w(x)\, dx \le 1 \right\}.$$

Moreover, the *weighted Orlicz Sobolev space* $W^{2,\Phi}_w(U)$ is defined as the set of all functions g in $L^\Phi_w(U)$ with weak derivatives $D^\alpha g \in L^\Phi_w(U)$ for $|\alpha| \le 2$, and its norm is given by

$$\|g\|_{W^{2,\Phi}_w(U)} := \|g\|_{L^\Phi_w(U)} + \|Dg\|_{L^\Phi_w(U)} + \|D^2 g\|_{L^\Phi_w(U)}.$$

Throughout this paper, we will assume that the N-function Φ satisfies $\Delta_2 \cap \nabla_2$-condition. This condition is required for regularity results for solutions to PDEs and variational problems, for instance, in [1, 2, 5, 17, 25, 27, 37], as well as the study of free boundary problems and the construction of fractional Orlicz spaces, for instance, in [19, 35, 36]. A typical example of the N-function Φ satisfying the $\Delta_2 \cap \nabla_2$-condition is $\Phi(t) = t^q$ with $q > 1$. In this case, the weighted Orlicz space $L^\Phi_w(U)$ and the weighted Orlicz Sobolev space $W^{2,\Phi}_w(U)$ coincide with the weighted Lebesgue space $L^q_w(U)$ and the weighted Sobolev space $W^{2,q}_w(U)$, respectively. In other words, the spaces $L^\Phi_w(U)$ and $W^{2,\Phi}_w(U)$ generalize the spaces $L^q_w(U)$ and

$W^{2,q}_w(U)$, respectively, in the sense that the power function t^q in the definition of the spaces $L^q_w(U)$ and $W^{2,q}_w(U)$ is replaced by a more general convex function, N-function Φ. In addition, when $w \equiv 1$ and $\Phi(t) = t^q$ with $q > 1$, it is clear that the spaces $L^\Phi_w(U)$ and $W^{2,\Phi}_w(U)$ are equivalent to the classical Lebesgue space $L^q(U)$ and the classical Sobolev space $W^{2,q}(U)$, respectively. We refer the reader to [31, 32, 39] for further properties of N-functions and weighted Orlicz spaces.

2.2. Assumption on a weight w. A weight w is called *an A_q (Muckenhoupt) weight* with $1 < q < \infty$, denoted by $w \in A_q$, if

$$[w]_q := \sup_{B\subset\mathbb{R}^n} \left(\fint_B w(x)\,dx\right)\left(\fint_B w(x)^{\frac{-1}{q-1}}\,dx\right)^{q-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. We identify the weight w with the measure

$$w(E) = \int_E w(x)\,dx$$

for the Lebesgue measurable set $E \subset \mathbb{R}^n$. The A_q weights are invariant under translation, dilation and multiplication by a positive scalar. Each A_q-weight has the doubling property and monotonicity, i.e, $A_{q_1} \subset A_{q_2}$ for $q_1 \le q_2$. Furthermore, the A_q weight has the self-improving property which means that if $w \in A_q$, then $w \in A_{q-\tilde{\epsilon}}$ for some small constant $\tilde{\epsilon} = \tilde{\epsilon}(n, q, [w]_q) > 0$. One more significant property of the A_q-weights is following:

Lemma 2.1. *Let $w \in A_q$ where $1 < q < \infty$. Let D be a measurable subset of a ball $B \subset \mathbb{R}^n$. There exist two positive constants s_1, s_2 depending only on n, q and $[w]_q$ such that*

$$[w]_q^{-1}\left(\frac{|D|}{|B|}\right)^q \le \frac{w(D)}{w(B)} \le s_1\left(\frac{|D|}{|B|}\right)^{s_2}.$$

We refer to [23, 42, 44] for further properties of A_q weights including the proof of Lemma 2.1.

Given the N-function Φ satisfying $\Phi \in \Delta_2 \cap \nabla_2$-condition, our principal assumption on the weight w is that w belongs to the $A_{i(\Phi)}$ class, where $i(\Phi)$ is the *lower index* of Φ given by

$$i(\Phi) = \lim_{\nu\to0^+} \frac{\log(h_\Phi(\nu))}{\log\nu} = \sup_{0<\nu<1} \frac{\log(h_\Phi(\nu))}{\log\nu},$$

with

$$h_\Phi(\nu) = \sup_{t>0} \frac{\Phi(\nu t)}{\Phi(t)} \quad \text{for } \nu > 0.$$

It is worth pointing out that this assumption guarantees the boundedness of the Hardy-Littlewood maximal function in the corresponding weighted Orlicz space; see Lemma 3.3 for more details. On the other hand, under the $\Delta_2 \cap \nabla_2$-condition of Φ, we notice that there exist two constants γ_1, γ_2 with $1 < \gamma_1 \le \gamma_2 < \infty$ such that

$$c^{-1}\min\{\nu^{\gamma_1}, \nu^{\gamma_2}\}\,\Phi(t) \le \Phi(\nu t) \le c\,\max\{\nu^{\gamma_1}, \nu^{\gamma_2}\}\,\Phi(t) \quad \text{for } \nu, t \ge 0, \tag{2.1}$$

where the constant c is independent of ν and t (see [24, 27, 31]), and that

$$\int_U \Phi\left(\frac{|g(x)|}{\|g\|_{L^\Phi_w(U)}}\right) w(x)\,dx = 1, \tag{2.2}$$

for nonzero function $g \in L^{\Phi}_{w}(U)$ (see [32, formula (9.22)]). Then using (2.1) and (2.2), it can be seen that

$$\|g\|_{L^{\Phi}_{w}(U)} - 1 \leq \int_U \Phi(|g(x)|)w(x)\,dx \leq c\left(\|g\|^{\gamma_2}_{L^{\Phi}_{w}(U)} + 1\right) \tag{2.3}$$

where the constant $c > 1$ is independent of g (see [32]). We also note that inclusions between weighted Orlicz spaces and weighted Lebesgue spaces hold as follows:

$$L^{\infty}(U) \subset L^{\gamma_2}_{w}(U) \subset L^{\Phi}_{w}(U) \subset L^{\gamma_1}_{w}(U) \subset L^{1}(U).$$

We further remark that $i(\Phi)$ is equal to the supremum of those γ_1 satisfying the above inequality (2.1) with $\nu \geq 1$, and then it is clear that $i(\Phi) > 1$ (or also see [20]). In the case that $\Phi(t) = t^q$ with $q > 1$, it is obvious that $A_{i(\Phi)}$ weights coincide with A_q weights.

2.3. Main result. Before stating our main result, we introduce some notation. We write $B_r(y)$ for the open ball in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$ with radius $r > 0$. We also denote $Q_r(y)$ as the open cube in $\mathbb{R}^n$ centered at $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ with side-length $r > 0$, i.e.,

$$Q_r(y) := \prod_{i=1}^{n}\left(y_i - \frac{r}{2}, y_i + \frac{r}{2}\right).$$

For simplicity, we set $B_r \equiv B_r(0)$ and $Q_r \equiv Q_r(0)$. In addition, we write

$$(g)_U := \fint_U g(x)\,dx = \frac{1}{|U|}\int_U g(x)\,dx$$

for a locally integrable function $g\colon U \to \mathbb{R}$ with a bounded set $U \subset \mathbb{R}^n$, where $|U|$ is the n-dimensional Lebesgue measure of U. From now on, the letter c denotes a positive universal constant that may vary at each appearance.

We recall the definition of viscosity solutions which will be treated throughout this paper. Let U be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. For a measurable function $f\colon U \to \mathbb{R}$, let us consider the fully nonlinear equations of the form

$$F(D^2u, x) = f(x) \quad \text{in } U, \tag{2.4}$$

where $F = F(X, x)$ is a Carathéodory function defined on $S(n) \times U$.

Definition 2.2. Let $q > \frac{n}{2}$ and assume that $f \in L^q_{\mathrm{loc}}(U)$. We say that $u \in C(U)$ is *an L^q-viscosity solution* of (2.4) if the following two conditions are satisfied:

(i) for all $\varphi \in W^{2,q}_{\mathrm{loc}}(U)$, whenever $\epsilon > 0$, $\mathcal{O} \subset U$ is open and
$$F\left(D^2\varphi(x), x\right) \leq f(x) - \epsilon \ \text{ a.e. in } \mathcal{O},$$
$u - \varphi$ cannot have a local maximum in $\mathcal{O}$,

(ii) for all $\varphi \in W^{2,q}_{\mathrm{loc}}(U)$, whenever $\epsilon > 0$, $\mathcal{O} \subset U$ is open and
$$F\left(D^2\varphi(x), x\right) \geq f(x) + \epsilon \ \text{ a.e. in } \mathcal{O},$$
$u - \varphi$ cannot have a local minimum in $\mathcal{O}$.

Under the assumption that F and f are continuous in all variables, $u \in C(U)$ is said to be *a C-viscosity solution* of (2.4) if the test function φ belongs to $C^2(U)$ in Definition 6.1. Note that C-viscosity solutions of (2.4) are L^q-viscosity solutions whenever F and f are continuous in all variables, see [10, Proposition 2.9]. We refer to [9, 10, 14, 15] and also the references therein for further properties of viscosity solutions to (2.4).

Remark 2.3. By virtue of the self improving property of N-function Φ, we note that if $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}$, $L^{\Phi}_w(U)$ is continuously embedded in $L^{\tilde{q}}(U)$ for some constant $\tilde{q} = \tilde{q}(\Phi, w)$ with $1 < \tilde{q} < i(\Phi)$, see [3, Lemma 2.5] for its proof. In this way, the notion of $L^{\tilde{q}\,n}$-viscosity solutions can be treated for the main problem (1.1) provided that $|f|^n \in L^{\Phi}_w(B_1)$. Hereafter, we let $\gamma := \tilde{q}\,n$.

As mentioned before, we shall employ the *recession* operator $F^\star$ associated with the nonlinear operator F which is given by

$$F^\star(X, x) := \lim_{\mu \to 0} F_\mu(X, x)$$

assuming its existence for any $X \in S(n)$ and $x \in B_1$, where we denote $F_\mu(X, x) := \mu F(\mu^{-1}X, x)$ for any $\mu > 0$. It can be easily checked that F_μ and $F^\star$ are uniformly elliptic with the same ellipticity constants as F. We refer to [38, 41] for an overview for the recession operator $F^\star$. In addition, let us point out that very recently the use of the recession operator has been extended to the context of fully nonlinear parabolic problems by Castillo and Pimentel in [12].

In this paper, we suppose that the recession function $F^\star$ associated with the original operator F exists, and $F^\star(D^2v, x_0) = 0$ has $C^{1,1}$ interior estimates with constant $c_\star$ for any $x_0 \in B_1$, that is, for any $v_0 \in C(\partial B_1)$ there exists a C-viscosity solution $v \in C^2(B_1) \cap C(\overline{B}_1)$ of

$$\begin{cases} F^\star(D^2v, x_0) &= 0 \quad \text{in} \quad B_1, \\ \qquad\qquad v &= v_0 \quad \text{on} \quad \partial B_1, \end{cases}$$

with

$$\|v\|_{C^{1,1}(\overline{B}_{1/2})} \le c_\star \|v_0\|_{L^\infty(\partial B_1)}$$

for some universal constant $c_\star > 0$. We remark that if $F^\star(X, x)$ has the uniform convexity with respect to X, the Evans-Krylov $C^{2,\alpha}$ regularity theorem (cf. [18, 33] or Chapter 6 in [9]) yields that $F^\star(D^2v, x_0) = 0$ has $C^{1,1}$ interior estimates.

Without loss of generality, we further assume that $F(0, \cdot) \equiv 0$ in B_1. Indeed, this assumption is not essential because Eq.(1.1) can be written as

$$\tilde{F}(D^2u, x) := F(D^2u, x) - F(0, x) = f(x) - F(0, x)$$

and then it is obvious that $\tilde{F}(0, \cdot) \equiv 0$ in B_1.

For any $x, y \in U$, we denote

$$\beta_G(x, y) := \sup_{X \in S(n) \setminus \{0\}} \frac{|G(X, x) - G(X, y)|}{\|X\|} \tag{2.5}$$

which will be utilized for measuring the oscillation of the operator $G = G(X, x) : S(n) \times U \to \mathbb{R}$ with respect to x. We then notice that $\beta_F(x, y) = \beta_{F_\mu}(x, y)$ and $\beta_{F^\star}(x, y) \le \beta_F(x, y)$.

The main theorem of this paper is following:

Theorem 2.4 (Main Theorem)**.** *Assume that Φ is an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and $w \in A_{i(\Phi)}$. Let u be an L^γ-viscosity solution to* (1.1)*, where γ is defined in Remark 2.3. Suppose that $F^\star(X, x)$ exists and $F^\star(D^2v, x_0) = 0$ has $C^{1,1}$ interior estimates with constant $c_\star$ for any $x_0 \in B_1$. Assume further* *that* $f \in L^{\Psi}_w(B_1)$ *with* $\Psi(t) := \Phi(t^n)$. *Then there exists a positive constant* $\delta = \delta(n, \lambda, \Lambda, \Phi, w, c_\star)$ *such that if*

$$\left(\fint_{B_r(x_0)} \beta_{F^\star}(x, x_0)^n \, dx \right)^{1/n} \leq \delta$$

for any ball $B_r(x_0) \subset B_1$ *with* $r > 0$, *then we have* $u \in W^{2,\Psi}_w(B_{\frac{1}{2}})$ *with the estimate*

$$\|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} \leq c \left(\|f\|_{L^{\Psi}_w(B_1)} + \|u\|^n_{L^\infty(B_1)} \right) \tag{2.6}$$

for some $c = c(n, \lambda, \Lambda, \Phi, w, c_\star) > 0$, *where*

$$\|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} := \||u|^n\|_{L^{\Phi}_w(B_{\frac{1}{2}})} + \||Du|^n\|_{L^{\Phi}_w(B_{\frac{1}{2}})} + \||D^2u|^n\|_{L^{\Phi}_w(B_{\frac{1}{2}})}.$$

Remark 2.5. Recall that the space BMO (bounded mean oscillation) can be defined to be the collection of all locally integrable function g in $\mathbb{R}^n$ such that

$$\|g\|_{BMO} := \sup_{B \subset \mathbb{R}^n} \frac{\|(g - (g)_B)\chi_B\|_{L^1}}{\|\chi_B\|_{L^1}} < \infty, \tag{2.7}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. It is well known that $f \in L^\infty$ cannot imply $D^2u \in L^\infty$ for the fully nonlinear equations (1.1) in general, even for poisson equations. However, it is possible to replace L^∞ space by BMO space from the regularity results in [38, Theorem 7.1] (also see [11]) that can be regarded as an endpoint case of L^p estimates. As a generalization of such estimates, we can further consider the BMO type estimates in the framework of weighted Orlicz spaces L^{Φ}_w, namely that $f \in BMO_{L^{\Phi}_w}(B_1)$ implies $D^2u \in BMO_{L^{\Phi}_w}(B_{\frac{1}{2}})$ with

$$\|D^2u\|_{BMO_{L^{\Phi}_w}(B_{\frac{1}{2}})} \leq c \left(\|f\|_{BMO_{L^{\Phi}_w}(B_1)} + \|u\|_{L^\infty(B_1)} \right), \tag{2.8}$$

under the same hypotheses of Theorem 2.4. Here the space $BMO_{L^{\Phi}_w}(U)$ is defined to be the collection of all locally integrable functions g on $\mathbb{R}^n$ such that

$$\|g\|_{BMO_{L^{\Phi}_w}(U)} := \sup_{B \subset U} \frac{\|(g - (g)_B)\chi_B\|_{L^{\Phi}_w(U)}}{\|\chi_B\|_{L^{\Phi}_w(U)}} < \infty,$$

where the supremum is taken over all balls B in a bounded domain $U \subset \mathbb{R}^n$. Due to [26, Theorem 2.3], it turns out that BMO is equal to $BMO_{L^{\Phi}_w}$ because of the boundedness of the maximal function on the weighted Orlicz space L^{Φ}_w under our main assumption that $w \in A_{i(\Phi)}$ with $\Phi \in \Delta_2 \cap \nabla_2$ (cf. Lemma 3.3). Moreover, from the famous John-Nirenberg inequality in [28], note that the space BMO is equivalent to the space BMO_{L^p} which is defined in (2.7) with L^p-norm instead of L^1-norm. Therefore, the estimates (2.8) can be easily obtained from the results in [12, Theorem 7.1] and [38, Theorem 7.1].

On the other hand, the John-Nirenberg inequality in [28] implies that the BMO space is contained in the Orlicz space L^{exp} generated by an N-function with exponential growth which does not satisfy Δ_2-condition. Besides, the L^∞ space is close to the Orlicz space generated by an N-function not satisfying Δ_2-condition. Therefore the natural question arises whether Hessian estimates like (2.8) for (1.1) hold in the framework of Orlicz spaces $L^{\tilde{\Phi}}$ or weighted Orlicz spaces $L^{\tilde{\Phi}}_w$ (with $\tilde{\Phi} \notin \Delta_2$) such that $L^\infty \hookrightarrow L^{\tilde{\Phi}} \hookrightarrow BMO \hookrightarrow L^{exp}$. This will be our next project in the near future.

3. Preliminaries

We first present the approximation lemma which will play an essential role in proving our main result, Theorem 2.4, since our approach is based on the perturbation argument following the idea of Caffarelli [8].

Lemma 3.1. *Let u be a C-viscosity solution of*

$$F_\mu(D^2u,x)=f(x) \quad \textit{in } B_{8\sqrt{n}}$$

with $\|u\|_{L^\infty(B_{8\sqrt{n}})}\le 1$. Assume that $F^\star(D^2\psi,0)=0$ has $C^{1,1}$ interior estimates with constant $c_\star$. For any $\epsilon>0$, there exists a small $\delta=\delta(\epsilon,n,\lambda,\Lambda)>0$ such that if

$$\mu<\delta \ \textit{and} \ \ \|f\|_{L^n(B_{8\sqrt{n}})},\ \|\beta_{F^\star}(\cdot,0)\|_{L^n(B_{7\sqrt{n}})}\le\delta,$$

then for any solution $v\in C^2(\overline{B}_{6\sqrt{n}})$ of

$$\begin{cases} F^\star(D^2v,0) = 0 & \textit{in} \quad B_{7\sqrt{n}}, \\ \qquad\qquad v = u & \textit{on} \quad \partial B_{7\sqrt{n}}, \end{cases}$$

we have

$$\|u-v\|_{L^\infty(B_{6\sqrt{n}})}\le\epsilon.$$

Proof. We argue by contradiction. Suppose that the lemma is not true. Then there exist $\epsilon_0>0,\{F^k\}_{k=1}^\infty,\{u_k\}_{k=1}^\infty,\{f_k\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$ such that

$$F^k_{\mu_k}(D^2u_k,x):=\mu_kF^k(\mu_k^{-1}D^2u_k,x)=f_k(x) \ \text{ in } B_{8\sqrt{n}}$$

with

$$\mu_k<\frac{1}{k},\ \|f_k\|_{L^n(B_{8\sqrt{n}})}\le\frac{1}{k},\ \|\beta_{(F^k)^\star}(\cdot,0)\|_{L^n(B_{7\sqrt{n}})}\le\frac{1}{k}, \tag{3.1}$$

and

$$\|u_k-v_k\|_{L^\infty(B_{6\sqrt{n}})}>\epsilon_0 \tag{3.2}$$

for the solution $v_k\in C^2(\overline{B}_{6\sqrt{n}})$ to

$$\begin{cases} (F^k)^\star(D^2v_k,0) = 0 & \text{in} \quad B_{7\sqrt{n}}, \\ \qquad\qquad\quad v_k = u_k & \text{on} \quad \partial B_{7\sqrt{n}}. \end{cases}$$

By the standard Hölder regularity theory (cf. [9, Proposition 4.10]), we note that $u_k\in C^{\alpha_1}(\overline{B}_{7\sqrt{n}})$ with

$$\|u_k\|_{C^{\alpha_1}(\overline{B}_{7\sqrt{n}})}\le c\left(\|f_k\|_{L^n(B_{8\sqrt{n}})}+\|u_k\|_{L^\infty(B_{8\sqrt{n}})}\right)\le c\left(\frac{1}{k}+1\right)\le 2c$$

for some $\alpha_1\in(0,1)$. Then there exists a subsequence of $\{u_k\}_{k=1}^\infty$ which is still denoted by $\{u_k\}_{k=1}^\infty$, and a function $u_\infty\in C^{\alpha_2}(\overline{B}_{7\sqrt{n}})$ such that

$$u_k\longrightarrow u_\infty \text{ locally in } C^{\alpha_2}(\overline{B}_{7\sqrt{n}})$$

for some $0<\alpha_2<\alpha_1$. Moreover, applying [9, Proposition 4.13], we obtain that $v_k\in C^{\frac{\alpha_1}{2}}(\overline{B}_{7\sqrt{n}})$ with

$$\|v_k\|_{C^{\frac{\alpha_1}{2}}(\overline{B}_{7\sqrt{n}})}\le c\|u_k\|_{C^{\alpha_1}(\partial B_{7\sqrt{n}})}\le 2c.$$

Then there there exists a subsequence of $\{v_k\}_{k=1}^\infty$ which is still denoted by $\{v_k\}_{k=1}^\infty$, and a function $v_\infty\in C^{\alpha_3}(\overline{B}_{7\sqrt{n}})$ such that

$$v_k\longrightarrow v_\infty \text{ locally in } C^{\alpha_3}(\overline{B}_{7\sqrt{n}})$$

for some $0 < \alpha_3 < \frac{\alpha_1}{2}$. It is obvious that $f_k \to 0$ as $k \to \infty$. From [38, Lemma 4.1] along with (3.1), it follows that for any $\tilde{\epsilon} > 0$, there exists $N = N(\tilde{\epsilon}) > 0$ such that

$$|F^k_{\mu_\ell}(X,x) - (F^k)^\star(X,x)| \le \tilde{\epsilon}\,(1 + \|X\|)$$

for every $X \in S(n)$, $x \in B_{7\sqrt{n}}$ and every $k \ge 1$, whenever $\ell \ge N$. In addition, since $(F^k)^\star$ are uniformly elliptic operators, one can check that $(F^k)^\star(\cdot,0)$ converges uniformly to $G(\cdot,0)$ on compact sets of $S(n)$, for some uniformly elliptic operator $G : S(n) \longrightarrow \mathbb{R}$. Then it is clear that v_∞ is a C-viscosity solution of

$$\begin{cases} G(D^2 v_\infty, 0) = 0 & \text{in } B_{7\sqrt{n}}, \\ v_\infty = u_\infty & \text{on } \partial B_{7\sqrt{n}}. \end{cases} \tag{3.3}$$

We note that for any $\varphi \in C^2(B_{8\sqrt{n}})$,

$$\begin{aligned}
&|F^k_{\mu_k}(D^2\varphi(x),x) - f_k(x) - G(D^2\varphi(x),0)| \\
&\le |F^k_{\mu_k}(D^2\varphi(x),x) - F^k_{\mu_k}(D^2\varphi(x),0)| + |f_k(x)| \\
&\quad + |F^k_{\mu_k}(D^2\varphi(x),0) - (F^k)^\star(D^2\varphi(x),0)| + |(F^k)^\star(D^2\varphi(x),0) - G(D^2\varphi(x),0)| \\
&\le \left(\sup_{X\in S(n)} \frac{|F^k_{\mu_k}(X,x) - F^k_{\mu_k}(X,0)|}{\|X\|+1}\right)\big(\|D^2\varphi\|+1\big) + |f_k(x)| \\
&\quad + |F^k_{\mu_k}(D^2\varphi(x),0) - (F^k)^\star(D^2\varphi(x),0)| + |(F^k)^\star(D^2\varphi(x),0) - G(D^2\varphi(x),0)|.
\end{aligned}$$

Here, we deduce that

$$\begin{aligned}
&\int_{B_{7\sqrt{n}}} \left(\sup_{X\in S(n)} \frac{|F^k_{\mu_k}(X,x) - F^k_{\mu_k}(X,0)|}{\|X\|+1}\right)^n dx \\
&\le c(n)\Bigg\{ \int_{B_{7\sqrt{n}}} \left(\sup_{X\in S(n)} \frac{|F^k_{\mu_k}(X,x) - (F^k)^\star(X,x)|}{\|X\|+1}\right)^n dx \\
&\qquad + \int_{B_{7\sqrt{n}}} \left(\sup_{X\in S(n)} \frac{|(F^k)^\star(X,0) - F^k_{\mu_k}(X,0)|}{\|X\|+1}\right)^n dx \\
&\qquad + \int_{B_{7\sqrt{n}}} \left(\sup_{X\in S(n)} \frac{|(F^k)^\star(X,x) - (F^k)^\star(X,0)|}{\|X\|+1}\right)^n dx \Bigg\} \\
&\le c(n) \int_{B_{7\sqrt{n}}} \left(\sup_{X\in S(n)} \frac{|F^k_{\mu_k}(X,x) - (F^k)^\star(X,x)|}{\|X\|+1}\right)^n dx \\
&\quad + c(n)\left\{ \sup_{X\in S(n)} \frac{|(F^k)^\star(X,0) - F^k_{\mu_k}(X,0)|^n}{(\|X\|+1)^n} + \|\beta_{(F^k)^\star}(x,0)\|^n_{L^n(B_{7\sqrt{n}})} \right\}.
\end{aligned}$$

Therefore the L^n-norm of the right-hand side of (3.4) goes to 0 as $k \to \infty$. By using the similar argument in the proof of [43, Lemma 2.3], we then apply [43, Lemma 1.7] (or [10, Theorem 3.8]) to obtain that u_∞ is a C-viscosity solution of (3.3). We notice that the solution u_∞ belongs to $C^2(\overline{B}_{6\sqrt{n}})$, since $G(D^2\psi,0)$ has $C^{1,1}$ interior estimates as $(F^k)^\star(D^2\psi,0)$. Hence, from the uniqueness of solutions of (3.3) in [10, Theorem 2.10], we conclude $v_\infty = u_\infty$, which is a contradiction to (3.2). This completes the proof. □

Let U be an open bounded set in $\mathbb{R}^n$ and M be a positive constant. A concave paraboloid P with opening M is given by

$$P(x) = l_0 + l(x) - \frac{M}{2}|x|^2, \tag{3.4}$$

where M is a positive constant, l_0 is a constant and l is a linear function. If M is replaced by $-M$ in (3.4), P is called a convex paraboloid with opening M. Consider a continuous function $u\colon U \to \mathbb{R}$. We define

$$\underline{\mathcal{G}}_M(u,U) := \left\{ x_0 \in U\colon \begin{array}{c} \text{there is a concave paraboloid } P \text{ with opening } M \text{ such} \\ \text{that } P(x_0) = u(x_0) \text{ and } P(x) \le u(x) \text{ for any } x \in U \end{array} \right\}$$

and $\underline{\mathcal{A}}_M(u,U) := U \setminus \underline{\mathcal{G}}_M(u,U)$. We also define $\overline{\mathcal{G}}_M(u,U)$ and $\overline{\mathcal{A}}_M(u,U)$ in the same way as $\underline{\mathcal{G}}_M(u,U)$ and $\underline{\mathcal{A}}_M(u,U)$ respectively, by using convex paraboloids instead of concave paraboloids. We denote

$$\mathcal{G}_M(u,U) := \underline{\mathcal{G}}_M(u,U) \cap \overline{\mathcal{G}}_M(u,U) \text{ and } \mathcal{A}_M(u,U) := \underline{\mathcal{A}}_M(u,U) \cap \overline{\mathcal{A}}_M(u,U).$$

Given a function $u \in C(U)$, we define

$$\Theta(u,U)(x) := \sup\{\underline{\Theta}(u,U)(x), \overline{\Theta}(u,U)(x)\},$$

where

$$\begin{aligned} \underline{\Theta}(u,U)(x) &:= \inf\{M > 0\colon x \in \underline{\mathcal{G}}_M(u,U)\}, \\ \overline{\Theta}(u,U)(x) &:= \inf\{M > 0\colon x \in \overline{\mathcal{G}}_M(u,U)\}. \end{aligned}$$

The following lemma is the modified version of Proposition 1.1 in [9] which will be used for proving our main theorem; see [3, Lemma 3.4] for its proof and more details.

Lemma 3.2. *Assume that Φ is an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and $w \in A_{i(\Phi)}$. Let u be a continuous function in a bounded domain $U \subset \mathbb{R}^n$. For a number $r > 0$, we denote*

$$\Theta(u,r)(x) := \Theta(u, U \cap B_r(x))(x) \quad \text{for } x \in U.$$

If $\Theta(u,r) \in L^\Phi_w(U)$, then we have $D^2u \in L^\Phi_w(U)$ and

$$\|D^2 u\|_{L^\Phi_w(U)} \le 8\|\Theta(u,r)\|_{L^\Phi_w(U)}.$$

One of the main tools for proving the main result is the Hardy-Littlewood maximal function which is defined on the Lebesgue space $L^1_{loc}(\mathbb{R}^n)$ by

$$\mathcal{M}g(y) = \sup_{r>0} \fint_{B_r(y)} |g(x)|\, dx$$

at each point $y \in \mathbb{R}^n$, and for a locally integrable function $g\colon \mathbb{R}^n \to \mathbb{R}$. We will use the fundamental properties of the Hardy-Littlewood maximal function as follows:

(a) (strong $p-p$ estimate)

$$\|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \le c\|g\|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad 1 < p \le \infty,$$

where a constant $c > 0$ depends only on n and p.

(b) (weak $1-1$ estimate)

$$|\{x \in \mathbb{R}^n : \mathcal{M}g(x) \ge t\}| \le \frac{c}{t}\|g\|_{L^1(\mathbb{R}^n)} \quad \text{for all } t > 0,$$

where a constant $c > 0$ depends only on n.

We also need the boundedness of the maximal function $\mathcal{M}$ on the weighted Orlicz spaces; see [30] and [31, Theorem 2.1.1] for its proof and more details.

Lemma 3.3. *Let Φ be an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and $w \in A_{i(\Phi)}$. Then*

$$\int_{\mathbb{R}^n} \Phi(|g(x)|)w(x)\,dx \le \int_{\mathbb{R}^n} \Phi\left(\mathcal{M}g(x)\right)w(x)\,dx \le c\int_{\mathbb{R}^n} \Phi(|g(x)|)w(x)\,dx$$

holds for all $g \in L^{\Phi}_w(\mathbb{R}^n)$, where a constant $c > 0$ is independent of g .

The following Lemma comes from classical measure theory and basic properties of the N-function Φ; see [5, Lemma 4.6] for its proof and more details.

Lemma 3.4. *Let Φ be an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and suppose $w \in A_q$ for some $1 < q < \infty$. Let $\eta > 0$ and $M > 1$ be constants. Then for any nonnegative measurable function g in U, we have that*

$$g \in L^{\Phi}_w(U) \text{ if and only if } \mathcal{S} := \sum_{j\ge 1} \Phi(M^j)w\left(\{x \in U : g(x) > \eta M^j\}\right) < \infty$$

and moreover,

$$\frac{1}{c}\,\mathcal{S} \le \int_U \Phi(|g(x)|)w(x)\,dx \le c\,(w(U) + \mathcal{S}),$$

where $c > 0$ is a constant depending only on η, M, Φ and $[w]_q$.

We end this section by introducing a corollary of the Calderón-Zygmund decomposition that will be used later; see [9, Lemma 4.2] for its proof and more details.

Lemma 3.5. *Let $\epsilon \in (0,1)$, and set D and E as measurable sets with $D \subset E \subset Q_1$ such that*

(i) $|D| \le \epsilon$, and

(ii) for every dyadic cube Q such that $|D \cap Q| > \epsilon|Q|$, $\tilde{Q} \subset E$,

where $\tilde{Q}$ is the predecessor of Q. Then $|D| \le \epsilon|E|$.

Here, $\tilde{Q}$ is called the predecessor of Q if Q is one of the 2^n cubes obtained from dividing $\tilde{Q}$.

4. $W^{2,p}$ ESTIMATES

In this section, we obtain the *a priori* interior $W^{2,p}$ regularity estimates for the viscosity solutions of (1.1), slightly relaxing the conditions imposed on the operator $F^\star$ in [38, Theorem 6.1].

Theorem 4.1. *Let u be an L^p-viscosity solution of (1.1) with $n < p < \infty$. Suppose that $F^\star(X,x)$ exists and $F^\star(D^2v, x_0) = 0$ has $C^{1,1}$ interior estimates with constant $c_\star$ for any $x_0 \in B_1$. Assume further that $F(0,\cdot) \equiv 0$ in B_1 and $f \in L^p(B_1)$. Then there exists a positive constant $\delta = \delta(n,\lambda,\Lambda,p,c_\star)$ such that if*

$$\left(\fint_{B_r(x_0)} \beta_{F^\star}(x,x_0)^n\,dx\right)^{1/n} \le \delta \tag{4.1}$$

for any ball $B_r(x_0) \subset B_1$ with $r > 0$, then we have $u \in W^{2,p}(B_{\frac{1}{2}})$ with the estimate

$$\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le c\left(\|f\|_{L^p(B_1)} + \|u\|_{L^\infty(B_1)}\right) \tag{4.2}$$

for some $c = c(n,\lambda,\Lambda,p,c_\star) > 0$.

Remark 4.2. The regularity results in Theorem 4.1 can be extended to a more general equations of the form $F(D^2u, Du, u, x) = f$, under an appropriate structure condition on the nonlinearity F by using the same argument as in the proof of [12, Theorem 1.1] (also see [43, Theorem 3.1]). As mentioned before, we will deal with these results in the setting of weighted Orlicz spaces in Section 6.

The proof of [38, Theorem 6.1] was described briefly, and so we will give the proof of Theorem 4.1 in more detail. Besides, a series of lemmas appeared in the process of its proof will be needed for proving our main result in the next section.

Lemma 4.3. *Let $\epsilon \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $B_{8\sqrt{n}} \subset \Omega$. Let $u \in C(\Omega)$ be a C-viscosity solution to $F_\mu(D^2u, x) = f(x)$ in $B_{8\sqrt{n}}$ with $\|u\|_{L^\infty(B_{8\sqrt{n}})} \leq 1$ and $-|x|^2 \leq u(x) \leq |x|^2$ in $\Omega \backslash B_{6\sqrt{n}}$, assuming that F and f are continuous in all variables. Suppose that $F^\star(X, x)$ exists and $F^\star(D^2v, 0) = 0$ has $C^{1,1}$ interior estimates with constant $c_\star$. Then there exists $\delta = \delta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$ such that if*

$$\mu < \delta \text{ and } \|f\|_{L^n(B_{8\sqrt{n}})},\ \|\beta_{F^\star}(\cdot, 0)\|_{L^n(B_{7\sqrt{n}})} \leq \delta,$$

then we have that

$$|\mathcal{G}_M(u, \Omega) \cap Q_1| \geq 1 - \epsilon$$

where $M = M(n, c_\star) > 1$.

Proof. The proof is similar to that of [9, Lemma 7.10], taking into account the approximation lemma, Lemma 3.1, in place of [9, Lemma 7.9] (also see [8]). □

Lemma 4.4. *Let $\epsilon \in (0,1)$. Under the same assumptions as in Lemma 4.3, $\mathcal{G}_1(u, \Omega) \cap Q_3 \neq \emptyset$ implies*

$$|\mathcal{G}_M(u, \Omega) \cap Q_1| \geq 1 - \epsilon$$

with M and δ as in Lemma 4.3 (also see [8]).

Proof. The proof is similar to that of [9, Lemma 7.11] (also see [8]) applying the above Lemma 4.3 instead of [9, Lemma 7.10].

□

Using Lemma 3.5, we derive the following power decay estimates.

Lemma 4.5. *Let Ω be a bounded domain with $B_{8\sqrt{n}} \subset \Omega$. Consider a C-viscosity solution $u \in C(\Omega)$ of $F_\mu(D^2u, x) = f(x)$ in $B_{8\sqrt{n}}$ with $\|u\|_{L^\infty(B_{8\sqrt{n}})} \leq 1$, assuming that F and f are continuous in all variables. Suppose that $F^\star(X, x)$ exists and $F^\star(D^2v, x_0) = 0$ has $C^{1,1}$ interior estimates with constant $c_\star$ for any $x_0 \in B_{8\sqrt{n}}$. For any $\epsilon \in (0,1)$, there exist $M = M(n, c_\star) > 1$ and $\delta = \delta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$ such that if $\mu < \delta$, $\|f\|_{L^n(B_{8\sqrt{n}})} \leq \delta$ and*

$$\left(\fint_{B_r(x_0)} \beta_{F^\star}(x, x_0)^n \, dx \right)^{1/n} \leq \delta$$

for any ball $B_r(x_0) \subset B_{8\sqrt{n}}$ with $r > 0$, then extending f by zero outside $B_{8\sqrt{n}}$, for $j = 0, 1, 2, \ldots$, we have

$$\begin{aligned} &\left|\mathcal{A}_{M^{j+1}}(u, B_{8\sqrt{n}}) \cap Q_1\right| \\ &\leq \epsilon \left| \left(\mathcal{A}_{M^j}(u, B_{8\sqrt{n}}) \cap Q_1\right) \cup \left\{x \in Q_1 : \mathcal{M}(f^n)(x) \geq (\eta M^i)^n\right\} \right| \end{aligned} \tag{4.3}$$

for some constant $\eta = \eta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$. Furthermore, we have that

$$\left|\mathcal{A}_{M^j}(u, B_{8\sqrt{n}}) \cap Q_1\right| \le \epsilon^j + \sum_{i=0}^{j-1} \epsilon^{j-i} \left|\left\{x \in Q_1 \colon \mathcal{M}(f^n)(x) \ge (\eta M^i)^n\right\}\right| \tag{4.4}$$

for some constant $\eta = \eta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$.

Proof. Let $\epsilon \in (0, 1)$ be given. Here, we take δ to be a positive constant small enough so that Lemma 4.3 and Lemma 4.4 can be applied. Using the same argument as in the proof of [9, Lemma 7.12] (or see [38, Lemma 5.3]), we can obtain the desired estimates (4.3). More precisely, we will employ Lemma 3.5 with

$$\begin{aligned} D &:= \mathcal{A}_{M^{j+1}}(u, B_{8\sqrt{n}}) \cap Q_1, \\ E &:= \left(\mathcal{A}_{M^j}(u, B_{8\sqrt{n}}) \cap Q_1\right) \cup \left\{x \in Q_1 \colon \mathcal{M}(f^n)(x) \ge (\eta M^i)^n\right\} \end{aligned}$$

for $j = 0, 1, 2, \dots$. We first note that $|u(x)| \le 1 \le |x|^2$ for any $x \in B_{8\sqrt{n}} \backslash B_{6\sqrt{n}}$. Moreover, it is clear that $\|\beta_{F^\star}(\cdot, 0)\|_{L^n(B_{7\sqrt{n}})} \le c(n)\,\delta$. Then Lemma 4.3 gives that

$$\left|\mathcal{G}_{M^{j+1}}(u, B_{8\sqrt{n}}) \cap Q_1\right| \ge \left|\mathcal{G}_M(u, B_{8\sqrt{n}}) \cap Q_1\right| \ge (1-\epsilon)|Q_1|$$

which implies that $|D| = \left|\mathcal{A}_{M^{j+1}}(u, B_{8\sqrt{n}}) \cap Q_1\right| < \epsilon\,|Q_1| = \epsilon$.

In order to apply Lemma 3.5, it remains to show the condition (ii), i.e., for any dyadic cube Q of Q_1 such that

$$|D \cap Q| > \epsilon |Q|, \tag{4.5}$$

we have that $\tilde{Q} \subset E$. To do this, letting $Q = Q_{\frac{1}{2^i}}(x_0)$ for some $i \ge 0$ and $x_0 \in Q_1$, we suppose that $\tilde{Q} \not\subset E$. Then there exists $x_1 \in \tilde{Q}$ such that

$$x_1 \in \tilde{Q} \cap \mathcal{G}_{M^j}(u, B_{8\sqrt{n}}) \quad \text{and} \quad \mathcal{M}(f^n)(x_1) \le (\eta M^i)^n.$$

We define $\tilde{u}(y) := \frac{4^i}{M^j} u\left(x_0 + \frac{1}{2^i} y\right)$ and let $\tilde{\Omega}$ be the image of Ω under transformation $x = x_0 + \frac{1}{2^i} y$. Note that $B_{8\sqrt{n}} \subset \tilde{\Omega}$ from the fact that $B_{\frac{8\sqrt{n}}{2^i}}(x_0) \subset B_{8\sqrt{n}}$. Then we observe that $\tilde{u}$ is a viscosity solution of

$$\tilde{F}_\mu(D^2\tilde{u}(y), y) = \tilde{f}(y) \text{ in } B_{8\sqrt{n}}$$

where

$$\tilde{F}_\mu(X, y) := \frac{1}{M^j} F_\mu\left(X, x_0 + \frac{1}{2^i} y\right) \quad \text{and} \quad \tilde{f}(y) = \frac{1}{M^j} f\left(x_0 + \frac{1}{2^i} y\right).$$

Using the same way as in the proof of [9, Lemma 7.12] (or see [38, Lemma 5.3]), one can check that all the assumptions of Lemma 4.4 with the nonlinear operator F replaced by $\tilde{F}$ are satisfied. In particular, since $B_{\frac{7\sqrt{n}}{2^i}}(x_0) \subset B_{8\sqrt{n}}$, we infer that

$$\|\beta_{\tilde{F}^\star}(\cdot, 0)\|_{L^n(B_{7\sqrt{n}})} \le \left(2^{in} \left|B_{\frac{7\sqrt{n}}{2^i}}(x_0)\right| \fint_{B_{\frac{7\sqrt{n}}{2^i}}(x_0)} \beta_{F^\star}(x, x_0)^n \, dx\right)^{\frac{1}{n}} \le c(n)\delta.$$

Therefore Lemma 4.4 yields that

$$\left|\mathcal{G}_M(\tilde{u}, \tilde{\Omega}) \cap Q_1\right| \ge (1-\epsilon)|Q_1|,$$

which means

$$|D \cap Q| = \left|\mathcal{A}_{M^{j+1}}(u, B_{8\sqrt{n}}) \cap Q\right| < \epsilon |Q|.$$

This contradiction of (4.5) leads us to verify the condition (ii) of Lemma 3.5.

Consequently, Lemma 3.5 gives the desired estimates (4.3). Furthermore, by iterating the estimates (4.3), we ultimately obtain (4.4).

□

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. By the same approximation procedure as in the proof of [43, Theorem 3.1], it suffices to derive the interior $W^{2,p}$ estimates for C-viscosity solutions u of (1.1) provided that F and f are continuous in all variables.

For $x_0 \in B_{\frac{1}{2}}$, we set

$$K := \rho^{-n/p}\delta^{-1}\|f\|_{L^p(B_{8\rho\sqrt{n}}(x_0))} + \rho^{-2}\|u\|_{L^\infty(B_{8\rho\sqrt{n}}(x_0))},$$

where $\rho \in \left(0, \frac{1}{16\sqrt{n}}\right)$ is a small constant which will be determined later and $\delta = \delta(n, \lambda, \Lambda, c_\star, \epsilon) \in (0,1)$ is the same as in Lemma 4.5. Here, ϵ will also be determined later. Defining $v(y) = \frac{\mu}{\rho^2 K}u(\rho y + x_0)$, we observe that v is a viscosity solution to

$$G_\mu\left(D^2 v(y), y\right) = g(y) \quad \text{in } B_{8\sqrt{n}},$$

where

$$G_\mu\left(X, y\right) = \mu\, G\left(\mu^{-1}X, y\right) := \frac{\mu}{K}\, F\left(K\mu^{-1}X, \rho y + x_0\right), \; g(y) = \frac{\mu}{K}\, f\left(\rho y + x_0\right)$$

for $\mu := \frac{\delta}{2} < \delta$. Then we have that

$$\|v\|_{L^\infty(B_{8\sqrt{n}})} = \frac{\mu}{\rho^2 K}\|u\|_{L^\infty(B_{8\rho\sqrt{n}}(x_0))} < \mu < \delta \le 1,$$

and

$$\|g\|_{L^p(B_{8\sqrt{n}})} = \frac{\mu}{\rho^{n/p}K}\|f\|_{L^p(B_{8\rho\sqrt{n}}(x_0))} \le \mu\,\delta < \delta^2$$

which implies that

$$\|g\|_{L^n(B_{8\sqrt{n}})} \le c(n)\|g\|_{L^p(B_{8\sqrt{n}})} \le c(n)\delta^2$$

by using Hölder inequality since $n < p < \infty$. It is clear that $G_\mu(0,\cdot) \equiv 0$ in $B_{8\sqrt{n}}$. It is easy to check that G and G_μ have the same ellipticity constants λ and Λ as F. By the definition of G, i.e. $G(X,y) := \frac{1}{K}F(KX, \rho y + x_0)$, we see that

$$G^\star(X,y) := \lim_{\mu\to 0} G_\mu(X,y) = \frac{1}{K}F^\star(KX, \rho y + x_0),$$

which implies that $\beta_{G^\star}(y, y_0) = \beta_{F^\star}(\rho\, y + x_0, \rho\, y_0 + x_0)$. Then we have that

$$\left(\fint_{B_r(y_0)} \beta_{G^\star}(y, y_0)^n \, dy\right)^{\frac{1}{n}} = \left(\fint_{B_{\rho r}(\rho y_0 + x_0)} \beta_{F^\star}(x, \rho y_0 + x_0)^n \, dx\right)^{\frac{1}{n}} \le \delta$$

for any $B_r(y_0) \subset B_{8\sqrt{n}}$ with $r > 0$, by applying (4.1) from the fact that $B_{\rho r}(\rho y_0 + x_0) \subset B_1$. Furthermore, one can easily check that $G^\star(D^2 v, y_0) = 0$ has $C^{1,1}$ interior estimates for any $y_0 \in B_{8\sqrt{n}}$. Then all the hypotheses of Lemma 4.5 are fullfiled, and hence, Lemma 4.5 yields that

$$\left|\mathcal{A}_{M^j}(v, B_{8\sqrt{n}}) \cap Q_1\right| \le \epsilon^j + \sum_{i=0}^{j-1} \epsilon^{j-i}\left|\left\{y \in Q_1 \colon \mathcal{M}(g^n)(y) \ge (\eta M^i)^n\right\}\right| \tag{4.6}$$

for some constants $M = M(n, c_\star) > 1$ and $\eta = \eta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$. Since $g^n \in L^{p/n}(B_{8\sqrt{n}})$, it follows that $\mathcal{M}(g^n) \in L^{p/n}(B_{8\sqrt{n}})$ with

$$\|\mathcal{M}(g^n)\|_{L^{p/n}(B_{8\sqrt{n}})} \leq c\|g\|^n_{L^p(B_{8\sqrt{n}})} < c\,\delta^{2n} < c,$$

by virtue of the strong $p-p$ estimate of the maximal function. Therefore according to [9, Lemma 7.3], we obtain that

$$\sum_{j\geq 0} M^{pj}\left|\left\{y \in Q_1 \colon \mathcal{M}(g^n)(y) \geq (\eta M^j)^n\right\}\right| \leq c.$$

Then combining this estimates with (4.6), we derive that

$$\begin{aligned}
&\sum_{j\geq 1} M^{pj}\left|\mathcal{A}_{M^j}(v, B_{8\sqrt{n}}) \cap Q_1\right| \\
&\leq \sum_{j\geq 1} M^{pj}\epsilon^j + \sum_{j\geq 1} M^{pj}\sum_{i=0}^{j-1}\epsilon^{j-i}\left|\left\{y \in Q_1 \colon \mathcal{M}(g^n)(y) \geq (\eta M^i)^n\right\}\right| \\
&= \sum_{j\geq 1}(M^p\epsilon)^j + \left(\sum_{l\geq 1}(M^p\epsilon)^l\right)\left(\sum_{j\geq 0} M^{pj}\left|\left\{y \in Q_1 \colon \mathcal{M}(g^n)(y) \geq (\eta M^j)^n\right\}\right|\right) \\
&\leq \sum_{j\geq 1} 2^{-j}\,(1+c) \leq c, \qquad (4.7)
\end{aligned}$$

by taking ϵ small enough such that $M^p\epsilon \leq \frac{1}{2}$. Here, δ is also determined.

From the definition of Θ, together with (4.7), we discover that

$$\begin{aligned}
&\sum_{j\geq 1} M^{pj}\left|\{x \in B_{\frac{1}{2}} \colon \Theta(v, B_{\frac{1}{2}})(x) > M^j\}\right| \\
&\leq \sum_{j\geq 1} M^{pj}\left|\mathcal{A}_{M^j}(v, B_{\frac{1}{2}})\right| \leq \sum_{j\geq 1} M^{pj}\left|\mathcal{A}_{M^j}(v, B_{8\sqrt{n}}) \cap Q_1\right| \leq c
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, p) > 0$, which yields that

$$\|\Theta(v, B_{\frac{1}{2}})\|_{L^p(B_{\frac{1}{2}})} \leq c$$

by applying [9, Lemma 7.3]. Hence, by virtue of [9, Proposition 1.1], we conclude that

$$\|D^2 v\|_{L^p(B_{\frac{1}{2}})} \leq c.$$

Therefore we get that for any $x_0 \in B_{\frac{1}{2}}$,

$$\|D^2 u\|_{L^p(B_{\frac{\rho}{2}}(x_0))} \leq c\,\rho^{n/p}\delta^{-1}\left(\rho^{-n/p}\delta^{-1}\|f\|_{L^p(B_{8\rho\sqrt{n}}(x_0))} + \rho^{-2}\|u\|_{L^\infty(B_{8\rho\sqrt{n}}(x_0))}\right),$$

for some $c = c(n, \lambda, \Lambda, p, c_\star) > 0$, since we let $\mu = \frac{\delta}{2}$.

By the standard covering argument, we finally derive that

$$\|D^2 u\|_{L^p(B_{\frac{1}{2}})} \leq c\left(\|f\|_{L^p(B_1)} + \|u\|_{L^\infty(B_1)}\right)$$

after taking $\rho \in \left(0, \frac{1}{16\sqrt{n}}\right)$ small enough so that $B_{\frac{1}{2}}$ is covered by finite number of balls $B_{\frac{\rho}{2}}(x_0)$ with $x_0 \in B_{\frac{1}{2}}$. In turn, the desired estimates (4.2) can be obtained by using the interpolation inequality in [22, Theorem 7.28].

□

5. Proof of main theorem

In this section, we are going to prove our main result, Theorem 2.4.

Lemma 5.1. *Under the same assumptions as in Lemma 4.5, we further suppose that* $w \in A_q$ *for some* $q > 1$. *For any* $\epsilon \in (0,1)$, *there exist* $M = M(n, c_\star) > 1$ *and* $\delta = \delta(n, \lambda, \Lambda, q, c_\star, w, \epsilon) \in (0,1)$ *such that if* $\mu < \delta$, $\|f\|_{L^n(B_{8\sqrt{n}})} \leq \delta$ *and*

$$\left(\fint_{B_r(x_0)} \beta_{F^\star}(x, x_0)^n \, dx \right)^{1/n} \leq \delta$$

for any $B_r(x_0) \subset B_{8\sqrt{n}}$ *with* $r > 0$, *then extending* f *by zero outside* $B_{8\sqrt{n}}$, *for* $j = 0, 1, 2, \ldots$, *we have*

(5.1)

$$w\left(\mathcal{A}_{M^j}(u, B_{8\sqrt{n}}) \cap Q_1\right) \leq \epsilon^j w(Q_1) + \sum_{i=0}^{j-1} \epsilon^{j-i} w\left(\left\{x \in Q_1 \colon \mathcal{M}(f^n)(x) \geq (\eta M^i)^n\right\}\right)$$

for some constant $\eta = \eta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$.

Proof. Let $\epsilon \in (0,1)$ be given. We choose $\delta = \delta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$ as in Lemma 4.5 with ϵ replaced by $\left(\frac{\epsilon}{s_1}\right)^{\frac{1}{s_2}}$ where s_1, s_2 are the constants depending only on n, q and $[w]_q$ in Lemma 2.1. Then letting D and E as in the proof of Lemma 4.5, Lemma 4.5 gives that $|D| < \left(\frac{\epsilon}{s_1}\right)^{\frac{1}{s_2}} |E|$. By virtue of Lemma 2.1, we obtain that

$$\frac{w(D)}{w(E)} \leq s_1 \left(\frac{|D|}{|E|}\right)^{s_2} < s_1 \left(\frac{\epsilon}{s_1}\right)^{\frac{s_2}{s_2}} = \epsilon,$$

which implies that

$$\begin{aligned} & w\left(\mathcal{A}_{M^{j+1}}(u, B_{8\sqrt{n}}) \cap Q_1\right) \\ & \quad \leq \epsilon\, w\left(\mathcal{A}_{M^j}(u, B_{8\sqrt{n}}) \cap Q_1\right) + \epsilon w\left(\left\{x \in Q_1 \colon \mathcal{M}(f^n)(x) \geq (\eta M^i)^n\right\}\right) \end{aligned}$$

for $j = 0, 1, \ldots$. Therefore by iterating these estimates, we conclude the desired estimates (5.1). □

Now we prove the main theorem in this paper.

Proof of Theorem 2.4. In the same argument as the proof of Theorem 4.1, from the approximation procedure, it is enough to derive the desired estimates (2.6) for C-viscosity solutions u to Eq.(1.1) provided that F and f are continuous in all variables.

Given the N-function Φ, we denote $\Psi(t) = \Phi(t^n)$ for $t \in [0, \infty)$. Then it can be easily seen that Ψ becomes an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and that $i(\Psi) = n\, i(\Phi)$. We further note that $w \in A_{i(\Phi)} \subset A_{i(\Psi)}$ by virtue of the monotonicity property for the A_q weight. Fix any point $x_0 \in B_{\frac{1}{2}}$ and choose a small constant $r \in \left(0, \frac{1}{16\sqrt{n}}\right)$ that will be determined later. We define $\tilde{w}(x) := w(rx + x_0)$ and then it is clear that $\tilde{w} \in A_{i(\Phi)}$. We also define $\tilde{u}(x) := \frac{\mu}{Lr^2} u(rx + x_0)$, where

$$L := \left(r^{-n} \delta^{-n} \|f\|_{L^\Psi_w(B_{8r\sqrt{n}}(x_0))} + r^{-2n} \|u\|^n_{L^\infty(B_{8r\sqrt{n}}(x_0))} \right)^{1/n}.$$

Here, $\delta = \delta(n, \lambda, \Lambda, \Phi, w, \epsilon) \in (0,1)$ is the same as in Lemma 5.1 and ϵ will be determined later. We proceed as in the proof of Theorem 4.1. One can check that $\tilde{u}$ is a viscosity solution to

$$\tilde{F}_\mu(D^2\tilde{u}, x) = \tilde{f}(x) \text{ in } B_{8\sqrt{n}}$$

where

$$\tilde{F}_\mu(X,x) = \mu\,\tilde{F}(\mu^{-1}X, x) := \frac{\mu}{L}F(L\mu^{-1}X, rx+x_0) \text{ and } \tilde{f}(x) := \frac{\mu}{L}f(rx+x_0)$$

for $\mu := \frac{\delta}{2} < \delta$. We get that $\|\tilde{u}\|_{L^\infty(B_{8\sqrt{n}})} \le \mu < \delta \le 1$. Applying Hölder inequality, we infer that

$$\begin{aligned} \|\tilde{f}\|_{L^n(B_{8\sqrt{n}})} &= \frac{\mu}{L\,r}\left(\int_{B_{8r\sqrt{n}}(x_0)} |f|^n\,dx\right)^{\frac{1}{n}} \\ &\le \frac{c\,\mu}{L\,r}\,\|f\|^{\frac{1}{n}}_{L^\Psi_w(B_{8r\sqrt{n}}(x_0))} \le c\,\mu\,\delta < c\,\delta^2 \end{aligned}$$

for some constant $c = c(n, \Phi, w) > 0$. By the same way as in the proof of Theorem 4.1, we see that the operator $\tilde{F}$ satisfies all the hypotheses of Lemma 5.1. Therefore, Lemma 5.1 allows us to discover that

$$\tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, B_{8\sqrt{n}}) \cap Q_1\right) \le \epsilon^j\tilde{w}(Q_1) + \sum_{i=0}^{j-1}\epsilon^{j-i}\tilde{w}\left(\{x \in Q_1 \colon \mathcal{M}(\tilde{f}^n)(x) \ge (\eta M^i)^n\}\right) \tag{5.2}$$

for some constants $M = M(n, c_\star) > 1$ and $\eta = \eta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$.

By the assumption $\Phi \in \Delta_2$, there exists a positive constant $\kappa = \kappa(M^n)$ such that $\Phi(M^n t) \le \kappa\,\Phi(t)$ for all $t > 0$. By iterating this inequality, we obtain that $\Phi(M^{jn}) \le \kappa^j\Phi(1)$ for each $j \ge 1$. We also see that $\Phi(M^{jn}) \le \kappa^{j-i}\Phi(M^{in})$ for any $0 \le i \le j-1$. Therefore, from the above estimates (5.2), we infer that

$$\begin{aligned} &\sum_{j\ge1}\Phi(M^{jn})\tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, B_{8\sqrt{n}}) \cap Q_1\right) \\ &\quad\le \sum_{j\ge1}\Phi(M^{jn})\epsilon^j\left(\tilde{w}(Q_1) + \sum_{i=0}^{j-1}\epsilon^{j-i}\tilde{w}\left(\{x \in Q_1 \colon \mathcal{M}(\tilde{f}^n)(x) \ge (\eta M^i)^n\}\right)\right) \\ &\quad\le \Phi(1)\tilde{w}(Q_1)\sum_{j\ge1}(\kappa\epsilon)^j \\ &\qquad+\left(\sum_{j\ge1}(\kappa\epsilon)^j\right)\left(\sum_{i\ge0}\Phi(M^{in})\tilde{w}\left(\{x \in Q_1 \colon \mathcal{M}(\tilde{f}^n)(x) \ge (\eta M^i)^n\}\right)\right). \end{aligned} \tag{5.3}$$

On the other hand, we see that $|\tilde{f}|^n \in L^\Phi_{\tilde{w}}(B_{8\sqrt{n}})$ because $|f|^n \in L^\Phi_w(B_1)$. Then by virtue of Lemma 3.3, taking account with (2.3) and the definition of $\tilde{f}$, we have that $\mathcal{M}(\tilde{f}^n) \in L^\Phi_{\tilde{w}}(B_{8\sqrt{n}})$ and

$$\begin{aligned}
&\int_{B_{8\sqrt{n}}} \Phi\left(\mathcal{M}(\tilde{f}^n)(x)\right) \tilde{w}(x)\, dx \\
&\quad\le c \int_{B_{8\sqrt{n}}} \Phi\left(|\tilde{f}(x)|^n\right) \tilde{w}(x)\, dx = \frac{c}{r^n} \int_{B_{8r\sqrt{n}}(x_0)} \Phi\left(\left|\frac{\mu\, f(y)}{L}\right|^n\right) w(y)\, dy \\
&\quad\le \frac{c}{r^n}\left(\left\|\frac{\mu^n |f|^n}{L^n}\right\|^{\tilde{\gamma}_2}_{L^\Phi_w\left(B_{8r\sqrt{n}}(x_0)\right)} + 1\right) \le \frac{c}{r^n}\left(\mu^{n\tilde{\gamma}_2} r^{n\tilde{\gamma}_2} \delta^{n\tilde{\gamma}_2} + 1\right) \le c,
\end{aligned}$$

for some constant $\tilde{\gamma}_2 > 1$. Hence, Lemma 3.4 leads us to find that

$$\begin{aligned}
&\sum_{i\ge 0} \Phi(M^{in}) \tilde{w}\left(\{x \in Q_1 \colon \mathcal{M}(\tilde{f}^n)(x) \ge (\eta M^i)^n\}\right) \\
&\quad\le c \int_{Q_1} \Phi\left(\mathcal{M}(\tilde{f}^n)(x)\right) \tilde{w}(x)\, dx \le c \int_{B_{8\sqrt{n}}} \Phi\left(\mathcal{M}(\tilde{f}^n)(x)\right) \tilde{w}(x)\, dx \le c. \qquad (5.4)
\end{aligned}$$

In turn, inserting (5.4) into (5.3), we infer that

$$\begin{aligned}
&\sum_{j\ge 1} \Psi\left(M^j\right) \tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, B_{8\sqrt{n}}) \cap Q_1\right) = \sum_{j\ge 1} \Phi(M^{jn}) \tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, B_{8\sqrt{n}}) \cap Q_1\right) \\
&\quad\le (\Phi(1)\tilde{w}(Q_1) + c) \sum_{j\ge 1} (\kappa\epsilon)^j \le c,
\end{aligned}$$

by taking ϵ so that $\kappa\epsilon \le \frac{1}{2}$. Then taking into account the definition of Θ, it follows that

$$\begin{aligned}
&\sum_{j\ge 1} \Psi(M^j)\, \tilde{w}\left(\{x \in B_{\frac{1}{2}} \colon \Theta(\tilde{u}, B_{\frac{1}{2}})(x) > M^j\}\right) \\
&\le \sum_{j\ge 1} \Psi(M^j)\, \tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, B_{\frac{1}{2}})\right) \le \sum_{j\ge 1} \Psi(M^j)\, \tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, B_{8\sqrt{n}}) \cap Q_1\right) \le c
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, c_\star, \Psi, w) > 0$. Applying Lemma 3.4, together with (2.3), we consequently obtain that

$$\begin{aligned}
&\|\Theta(\tilde{u}, B_{\frac{1}{2}})\|_{L^\Psi_{\tilde{w}}(B_{\frac{1}{2}})} \le \int_{B_{\frac{1}{2}}} \Psi\left(\Theta(\tilde{u}, B_{\frac{1}{2}})(x)\right) \tilde{w}(x)\, dx + 1 \\
&\quad\le c\left(\tilde{w}(B_{\frac{1}{2}}) + \sum_{j\ge 1} \Psi(M^j)\, \tilde{w}\left(\{x \in B_{\frac{1}{2}} \colon \Theta(\tilde{u}, B_{\frac{1}{2}})(x) > M^j\}\right)\right) + 1 \le c
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, c_\star, \Psi, w) > 0$. Therefore, thanks to Lemma 3.2, we get $\|D^2\tilde{u}\|_{L^\Psi_{\tilde{w}}(B_{\frac{1}{2}})} \le 8\|\Theta(\tilde{u}, B_{\frac{1}{2}})\|_{L^\Psi_{\tilde{w}}(B_{\frac{1}{2}})} \le c$, which implies

$$\|D^2 u\|_{L^\Psi_w(B_{\frac{r}{2}}(x_0))} \le c\, r^n \delta^{-n}\left(r^{-n}\delta^{-n}\|f\|_{L^\Psi_{\tilde{w}}(B_{8r\sqrt{n}}(x_0))} + r^{-2n}\|u\|^n_{L^\infty(B_{8r\sqrt{n}}(x_0))}\right).$$

Accordingly, we have

$$\|D^2 u\|_{L^\Psi_{\tilde{w}}(B_{\frac{1}{2}})} \le c\left(\|f\|_{L^\Psi_{\tilde{w}}(B_1)} + \|u\|^n_{L^\infty(B_1)}\right)$$

from the standard covering argument, by choosing r sufficiently small so that $B_{\frac{1}{2}}$ is covered by finite number of balls $B_r(x_0)$ for $x_0 \in B_{\frac{1}{2}}$. The desired estimate (2.6) is thus obtained by applying the interpolation inequality for weighted Orlicz spaces in [29, Theorem 4.3-4.4 or Example 6.1]. □

6. Regularity for fully nonlinear equations with measurable terms

In this section, we further extend regularity result in Theorem 2.4 to fully nonlinear elliptic equations of the form

$$F(D^2u, Du, u, x) = f(x) \quad \text{in } B_1, \tag{6.1}$$

provided that the nonlinear operator F is uniformly elliptic and Lipschitz with respect to Du and u, by the same argument as in the proof of [12, Theorem 1.1] (see also [43, Theorem 3.1]).

Let us consider the nonlinear operator $F = F(X, z, s, x)$ which is a real valued Carathéodory function defined on $S(n) \times \mathbb{R}^n \times \mathbb{R} \times B_1$. We assume that the operator F satisfies that there exist constants $\Lambda \geq \lambda > 0,\ \kappa, \nu \geq 0$ such that

$$\begin{aligned} &\mathcal{P}^-_{\lambda,\Lambda}(X - Y) - \kappa|z - \tilde{z}| - \nu|s - \tilde{s}| \\ &\quad \leq F(X, z, s, x) - F(Y, \tilde{z}, \tilde{s}, x) \\ &\qquad \leq \mathcal{P}^+_{\lambda,\Lambda}(X - Y) + \kappa|z - \tilde{z}| + \nu|s - \tilde{s}|, \end{aligned} \tag{6.2}$$

for all $X, Y \in S(n), Y \geq 0, z \in \mathbb{R}^n, s \in \mathbb{R}$ and almost all $x \in B_1$, where $\mathcal{P}^+_{\lambda,\Lambda}, \mathcal{P}^-_{\lambda,\Lambda}$ are the *Pucci extremal operators* defined by

$$\mathcal{P}^-_{\lambda,\Lambda}(X) := \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i \quad \text{and} \quad \mathcal{P}^+_{\lambda,\Lambda}(X) := \lambda \sum_{\mu_i < 0} \mu_i + \Lambda \sum_{\mu_i > 0} \mu_i$$

for the eigenvalues μ_i of X. Note that the above condition (6.2) implies the uniform ellipticity of F. We further suppose that $F(0, 0, 0, \cdot) \equiv 0$ in B_1. Similar to (2.5), let the oscillation of $F^\star(X, z, s, x)$ in x be measured by

$$\tilde{\beta}_{F^\star}(x, x_0) := \sup_{X \in S(n) \setminus \{0\}} \frac{|F^\star(X, 0, 0, x) - F^\star(X, 0, 0, x_0)|}{\|X\|}.$$

The definition of viscosity solutions to (6.1) that we are treating in this section is following:

Definition 6.1. Let $q > \frac{n}{2}$ and $f \in L^q_{\mathrm{loc}}(B_1)$. A function $u \in C(B_1)$ is called *an L^q-viscosity solution* of (6.1) if the following two conditions hold:

(i) For all $\varphi \in W^{2,q}_{\mathrm{loc}}(B_1)$ whenever $\epsilon > 0$, $\mathcal{O} \subset B_1$ is open and

$$F\left(D^2\varphi(x), D\varphi(x), u(x), x\right) \leq f(x) - \epsilon \quad \text{a.e. in } \mathcal{O},$$

$u - \varphi$ cannot attain a local maximum in $\mathcal{O}$.

(ii) For all $\varphi \in W^{2,q}_{\mathrm{loc}}(B_1)$ whenever $\epsilon > 0$, $\mathcal{O} \subset B_1$ is open and

$$F\left(D^2\varphi(x), D\varphi(x), u(x), x\right) \geq f(x) + \epsilon \quad \text{a.e. in } \mathcal{O},$$

$u - \varphi$ cannot attain a local minimum in $\mathcal{O}$.

We refer to [10, 14] for a detailed account for the theory of the viscosity solutions.

Under the above structure condition (6.2) on the nonlinearity F, we have the following result.

Theorem 6.2. *Assume that Φ is an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and $w \in A_{i(\Phi)}$. Let u be an L^γ-viscosity solution to (6.1), where γ is defined in Remark 2.3. Suppose that $F^\star$ exists and $F^\star(D^2v, 0, 0, x_0) = 0$ has $C^{1,1}$ interior estimates with constant $\tilde{c}_\star$ for any $x_0 \in B_1$. Assume that $f \in L^\Psi_w(B_1)$ with $\Psi(t) := \Phi(t^n)$. Then there exists a positive constant $\delta = \delta(n, \lambda, \Lambda, \Phi, w, \tilde{c}_\star)$ such that if*

$$\left(\fint_{B_r(x_0)} \tilde{\beta}_{F^\star}(x, x_0)^n \, dx \right)^{1/n} \leq \delta \tag{6.3}$$

holds for any ball $B_r(x_0) \subset B_1$ with $r > 0$, then we have $u \in W^{2,\Psi}_w(B_{\frac{1}{2}})$ with the estimate

$$\|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} \leq c \left(\|f\|_{L^\Psi_w(B_1)} + \|u\|^n_{L^\infty(B_1)} \right) \tag{6.4}$$

for some $c = c(n, \lambda, \Lambda, \kappa, \nu, \Phi, w, \tilde{c}_\star) > 0$.

To prove Theorem 6.2, we need the following gradient estimates.

Lemma 6.3. *Under the same assumptions of Theorem 6.2, there exists a small $\delta = \delta(n, \lambda, \Lambda, \Phi, w) > 0$ such that if (6.3) holds for any ball $B_r(x_0) \subset B_1$ with $r > 0$, we have that $Du \in L^\Psi_w(B_{\frac{1}{2}})$ and*

$$\|Du\|_{L^\Psi_w(B_{\frac{1}{2}})} \leq c \left(\|f\|_{L^\Psi_w(B_1)} + \|u\|^n_{L^\infty(B_1)} \right)$$

for some constant $c = c(n, \lambda, \Lambda, \kappa, \nu, \Phi, w, \tilde{c}_\star) > 0$.

We omit the proof of Lemma 6.3 since it is almost literally the same as in the proof of [3, Lemma 3.2] when the approximation lemma, Lemma 3.1 is taken into account.

Proof of Theorem 6.2. let us set

$$g(x) := F(D^2u, 0, 0, x) - F(D^2u, Du, u, x) + f(x) = F(D^2u, 0, 0, x).$$

Then from the structure condition (6.2) of F, it follows that $|g(x)| \leq \kappa |Du(x)| + \nu |u(x)| + |f(x)|$ for a.e. $x \in B_1$. Due to Lemma 6.3, we note that g belongs to the space $L^\Psi_w(B_1)$ locally. Then for any L^γ-viscosity solution u of

$$F(D^2u, 0, 0, x) = g \text{ in } B_1,$$

we apply Theorem 2.4 with g instead of f to discover that

$$\begin{aligned} \|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} &\leq c \left(\|g\|_{L^\Psi_w(B_{\frac{2}{3}})} + \|u\|_{L^\infty(B_{\frac{2}{3}})} \right) \\ &\leq c \left(\|f\|_{L^\Psi_w(B_{\frac{2}{3}})} + \|Du\|_{L^\Psi_w(B_{\frac{2}{3}})} + \|u\|_{L^\infty(B_{\frac{2}{3}})} \right). \end{aligned}$$

In turn, from Lemma 6.3, we obtain the desired estimates (6.4). □

7. Regularity for asymptotically elliptic equations

In this section, we derive weighted Orlicz estimates for fully nonlinear, asymptotically elliptic equations as an outgrowth of our main result, Theorem 2.4. In order to descirbe our results, let us recall the definition of asymptotically elliptic operators.

Definition 7.1. The operator $F = F(X,x) : S(n) \times U \to \mathbb{R}$ is asymptotically elliptic if there exist a uniformly elliptic operator $\bar{F} = \bar{F}(X,x)$ and a bounded function $\theta : \mathbb{R}^+ \to \mathbb{R}^n$ with $\lim_{r\to\infty} \theta(r) = 0$ such that

$$0 \le F(X,x) - \bar{F}(X,x) \le \theta(\|X\|)(1 + \|X\|) \tag{7.1}$$

for all $X \in S(n)$ and all $x \in U$.

The following is our result of this section.

Theorem 7.2. *Assume that Φ is an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and $w \in A_{i(\Phi)}$. Let u be an L^γ-viscosity solution to (1.1), where γ is defined in Remark 2.3. Suppose that $F(\cdot,0) \equiv 0$, and $F(X,x)$ is asymptotically elliptic with $\bar{F}$ satisfying that its recession function $\bar{F}^\star(X,x)$ exists and has uniform convexity with respect to X. Assume further that $f \in L^\Psi_w(B_1)$ with $\Psi(t) := \Phi(t^n)$. Then there exists a positive constant $\delta = \delta(n,\lambda,\Lambda,\Phi,w)$ such that if*

$$\left(\fint_{B_r(x_0)} \beta_{\bar{F}^\star}(x,x_0)^n \, dx \right)^{1/n} \le \delta \tag{7.2}$$

for any ball $B_r(x_0) \subset B_1$ with $r > 0$, then we have $u \in W^{2,\Psi}_w(B_{\frac{1}{2}})$ with the estimate

$$\|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} \le c \left(\|f\|_{L^\Psi_w(B_1)} + \|u\|^n_{L^\infty(B_1)} + 1 \right) \tag{7.3}$$

for some $c = c(n,\lambda,\Lambda,\Phi,w,\theta) > 0$.

Proof. The proof is similar to that of Theorem 2.7 in [7]. Let $F(X,x)$ be asymptotically elliptic with $\bar{F}$ satisfying that its recession function $\bar{F}^\star(X,x)$ exists and has uniform convexity with respect to X and satisfy (7.2) with $\delta = \delta_0$ which will be determined later. We now define $H(X,x)$ by

$$H(X,x) := \frac{F(X,x) - \bar{F}(X,x)}{\|X\|}$$

for $X \in S(n)\backslash\{0\}$. Since $F(X,x)$ is asymptotically elliptic with $\bar{F}$, (7.1) implies that there exists $N = N(\delta_0) > 0$ such that if $\|X\| \ge N$, then $|H(X,x)| \le \delta_0$ for any $x \in B_1$. Then we define $\mathcal{H}(X,x)$ by

$$\mathcal{H}(X,x) := \begin{cases} 0 & \text{if } X = 0, \\ \frac{\|X\|}{N} H\left(\frac{N}{\|X\|}X, x\right) & \text{if } 0 < \|X\| < N, \\ H(X,x) & \text{if } \|X\| \ge N. \end{cases}$$

It is clear that $\mathcal{H} = \mathcal{H}(X,x)$ is a Carathéodory function defined on $S(n) \times B_1$, and $|\mathcal{H}(X,x)| \le \delta_0$, uniformly with respect to $x \in B_1$ for any $X \in S(n)$. From the definitions of H and $\mathcal{H}$, we infer that

$$\begin{aligned} F(X,x) &= \bar{F}(X,x) + H(X,x)\|X\| \\ &= \bar{F}(X,x) + \mathcal{H}(X,x)\|X\| + \left(H(X,x) - \mathcal{H}(X,x)\right)\|X\|\,\chi_{\{X\in S(n):\|X\|<N\}} \end{aligned} \tag{7.4}$$

for any $X \in S(n)\backslash\{0\}$, where $\chi_{\{X\in S(n):\|X\|<N\}}$ is the characteristic function of the set $\{X \in S(n) : \|X\| < N\}$. If we let $H(X,x)\|X\| = F(0,x) - \bar{F}(0,x)$ when $X \equiv 0$, then (7.4) is satisfied for any $X \in S(n)$.

Let us consider an L^γ-viscosity solution u of (1.1). We denote

$$\mathcal{F}(X,x) := \bar{F}(X,x) + \mathcal{H}(D^2u,x)\|X\|$$

and then from (7.4), it is seen that

$$F(D^2u,x)=\mathcal{F}(D^2u,x)+\big(H(D^2u,x)-\mathcal{H}(D^2u,x)\big)\,\|D^2u\|\chi_{\{x\in B_1:\|D^2u\|<N\}}.$$

In turn, we observe that u is also an L^γ-viscosity solution of

$$\mathcal{F}(D^2u,x)=\tilde{f}(x) \quad \text{in } B_1, \tag{7.5}$$

where $\tilde{f}(x):=f(x)-\big(H(D^2u,x)-\mathcal{H}(D^2u,x)\big)\,\|D^2u\|\chi_{\{x\in B_1:\|D^2u\|<N\}}$.

Take $\delta_0=\min\{\frac{\delta}{3},\frac{\lambda}{2}\}$, where $\delta=\delta(n,\lambda,\Lambda,\Phi,w,c_\star)$ is the same as in Theorem 2.4. In order to apply Theorem 2.4 with Eq.(7.5), we need to check the following :

(a) the operator $\mathcal{F}$ is uniformly elliptic,
(b) the recession function $\mathcal{F}^\star$ associated with $\mathcal{F}$ exists and $\mathcal{F}^\star(D^2v,x_0)=0$ has $C^{1,1}$ interior estimates for any $x_0\in B_1$,
(c) the operator $\mathcal{F}$ satisfies that

$$\left(\fint_{B_r(x_0)}\beta_{\mathcal{F}^\star}(x,x_0)^n\,dx\right)^{1/n}\le 3\,\delta_0 \quad \text{for any ball } B_r(x_0)\subset B_1 \text{ with } r>0.$$

To show (a), we deduce that

$$\begin{aligned}&\mathcal{F}(X+Y,x)-\mathcal{F}(X,x)\\&\quad=\bar{F}(X+Y,x)-\bar{F}(X,x)+\mathcal{H}(D^2u,x)\,(\|X+Y\|-\|X\|)\\&\quad\ge\lambda\|Y\|-\delta_0\,|\|X+Y\|-\|X\||\ge(\lambda-\delta_0)\|Y\|\ge\frac{\lambda}{2}\|Y\|,\end{aligned}$$

and moreover,

$$\begin{aligned}&\mathcal{F}(X+Y,x)-\mathcal{F}(X,x)\\&\quad\le\Lambda\|Y\|+\delta_0\,|\|X+Y\|-\|X\||\le(\Lambda+\delta_0)\|Y\|\le\left(\Lambda+\frac{\lambda}{2}\right)\|Y\|.\end{aligned}$$

Consequently, we obtain that

$$\frac{\lambda}{2}\|Y\|\le\mathcal{F}(X+Y,x)-\mathcal{F}(X,x)\le\left(\Lambda+\frac{\lambda}{2}\right)\|Y\|,$$

which means that the operator $\mathcal{F}$ is uniformly elliptic with ellipticity constants $\frac{\lambda}{2}$ and $\Lambda+\frac{\lambda}{2}$.

On the other hand, we note that

$$\begin{aligned}\mathcal{F}_\mu(X,x)&=\mu\,\mathcal{F}(\mu^{-1}X,x)=\mu\,\bar{F}(\mu^{-1}X,x)+\mu\,\mathcal{H}(D^2u,x)\|\mu^{-1}X\|\\&=\bar{F}_\mu(X,x)+\mathcal{H}(D^2u,x)\|X\|\end{aligned}$$

for any $\mu>0$, and then it follows from the definition of the recession operator that

$$\mathcal{F}^\star(X,x)=\bar{F}^\star(X,x)+\mathcal{H}(D^2u,x)\|X\|.$$

This implies that the existence of $\bar{F}^\star$ guarantees the one of $\mathcal{F}^\star$. Furthermore, from the definition of H and (7.1), it is clear that $\mathcal{H}(D^2u,x)\ge 0$ for all $x\in B_1$ and so we see that $\mathcal{H}(D^2u,x)\|X\|$ is convex with respect to X. Then by the uniform convexity assumption on $\bar{F}^\star(X,x)$ with respect to X, $\mathcal{F}^\star(X,x)$ is also uniformly convex with respect to X. According to the Evans-Krylov $C^{2,\alpha}$ regularity theorem (cf. [18, 33] or Chapter 6 in [8]), it turns out that $\mathcal{F}^\star(D^2v, x_0) = 0$ has $C^{1,1}$ interior estimates for any $x_0 \in B_1$. Moreover, we have that

$$\begin{aligned}
\beta_{\mathcal{F}^\star}(x, x_0) &= \sup_{X\in S(n)\setminus\{0\}} \frac{|\mathcal{F}^\star(X,x) - \mathcal{F}^\star(X,x_0)|}{\|X\|} \\
&\le \sup_{X\in S(n)\setminus\{0\}} \frac{|\bar{F}^\star(X,x) - \bar{F}^\star(X,x_0)|}{\|X\|} + \left|\mathcal{H}(D^2u, x) - \mathcal{H}(D^2u, x_0)\right| \\
&\le \beta_{\bar{F}^\star}(x, x_0) + 2\delta_0.
\end{aligned}$$

Then it follows that

$$\left(\fint_{B_r(x_0)} \beta_{\mathcal{F}^\star}(x,x_0)^n \, dx\right)^{1/n} \le \left(\fint_{B_r(x_0)} \beta_{\bar{F}^\star}(x,x_0)^n \, dx\right)^{1/n} + 2\delta_0 \le 3\delta_0$$

and hence, (c) is satisfied.

Now, we claim that if $f \in L^\Psi_w(B_1)$, then $\tilde{f} \in L^\Psi_w(B_1)$ with the estimate

$$\|\tilde{f}\|_{L^\Psi_w(B_1)} \le c\left(\|f\|_{L^\Psi_w(B_1)} + 1\right)$$

for some constant $c = c(n, \theta, \Psi, w, \delta_0) > 0$. Indeed, from the definition of H and (7.1), it is easy to see that

$$\left|H(D^2u, x)\right| \|D^2u\| \chi_{\{\|D^2u\|<N\}} \le \|\theta\|_\infty \left(1 + \|D^2u\|\right) \chi_{\{\|D^2u\|<N\}} \le 2N\|\theta\|_\infty,$$

and then we infer that

$$\begin{aligned}
|\tilde{f}(x)| &\le |f(x)| + \left|H(D^2u, x) - \mathcal{H}(D^2u, x)\right| \|D^2u\| \chi_{\{\|D^2u\|<N\}} \\
&\le |f(x)| + \left|H(D^2u, x)\right| \|D^2u\| \chi_{\{\|D^2u\|<N\}} + \delta_0 N \\
&\le |f(x)| + 2N\|\theta\|_\infty + \delta_0 N \le |f(x)| + (2\|\theta\|_\infty + \delta_0)N,
\end{aligned}$$

which implies that the claim holds.

Accordingly, all the hypotheses in Theorem 2.4 with Eq.(7.5) are fulfilled and hence, Theorem 2.4 provides that $u \in W^{2,\Psi}_w(B_{\frac{1}{2}})$ with

$$\|u\|_{W^{2,\Psi}_w(B_{\frac{1}{2}})} \le c\left(\|\tilde{f}\|_{L^\Psi_w(B_1)} + \|\mathcal{F}(0,\cdot)\|_{L^\Psi_w(B_1)} + \|u\|^n_{L^\infty(B_1)}\right)$$

for some $c = c(n, \lambda, \Lambda, \Phi, w, c_\star) > 0$. We notice that $\mathcal{F}(0,\cdot) \equiv \bar{F}(0,\cdot)$. Moreover, (7.1) implies that

$$0 \le F(0,x) - \bar{F}(0,x) \le \theta(0).$$

Then it follows that

$$\|\mathcal{F}(0,\cdot)\|_{L^\Psi_w(B_1)} = \|\bar{F}(0,\cdot)\|_{L^\Psi_w(B_1)} \le c\left(\|F(0,\cdot)\|_{L^\Psi_w(B_1)} + 1\right) \le c$$

for some constant $c = c(n, \Psi, w, \theta) > 0$. Hence, the desired estimates (7.3) are obtained.

□

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