Cellular Automata and Powers of p/q *

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Abstract

We consider one-dimensional cellular automata $F_{p,q}$ which multiply numbers by p/q in base pq for relatively prime integers p and q. By studying the structure of traces with respect to $F_{p,q}$ we show that for $p \geq 2q-1$ (and then as a simple corollary for $p > q > 1$) there are arbitrarily small finite unions of intervals which contain the fractional parts of the sequence $\xi(p/q)^n$, $(n = 0, 1, 2, \dots)$ for some $\xi > 0$. To the other direction, by studying the measure theoretical properties of $F_{p,q}$, we show that for $p > q > 1$ there are finite unions of intervals approximating the unit interval arbitrarily well which don't contain the fractional parts of the whole sequence $\xi(p/q)^n$ for any $\xi > 0$.

Keywords: distribution modulo 1, Z-numbers, cellular automata, ergodicity, strongly mixing

Introduction

In [11] Weyl proved that for any $\alpha > 1$ the sequence of numbers $\{\xi\alpha^i\}$, $i \in \mathbb{N}$ is uniformly distributed in the interval $[0, 1)$ for almost every choice of $\xi > 0$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. In particular, $\{\{\xi\alpha^i\} \mid i \in \mathbb{N}\}$ is dense in $[0, 1)$ for almost every $\xi > 0$. However, this doesn't hold for every $\xi > 0$, and it would be interesting to know what other types of distribution the set $\{\{\xi\alpha^i\} \mid i \in \mathbb{N}\}$ can exhibit for different choices of ξ.

As a special case of this problem, in [8] Mahler posed the question of whether there exist so called Z-numbers, i.e. real numbers $\xi > 0$ such that

$$\left\{\xi\left(\frac{3}{2}\right)^i\right\} \in [0, 1/2)$$

for every $i \in \mathbb{N}$. We will work with the following generalization of the notion of Z-numbers: let $p > q > 1$ be relatively prime integers and let $S \subseteq [0, 1)$ be a

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finite union of intervals. Then if we denote by $Z_{p/q}(S)$ the set of real numbers $\xi > 0$ such that

$$\left\{ \xi \left(\frac{p}{q} \right)^i \right\} \in S$$

for every $i \in \mathbb{N}$, Z-numbers are the elements of the set $Z_{3/2}([0, 1/2))$ and Mahler's question can be reformulated as whether $Z_{3/2}([0, 1/2)) = \emptyset$ or not.

A natural approach to the emptiness problem of $Z_{3/2}([0, 1/2))$ is to seek sets S as small as possible such that $Z_{p/q}(S) \neq \emptyset$ and sets S as large as possible such that $Z_{p/q}(S) = \emptyset$ (for previous results, see e.g. [1–4]). In this paper we prove that for $p \geq 2q - 1$ and $k > 0$ there exists a union of q^{2k} intervals $I_{p,q,k}$ of total length at most $(q/p)^k$ such that $Z_{p/q}(I_{p,q,k})$ is non-empty. From this it follows as a simple corollary that for $p > q$ and $\epsilon > 0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon})$ is non-empty. On the other hand, for $p > q$ and $\epsilon > 0$ we prove that there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1 - \epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon})$ is empty. The proofs of emptiness and non-emptiness are based on the study of the cellular automaton $F_{p,q}$ that implements multiplication by p/q in base pq. This cellular automaton was introduced in [7] in relation with the problem of universal pattern generation and the connection to Mahler's problem was pointed out in [6].

1 Preliminaries

For a finite set A (an *alphabet*) the set $A^{\mathbb{Z}}$ is called a *configuration space* and its elements are called *configurations*. An element $c \in A^{\mathbb{Z}}$ is a bi-infinite sequence and the element at position i in the sequence is denoted by $c(i)$. A *factor* of c is any finite sequence $c(i)c(i-1)\dots c(j)$ where $i, j \in \mathbb{Z}$, and we interpret the sequence to be empty if $j < i$. Any finite sequence $a(1)a(2)\dots a(n)$ (also the empty sequence, which is denoted by λ) where $a(i) \in A$ is a *word* over A. The set of all words over A is denoted by A^*, and the set of non-empty words is $A^+ = A^* \setminus \{\lambda\}$. The set of words of length n is denoted by A^n. For a word $w \in A^*$, $|w|$ denotes its length, i.e. $|w| = n \iff w \in A^n$.

Definition 1.1. Any $w \in A^+$ and $i \in \mathbb{Z}$ determine a *cylinder*

$$\mathrm{Cyl}_A(w, i) = \{c \in A^{\mathbb{Z}} \mid c(i)c(i+1)\dots c(i+|w|-1) = w\}.$$

The collection of all cylinders over A is

$$\mathcal{C}_A = \{\mathrm{Cyl}_A(w, i) \mid w \in A^+, i \in \mathbb{Z}\}.$$

The subscript A is omitted when the used alphabet is clear from the context.

The configuration space $A^{\mathbb{Z}}$ becomes a topological space when endowed with the topology $\mathcal{T}$ generated by $\mathcal{C}$. It can be shown that this topology is metrizable, and that a set $S \subseteq A^{\mathbb{Z}}$ is compact if and only if it is closed. $A^{\mathbb{Z}}$ can also be endowed with measure theoretical structure: it is known that there is a unique probability space $(A^{\mathbb{Z}}, \Sigma(\mathcal{C}), \mu)$, where $\Sigma(\mathcal{C})$ is the sigma-algebra generated by $\mathcal{C}$ and $\mu : \Sigma(\mathcal{C}) \to \mathbb{R}$ is a measure such that $\mu(\mathrm{Cyl}(w, i)) = |A|^{-|w|}$ for every $\mathrm{Cyl}(w, i) \in \mathcal{C}$. Note that $\mathcal{T} \subseteq \Sigma(\mathcal{C})$ because $\mathcal{C}$ is a countable basis of $\mathcal{T}$, so $\Sigma(\mathcal{C})$ is actually the collection of Borel sets of $A^{\mathbb{Z}}$.

Definition 1.2. A one-dimensional cellular automaton (CA) is a 3-tuple (A, N, f), where A is a finite *state set*, $N = (n_1, \ldots, n_m) \in \mathbb{Z}^m$ is a *neighborhood vector* and $f : A^m \to A$ is a *local rule*. A given CA (A, N, f) is customarily identified with a corresponding *CA function* $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by

$$F(c)(i) = f(c(i + n_1), \ldots, c(i + n_m))$$

for every $c \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

To every configuration space $A^{\mathbb{Z}}$ is associated a *(left) shift* CA $(A, (1), \iota)$, where $\iota : A \to A$ is the identity function. Put in terms of the CA-function determined by this 3-tuple, the left shift is $\sigma_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma_A(c)(i) = c(i + 1)$ for every $c \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

For a given CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ and a configuration $c \in A^{\mathbb{Z}}$ it is often helpful to consider a *space-time diagram* of c with respect to F. A space-time diagram is a picture which depicts elements of the sequence $(F^i(c))_{i\in\mathbb{N}}$ (or possibly $(F^i(c))_{i\in\mathbb{Z}}$ in the case when F is reversible) in such a way that $F^{i+1}(c)$ is drawn below $F^i(c)$ for every i. As an example, Figure 1 contains a space-time diagram of $c = \ldots 01101001 \ldots$ with respect to the left shift on $A = \{0, 1\}$.

All CA-functions are continuous with respect to $\mathcal{T}$ and commute with the shift.

c	$\cdots$	0	1	1	0	1	0	0	1	$\cdots$
$\sigma_A(c)$	$\cdots$	1	1	0	1	0	0	1		$\cdots$
$\sigma_A^2(c)$	$\cdots$	1	0	1	0	0	1			$\cdots$

Figure 1: An example of a space-time diagram.

2 The cellular automata $G_{p,q}$ and $F_{p,q}$

In this section we define auxiliary CA $G_{p,q}$ for relatively prime $p, q \geq 2$ and show that they multiply numbers by p in base pq. Then we use $G_{p,q}$ in constructing the CA $F_{p,q}$ which multiply numbers by p/q in base pq, and cover some basic properties of $F_{p,q}$.

Let us denote by A_n the set of digits $\{0, 1, 2, \ldots, n-1\}$ for $n \in \mathbb{N}$, $n > 1$. To perform multiplication using a CA we need be able to represent a nonnegative real number as a configuration in $A_n^{\mathbb{Z}}$. If $\xi \geq 0$ is a real number and $\xi = \sum_{i=-\infty}^{\infty} \xi_i n^i$ is the unique base n expansion of ξ such that $\xi_i \neq n - 1$ for infinitely many $i < 0$, we define $\text{config}_n(\xi) \in A_n^{\mathbb{Z}}$ by

$$\text{config}_n(\xi)(i) = \xi_{-i}$$

for all $i \in \mathbb{Z}$. In reverse, whenever $c \in A_n^{\mathbb{Z}}$ is such that $c(i) = 0$ for all sufficiently small i, we define

$$\text{real}_n(c) = \sum_{i=-\infty}^{\infty} c(-i) n^i.$$

For words $w = w(1)w(2)\dots w(k) \in A_n^k$ we define analogously

$$\mathrm{real}_n(w) = \sum_{i=1}^{k} w(i)n^{-i}.$$

Clearly $\mathrm{real}_n(\mathrm{config}_n(\xi)) = \xi$ and $\mathrm{config}_n(\mathrm{real}_n(c)) = c$ for every $\xi \geq 0$ and every $c \in A_n^{\mathbb{Z}}$ such that $c(i) = 0$ for all sufficiently small i and $c(i) \neq n-1$ for infinitely many $i > 0$.

For relatively prime integers $p, q \geq 2$ let $g_{p,q} : A_{pq} \times A_{pq} \to A_{pq}$ be defined as follows. Digits $x, y \in A_{pq}$ are represented as $x = x_1 q + x_0$ and $y = y_1 q + y_0$, where $x_0, y_0 \in A_q$ and $x_1, y_1 \in A_p$: such representations always exist and they are unique. Then

$$g_{p,q}(x, y) = g_{p,q}(x_1 q + x_0, y_1 q + y_0) = x_0 p + y_1.$$

An example in the particular case $(p, q) = (3, 2)$ is given in Figure 2.

$x \backslash y$	0	1	2	3	4	5
0	0	0	1	1	2	2
1	3	3	4	4	5	5
2	0	0	1	1	2	2
3	3	3	4	4	5	5
4	0	0	1	1	2	2
5	3	3	4	4	5	5

Figure 2: The values of $g_{p,q}(x, y)$ in the case $(p, q) = (3, 2)$.

The CA function $G_{p,q} : A_{pq}^{\mathbb{Z}} \to A_{pq}^{\mathbb{Z}}$, $G_{p,q}(c)(i) = g_{p,q}(c(i), c(i+1))$ determined by $(A_{pq}, (0, 1), g_{p,q})$ implements multiplication by p in base pq in the sense of the following lemma.

Lemma 2.1. $\mathrm{real}_{pq}(G_{p,q}(\mathrm{config}_{pq}(\xi))) = p\xi$ for all $\xi \geq 0$.

Proof. Let $c = \mathrm{config}_{pq}(\xi)$. For every $i \in \mathbb{Z}$, denote by $c(i)_0$ and $c(i)_1$ the natural numbers such that $0 \leq c(i)_0 < q$, $0 \leq c(i)_1 < p$ and $c(i) = c(i)_1 q + c(i)_0$. Then

$$\begin{aligned}
&\mathrm{real}_{pq}(G_{p,q}(\mathrm{config}_{pq}(\xi))) = \mathrm{real}_{pq}(G_{p,q}(c)) = \sum_{i=-\infty}^{\infty} G_{p,q}(c)(-i)(pq)^i \\
=& \sum_{i=-\infty}^{\infty} g_{p,q}(c(-i), c(-i+1))(pq)^i = \sum_{i=-\infty}^{\infty} (c(-i)_0 p + c(-i+1)_1)(pq)^i \\
=& \sum_{i=-\infty}^{\infty} (c(-i)_0 p(pq)^i + c(-i+1)_1 pq(pq)^{i-1}) \\
=& \sum_{i=-\infty}^{\infty} (c(-i)_0 p(pq)^i + c(-i)_1 pq(pq)^i) \\
=& p \sum_{i=-\infty}^{\infty} (c(-i)_1 q + c(-i)_0)(pq)^i = p\,\mathrm{real}_{pq}(c) = p\,\mathrm{real}_{pq}(\mathrm{config}_{pq}(\xi)) = p\xi.
\end{aligned}$$

□

w	3	4	3	4	2	0	5
$F_{3,2}(w)$		3	5	3	3	1	
$F_{3,2}^2(w)$			5	2	1		
$F_{3,2}^3(w)$				0			

Figure 3: Iterated application of $F_{p,q}$ on w for $(p,q) = (3,2)$ and $w = 3434205$.

We also define $G_{p,q}(w)$ for words $w = w(1)w(2)\dots w(|w|)$ such that $|w| \geq 2$:

$$G_{p,q}(w) = u = u(1)\dots u(|w|-1) \in A_{pq}^{|w|-1},$$

where $u(i) = g_{p,q}(w(i), w(i+1))$ for $1 \leq i \leq |w|-1$. Inductively it is possible to define $G_{p,q}^t(w)$ for every $t > 0$ and word w such that $|w| \geq t+1$:

$$G_{p,q}^t(w) = G_{p,q}(G_{p,q}^{t-1}(w)) \in A_{pq}^{|w|-t}.$$

Clearly the shift CA $\sigma_{A_{pq}}$ multiplies by pq in base pq and its inverse divides by pq. This combined with Lemma 2.1 shows that the composition $F_{p,q} = \sigma_{A_{pq}}^{-1} \circ G_{p,q} \circ G_{p,q}$ implements multiplication by p/q in base pq. The value of $F_{p,q}(c)(i)$ is given by the local rule $f_{p,q}$ defined as follows:

$$\begin{aligned} F_{p,q}(c)(i) =& f_{p,q}(c(i-1), c(i), c(i+1)) \\ =& g_{p,q}(g_{p,q}(c(i-1), c(i)), g_{p,q}(c(i), c(i+1))). \end{aligned}$$

The CA function $F_{p,q}$ is reversible: if $c \in A_{pq}^{\mathbb{Z}}$ is a configuration with a finite number of non-zero coordinates, then

$$\begin{aligned} & F_{p,q}(F_{q,p}(c)) = F_{p,q}(F_{q,p}(\mathrm{config}_{pq}(\mathrm{real}_{pq}(c)))) \\ \overset{L2.1}{=} & \mathrm{config}_{pq}((p/q)(q/p)\,\mathrm{real}_{pq}(c)) = c. \end{aligned}$$

Since $F_{p,q} \circ F_{q,p}$ is continuous and agrees with the identity function on a dense set, it follows that $F_{p,q}(F_{q,p}(c)) = c$ for all configurations $c \in A_{pq}^{\mathbb{Z}}$. We will denote the inverse of $F_{p,q}$ interchangeably by $F_{q,p}$ and $F_{p,q}^{-1}$.

As for $G_{p,q}$, we define $F_{p,q}(w)$ for words $w = w(1)w(2)\dots w(|w|)$ such that $|w| \geq 3$:

$$F_{p,q}(w) = u = u(1)\dots u(|w|-2) \in A_{pq}^{|w|-2},$$

where $u(i) = f_{p,q}(w(i), w(i+1), w(i+2))$ for $1 \leq i \leq |w|-2$, and $F_{p,q}^t(w)$ for every $t > 0$ and word w such that $|w| \geq 2t+1$:

$$F_{p,q}^t(w) = F_{p,q}(F_{p,q}^{t-1}(w)) \in A_{pq}^{|w|-2t}$$

(see an example in Figure 3).

By the definition of $F_{p,q}$, for every $c \in A_{pq}^{\mathbb{Z}}$ and every $i \in \mathbb{Z}$ the value of $F_{p,q}(c)(i)$ is uniquely determined by $c(i-1), c(i)$ and $c(i+1)$, the three nearest digits above in the space-time diagram. Proposition 2.5 gives similarly that each digit in the space-time diagram is determined by the three nearest digits to the right (see Figure 4). Its proof is broken down into the following sequence of lemmas.

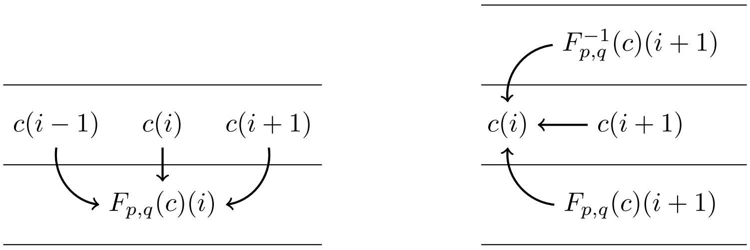

Figure 4: Determination of digits in the space-time diagram of c with respect to $F_{p,q}$.

Lemma 2.2. If $g_{p,q}(x,z) = g_{p,q}(y,w)$, then $x \equiv y \pmod q$.

Proof. Let $x = x_1 q + x_0$, $y = y_1 q + y_0$, $z = z_1 q + z_0$ and $w = w_1 q + w_0$. Then

$$\begin{aligned} g_{p,q}(x,z) = g_{p,q}(y,w) &\implies x_0 p + z_1 = y_0 p + w_1 \\ &\implies x_0 = y_0 \implies x \equiv y \pmod q. \end{aligned}$$

□

Lemma 2.3. $g_{p,q}(x,a) \equiv g_{p,q}(y,a) \pmod q \iff x \equiv y \pmod q$.

Proof. Let $x = x_1 q + x_0$, $y = y_1 q + y_0$ and $a = a_1 q + a_0$. Then

$$\begin{aligned} g_{p,q}(x,a) \equiv g_{p,q}(y,a) \pmod q &\iff x_0 p + a_1 \equiv y_0 p + a_1 \pmod q \\ &\iff x_0 = y_0 \iff x \equiv y \pmod q \end{aligned}$$

□

Lemma 2.4. If $f_{p,q}(x,a,y) = f_{p,q}(z,a,w)$, then $x \equiv z \pmod q$.

Proof.

$$\begin{aligned} & f_{p,q}(x,a,y) = f_{p,q}(z,a,w) \\ \implies & g_{p,q}(g_{p,q}(x,a), g_{p,q}(a,y)) = g_{p,q}(g_{p,q}(z,a), g_{p,q}(a,w)) \\ \overset{L2.2}{\implies} & g_{p,q}(x,a) \equiv g_{p,q}(z,a) \pmod q \overset{L2.3}{\implies} x \equiv z \pmod q. \end{aligned}$$

□

Proposition 2.5. For every $c \in A_{pq}^{\mathbb{Z}}$ and for all $k, i \in \mathbb{Z}$, the value of $F_{p,q}^k(c)(i)$ is uniquely determined by the values of $F_{p,q}^{k-1}(c)(i+1)$, $F_{p,q}^k(c)(i+1)$ and $F_{p,q}^{k+1}(c)(i+1)$.

Proof. Denote $e = \sigma_{A_{pq}}^i(F_{p,q}^k(c))$. It suffices to show that $e(0)$ is uniquely determined by $F_{q,p}(e)(1)$, $e(1)$ and $F_{p,q}(e)(1)$. Since $F_{p,q}(e)(1) = f_{p,q}(e(0), e(1), e(2))$, by Lemma 2.4 $e(1)$ and $F_{p,q}(e)(1)$ determine the value of $e(0)$ modulo q (see Figure 2.5, left). Similarly, because $F_{q,p}(e)(1) = f_{q,p}(e(0), e(1), e(2))$, by the same lemma $e(1)$ and $F_{q,p}(e)(1)$ determine the value of $e(0)$ modulo p (Fig. 2.5, middle). In total, $F_{q,p}(e)(1)$, $e(1)$ and $F_{p,q}(e)(1)$ determine the value of $e(0)$ both modulo q and modulo p (Fig. 2.5, right). Because $e(0) \in A_{pq}$, the value of $e(0)$ is uniquely determined. □

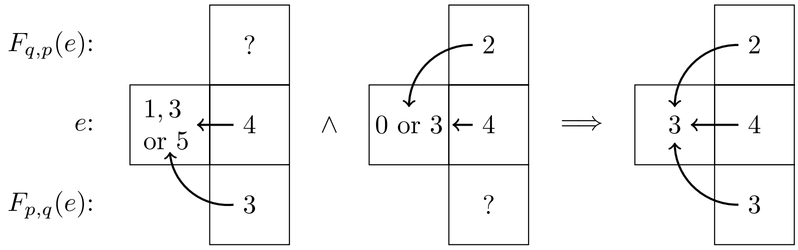

Figure 5: The proof of Proposition 2.5 (here $(p, q) = (3, 2)$).

3 Traces of configurations

For $\xi \geq 0$ we are interested in the values of $\{\xi(p/q)^i\}$ as i ranges over $\mathbb{N}$. In terms of the configuration $\mathrm{config}_{pq}(\xi)$ these correspond to the tails of the configurations $F^i_{p,q}(\mathrm{config}_{pq}(\xi))$, i.e. to the digits $F^i_{p,q}(\mathrm{config}_{pq}(\xi))(j)$ for $j > 0$. Partial information on the tails is preserved in the traces of a configuration. In this section we study traces with respect to $F_{p,q}$ to prove in the case $p \geq 2q - 1$ the existence of small sets S such that $Z_{p/q}(S)$ is non-empty, and then as a corollary for all $p > q > 1$.

Definition 3.1. For any $k \in \mathbb{Z}$, the *k-trace* of a configuration $c \in A^{\mathbb{Z}}_{pq}$ (with respect to $F_{p,q}$) is the sequence

$$\mathrm{Tr}_{p,q}(c, k) = (F^n_{p,q}(c)(k))_{n \in \mathbb{Z}}.$$

In the special case $k = 1$, we denote $\mathrm{Tr}_{p,q}(c, 1) = \mathrm{Tr}_{p,q}(c)$.

A k-trace of c is simply the sequence of digits in the k-th column of the space-time diagram of c with respect to $F_{p,q}$ (see Figure 6).

$F^{-2}_{3,2}(c)$	$\cdots$	5	4	0	1	5	3	4	$\cdots$
$F^{-1}_{3,2}(c)$	$\cdots$	2	3	0	2	5	2	3	$\cdots$
c	$\cdots$	3	4	3	4	2	0	5	$\cdots$
$F_{3,2}(c)$	$\cdots$	5	3	5	3	3	1	2	$\cdots$
$F^{2}_{3,2}(c)$	$\cdots$	5	2	5	2	1	5	1	$\cdots$

Figure 6: A trace of a configuration.

Definition 3.2. The set of allowed words of $\mathrm{Tr}_{p,q}$ is

$$L(p, q) = \{w \in A^*_{pq} \mid w \text{ is a factor of } \mathrm{Tr}_{p,q}(c) \text{ for some } c \in A^{\mathbb{Z}}_{pq}\},$$

i.e. the set of words that can appear in the columns of space-time diagrams with respect to $F_{p,q}$.

The following is a reformulation of Proposition 2.5 in terms of traces (see Figure 7).

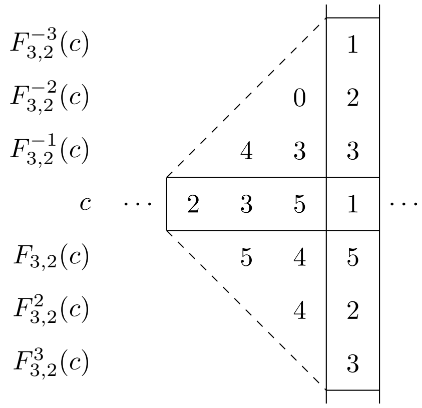

Figure 7: A trace determining part of the configuration.

Corollary 3.3. For every $c \in A_{pq}^{\mathbb{Z}}$ and $k > 0$, the values of $\mathrm{Tr}_{p,q}(c,k)(i)$ for $-(k-1) \leq i \leq (k-1)$ uniquely determine the values of $c(j)$ for $1 \leq j \leq k$.

Proof. The proof is by induction. The case $k = 1$ follows from the fact that $\mathrm{Tr}_{p,q}(c,1)(0) = c(1)$. Next assume that the claim holds for some $k > 0$ and consider the values of $\mathrm{Tr}_{p,q}(c,k+1)(i)$ for $-k \leq i \leq k$. By Proposition 2.5 these determine $\mathrm{Tr}_{p,q}(c,k)(i)$ for $-(k-1) \leq i \leq (k-1)$, which in turn determine $c(j)$ for $1 \leq j \leq k$ by the induction hypothesis. The value of $c(k+1)$ is determined by $\mathrm{Tr}_{p,q}(c,k+1)(0) = c(k+1)$. $\square$

Next we prove an important restriction on the words in the set $L(q,p)$ when $p \geq 2q-1$. Note that the words in $L(q,p)$ are mirror images of the words in $L(p,q)$ (traces with respect to $F_{p,q}$ are read "from bottom to top").

Lemma 3.4. Let $p > q \geq 2$ be relatively prime such that $p \geq 2q-1$, and for every $d \in A_q$ let $k_d \in A_p$ and $j_d \in A_q$ be the unique elements such that $k_d q \equiv d \pmod p$ and $k_d q = j_d p + d$. If $wab \in L(q,p)$ for some $w \in A_{pq}^*$, $a, b \in A_{pq}$ and $a \equiv k_d \pmod p$, then $b \equiv j_d \pmod q$.

Proof. From $wab \in L(q,p)$ it follows that $b = f_{q,p}(x,a,y)$ for some $x, y \in A_{pq}$. Let us write $a = a_1 p + a_0$, $y = y_1 p + y_0$, $g_{q,p}(x,a) = z = z_1 p + z_0$ and $g_{q,p}(a,y) = w = w_1 p + w_0$, where $a_0, y_0, z_0, w_0 \in A_p$ and $a_1, y_1, z_1, w_1 \in A_q$. Here $a_0 = k_d$ because $a \equiv k_d \pmod p$ and $w_1 = j_d$ because $g_{q,p}(a,y) = k_d q + y_1 = j_d p + (d + y_1)$ and $d + y_1 \leq (q-1) + (q-1) < p$. Now

$$f_{q,p}(x,a,y) = g_{q,p}(g_{q,p}(x,a), g_{q,p}(a,y)) = g_{q,p}(z,w) = z_0 q + j_d,$$

and thus $b \equiv j_d \pmod q$. $\square$

Based on the previous lemma, we define a special set of digits

$$D_{p,q} = \{a \in A_{pq} \mid a \equiv k_d \pmod p \text{ for some } d \in A_q\},$$

where the digits k_d are as above.

Example 3.5. Consider the case $p = 3$ and $q = 2$. Then $A_q = \{0,1\}$ and $D_{3,2} = \{0,2,3,5\}$ consists of the elements of A_6 which are congruent to either $k_0 = 0$ or $k_1 = 2 \pmod 3$.

Lemma 3.6. If $p \geq 2q-1$, then $|L(p,q) \cap D_{p,q}^n| \leq q^{n+1}$ for every $n > 0$.

Proof. The proof is by induction. The case $n = 1$ is clear because $|D_{p,q}| = q^2$. Next assume that the claim holds for some $n > 0$. It is sufficient to compute an upper bound for $|L(q,p) \cap D_{p,q}^{n+1}|$, because the words in $L(p,q)$ are mirror images of the words in $L(q,p)$. If $v \in L(q,p) \cap D_{p,q}^{n+1}$, by the previous lemma it can be written in the form $v = wab$, where $a \equiv k_d \pmod p$ and $b \equiv j_d \pmod q$ for some $d \in A_q$. Because $wa \in L(q,p) \cap D_{p,q}^n$, by the induction hypothesis there are at most q^{n+1} different choices for the word wa. Let us fix wa and $d \in A_q$ such that $a \equiv k_d \pmod p$. To prove the claim, it is enough to show that there are at most q choices for the digit b.

Let us assume to the contrary that there are distinct digits $b_1, b_2, \ldots b_{q+1} \in D_{p,q}$ such that $wab_i \in L(q,p) \cap D_{p,q}^{n+1}$ whenever $1 \leq i \leq q+1$. For every i the congruence $b_i \equiv k_{d_i} \pmod p$ holds for some $d_i \in A_q$. By pigeonhole principle we may assume that $d_1 = d_2$ and therefore $b_1 \equiv k_{d_1} \equiv b_2 \pmod p$. Because $wab_1, wab_2 \in L(q,p) \cap D_{p,q}^{n+1}$, we also have $b_1 \equiv j_d \equiv b_2 \pmod q$. Because $b_1, b_2 \in A_{pq}$ are congruent both modulo p and modulo q, they are equal, contradicting the distinctness of $b_1, b_2, \ldots b_{q+1}$. $\square$

As in the introduction, for relatively prime $p > q > 1$ and any $S \subseteq [0,1)$ we denote

$$Z_{p/q}(S) = \left\{ \xi > 0 \;\middle|\; \left\{ \xi \left(\frac{p}{q} \right)^i \right\} \in S \text{ for every } i \in \mathbb{N} \right\}.$$

In [1] it was proved that if $p, q > 1$ are relatively prime integers such that $p > q^2$, then for every $\epsilon > 0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$. We extend this result to the case $p > q > 1$, which in particular covers $p/q = 3/2$. The following theorem by Akiyama, Frougny and Sakarovitch is needed.

Theorem 3.7 (Akiyama, Frougny, Sakarovitch [2])**.** If $p \geq 2q-1$, then $Z_{p/q}(Y_{p,q}) \neq \emptyset$, where

$$Y_{p,q} = \bigcup_{d \in A_q} \left[\frac{1}{p} k_d, \frac{1}{p}(k_d + 1) \right)$$

and $k_d \in A_p$ are as in Lemma 3.4.

Corollary 3.8. If $p \geq 2q-1$, then $Z_{p/q}(X_{p,q}) \neq \emptyset$, where

$$X_{p,q} = \bigcup_{a \in D_{p,q}} \left[\frac{1}{pq} a, \frac{1}{pq}(a+1) \right).$$

Proof. If $\xi \in Z_{p/q}(Y_{p,q})$, then $\xi/q \in Z_{p/q}(X_{p,q})$. $\square$

Theorem 3.9. If $p \geq 2q-1$ and $k > 0$, then there exists a finite union of intervals $I_{p,q,k}$ of total length at most $(q/p)^k$ such that $Z_{p/q}(I_{p,q,k}) \neq \emptyset$.

Proof. Let $k > 0$ be fixed and choose any $\xi' \in Z_{p/q}(X_{p,q})$, where $X_{p,q}$ is the set in the previous corollary. Let $\xi = \xi'(pq)^{-(k-1)}(p/q)^{k-1}$ and denote $c = \text{config}_{pq}(\xi)$. Based on c we define a collection of words

$$W = \{ w = e(1)e(2)\ldots e(k) \mid e = F_{p,q}^n(c) \text{ for some } n \in \mathbb{N} \}.$$

The set W determines a finite union of intervals

$$I_{p,q,k} = \bigcup_{w \in W} \left[\mathrm{real}_{pq}(w), \mathrm{real}_{pq}(w) + (pq)^{-k}\right)$$

and $\xi \in Z_{p/q}(I_{p,q,k})$ by the definition of W. Each interval in $I_{p,q,k}$ has length $(pq)^{-k}$, so to prove that the total length of $I_{p,q,k}$ is at most $(q/p)^k$ it is sufficient to show that $|W| \leq q^{2k}$.

By the definition of $X_{p,q}$, $\mathrm{Tr}_{p,q}(\mathrm{config}_{pq}(\xi'))(i) \in D_{p,q}$ for every $i \geq 0$. For the k-trace of c

$$\begin{aligned}
&\mathrm{Tr}_{p,q}(c,k)(i) = \mathrm{Tr}_{p,q}(\mathrm{config}_{pq}(\xi'(pq)^{-(k-1)}(p/q)^{k-1}), k)(i) \\
= &\mathrm{Tr}_{p,q}(\sigma_{A_{pq}}^{-(k-1)}(F_{p,q}^{k-1}(\mathrm{config}_{pq}(\xi'))), k)(i) = \mathrm{Tr}_{p,q}(F_{p,q}^{k-1}(\mathrm{config}_{pq}(\xi')), 1)(i) \\
= &\mathrm{Tr}_{p,q}(\mathrm{config}_{pq}(\xi'))(i + (k-1)) \text{ for every } i \in \mathbb{N},
\end{aligned}$$

from which it follows that $\mathrm{Tr}_{p,q}(c,k)(i) \in D_{p,q}$ for every $i \geq -(k-1)$. Thus, the words in the set

$$V = \{\mathrm{Tr}_{p,q}(F_{p,q}^n(c), k)(-(k-1)) \ldots \mathrm{Tr}_{p,q}(F_{p,q}^n(c), k)(k-1) \mid n \in \mathbb{N}\}$$

also belong to $L(p,q) \cap D_{p,q}^{2k-1}$, and by Corollary 3.3 and Lemma 3.6

$$|W| \leq |V| \leq |L(p,q) \cap D_{p,q}^{2k-1}| \leq q^{2k}.$$

□

Remark 3.10. The set $I_{p,q,k}$ constructed in the proof of the previous theorem is a union of q^{2k} intervals, each of which is of length $(pq)^{-k}$.

Corollary 3.11. If $p > q > 1$ and $\epsilon > 0$, then there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$.

Proof. Choose some $n > 0$ such that $p^n \geq 2q^n - 1$. Then by the previous theorem there exists a finite union of intervals I_0 of total length at most $\eta = \epsilon(p-1)/(p^n-1)$ such that $Z_{p^n/q^n}(I_0) \neq \emptyset$. For $0 < i < n$ define inductively

$$I_i = \left\{ \left\{ \xi \frac{p}{q} \right\} \in [0,1) \;\middle|\; \xi \geq 0 \text{ and } \{\xi\} \in I_{i-1} \right\},$$

each of which is a finite union of intervals of total length at most $p^i \eta$. Then $J_{p,q,\epsilon} = \bigcup_{i=0}^{n-1} I_i$ is a finite union of intervals of total length at most

$$\sum_{i=0}^{n-1} (p^i)\eta = \frac{p^n - 1}{p - 1}\eta = \epsilon$$

and $Z_{p/q}(J_{p,q,\epsilon}) \supseteq Z_{p^k/q^k}(I_0) \neq \emptyset$. □

4 Ergodicity of $F_{p,q}$

In this section we study the measure theoretical properties of $F_{p,q}$ to prove the existence of large sets S such that $Z_{p/q}(S)$ is empty.

Definition 4.1. A CA function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is *measure preserving* if $\mu(F^{-1}(S)) = \mu(S)$ for every $S \in \Sigma(\mathcal{C})$.

Definition 4.2. A measure preserving CA function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is *ergodic* if for every $S \in \Sigma(\mathcal{C})$ with $F^{-1}(S) = S$ either $\mu(S) = 0$ or $\mu(S) = 1$.

The next lemma is a special case of a well known measure theoretical result (see e.g. Theorem 2.18 in [9]):

Lemma 4.3. For every $S \in \Sigma(\mathcal{C})$ and $\epsilon > 0$ there is an open set $U \subseteq A^{\mathbb{Z}}$ such that $S \subseteq U$ and $\mu(U \setminus S) < \epsilon$.

Lemma 4.4. If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is an ergodic CA, then for every $\epsilon > 0$ there is a finite collection of cylinders $\{U_i\}_{i \in I}$ such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and

$$\left\{ c \in A^{\mathbb{Z}} \mid F^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N} \right\} = A^{\mathbb{Z}}.$$

Proof. Let $C \in \mathcal{C}$ be such that $0 < \mu(C) < \epsilon/2$. By continuity of F, $B = \bigcup_{t \in \mathbb{N}} F^{-t}(C)$ is open and $\mu(B) = 1$ by ergodicity of F (see Theorem 1.5 in [10]). Equivalently, $B' = A^{\mathbb{Z}} \setminus B$ is closed (and compact) and $\mu(B') = 0$. Let V be an open set such that $B' \subseteq V$ and $\mu(V) < \epsilon/2$: such a set exists by Lemma 4.3. Because $\mathcal{C}$ is a basis of $\mathcal{T}$, there is a collection of cylinders $\{V_i\}_{i \in J}$ such that $V = \bigcup_{i \in J} V_i$. By compactness of B' there is a finite set $I' \subseteq J$ such that $B' \subseteq \bigcup_{i \in I'} V_i$. Now $\{U_i\}_{i \in I} = \{C\} \cup \{V_i\}_{i \in I'}$ is a finite collection of cylinders such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and

$$\left\{ c \in A^{\mathbb{Z}} \mid F^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N} \right\} \supseteq B \cup \bigcup_{i \in I'} V_i \supseteq B \cup B' = A^{\mathbb{Z}}.$$

□

To apply this lemma in our setup, we need to show that $F_{p,q}$ is ergodic for $p > q > 1$. In fact, it turns out that a stronger result holds.

Definition 4.5. A measure preserving CA function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is *strongly mixing* if

$$\lim_{t \to \infty} \mu(F^{-t}(U) \cap V) = \mu(U)\mu(V)$$

for every $U, V \in \Sigma(\mathcal{C})$.

We will prove that $F_{p,q}$ is strongly mixing. For the statement of the following lemmas, we define a function $\mathrm{int} : A_{pq}^{+} \to \mathbb{N}$ by

$$\mathrm{int}(w(1)w(2)\dots w(k)) = \sum_{i=0}^{k-1} w(k-i)(pq)^i,$$

i.e. $\mathrm{int}(w)$ is the integer having w as a base pq representation.

Lemma 4.6. Let $w_1, w_2 \in A_{pq}^{k}$ for some $k \geq 2$ and let $t > 0$ be a natural number. Then

1. $\mathrm{int}(w_1) < q^t \implies \mathrm{int}(G_{p,q}(w_1)) < q^{t-1}$ and

2. $\mathrm{int}(w_2) \equiv \mathrm{int}(w_1) + q^t \pmod{(pq)^k} \implies \mathrm{int}(G_{p,q}(w_2)) \equiv \mathrm{int}(G_{p,q}(w_1)) + q^{t-1} \pmod{(pq)^{k-1}}$.

Proof. Let $c_i \in A_{pq}^{\mathbb{Z}}$ $(i = 1, 2)$ be such that $c_i(-(k-1))c_i(-(k-1)+1)\dots c_i(0) = w_i$ and $c_i(j) = 0$ for $j < -(k-1)$ and $j > 0$. From this definition of c_i it follows that $\mathrm{int}(w_i) = \mathrm{real}_{pq}(c_i)$. Denote $e_i = G_{p,q}(c_i)$. We have

$$\sum_{j=-\infty}^{\infty} e_i(-j)(pq)^j = \mathrm{real}_{pq}(e_i) = p\,\mathrm{real}_{pq}(c_i) = p\,\mathrm{int}(w_i)$$

and

$$\begin{aligned}\mathrm{int}(G_{p,q}(w_i)) &= \mathrm{int}(e_i(-(k-1))\dots e_i(-1)) \\ &= \sum_{j=1}^{k-1} e_i(-j)(pq)^{j-1} \equiv \lfloor \mathrm{int}(w_i)/q \rfloor \pmod{(pq)^{k-1}}.\end{aligned}$$

Also note that $\mathrm{int}(G_{p,q}(w_i)) < (pq)^{k-1}$.

For the proof of the first part, assume that $\mathrm{int}(w_1) < q^t$. Combining this with the observations above yields $\mathrm{int}(G_{p,q}(w_i)) \leq \lfloor \mathrm{int}(w_i)/q \rfloor < q^{t-1}$.

For the proof of the second part, assume that $\mathrm{int}(w_2) \equiv \mathrm{int}(w_1) + q^t \pmod{(pq)^k}$. Then there exists $n \in \mathbb{Z}$ such that $\mathrm{int}(w_2) = \mathrm{int}(w_1) + q^t + n(pq)^k$ and

$$\begin{aligned}\mathrm{int}(G_{p,q}(w_2)) &\equiv \lfloor \mathrm{int}(w_2)/q \rfloor \equiv \lfloor \mathrm{int}(w_1)/q \rfloor + q^{t-1} + np(pq)^{k-1} \\ &\equiv \lfloor \mathrm{int}(w_1)/q \rfloor + q^{t-1} \equiv \mathrm{int}(G_{p,q}(w_1)) + q^{t-1} \pmod{(pq)^{k-1}}.\end{aligned}$$

□

Lemma 4.7. Let $t > 0$ and $w_1, w_2 \in A_{pq}^k$ for some $k \geq 2t + 1$. Then

1. $\mathrm{int}(w_1) < q^{2t} \implies \mathrm{int}(F_{p,q}^t(w_1)) = 0$ and
2. $\mathrm{int}(w_2) \equiv \mathrm{int}(w_1) + q^{2t} \pmod{(pq)^k} \implies \mathrm{int}(F_{p,q}^t(w_2)) \equiv \mathrm{int}(F_{p,q}^t(w_1)) + 1 \pmod{(pq)^{k-2t}}$.

Proof. First note that $F_{p,q}(w) = G_{p,q}^2(w)$ for every $w \in A_{pq}^*$ such that $|w| \geq 3$, because $F_{p,q} = \sigma_{A_{pq}}^{-1} \circ G_{p,q} \circ G_{p,q}$. Then both claims follow by repeated application of the previous lemma. □

The content of Lemma 4.7 is as follows. Assume that $\{w_i\}_{i=0}^{(pq)^k-1}$ is the enumeration of all the words in A_{pq}^k in the lexicographical order, meaning that $w_0 = 00\dots00$, $w_1 = 00\dots01$, $w_2 = 00\dots02$ and so on. Then let i run through all the integers between 0 and $(pq)^k - 1$. For the first q^{2t} values of i we have $F_{p,q}^t(w_i) = 00\dots00$, for the next q^{2t} values of i we have $F_{p,q}^t(w_i) = 00\dots01$, and for the following q^{2t} values of i we have $F_{p,q}^t(w_i) = 00\dots02$. Eventually, as i is incremented from $q^{2t}(pq)^{k-2t} - 1$ to $q^{2t}(pq)^{k-2t}$, the word $F_{p,q}^t(w_i)$ loops from $(pq-1)(pq-1)\dots(pq-1)(pq-1)$ back to $00\dots00$.

Theorem 4.8. If $p > q > 1$, then $F_{p,q}$ is strongly mixing.

Proof. Firstly, because $F_{p,q}$ is surjective, the fact that $F_{p,q}$ is measure preserving follows from Theorem 5.4 in [5]. Then, by Theorem 1.17 in [10] it is sufficient to verify the condition

$$\lim_{t\to\infty} \mu(F_{p,q}^{-t}(C_1) \cap C_2) = \mu(C_1)\mu(C_2)$$

for every $C_1, C_2 \in \mathcal{C}$. Without loss of generality we may consider cylinders $C_1 = \mathrm{Cyl}(v_1, 0)$ and $C_2 = \mathrm{Cyl}(v_2, i)$. Denote $l_1 = |v_1|$, $l_2 = |v_2|$ and let $t \geq i + l_2$ be a natural number.

Consider an arbitrary word $w \in A_{pq}^{2t+l_1}$ and its decomposition $w = w_1 w_2 w_3$, where $w_1 \in A_{pq}^{t+i}$, $w_2 \in A_{pq}^{l_2}$ and $w_3 \in A_{pq}^{t+l_1-i-l_2}$. The following conditions may or may not be satisfied by w (see Figure 8):

1. $F_{p,q}^t(w) = v_1$
2. $w_2 = v_2$.

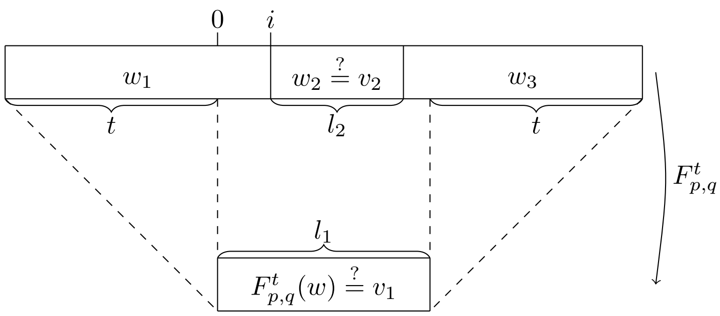

Figure 8: Relations between the words v_1, v_2 and $w_1 w_2 w_3$.

Note that if w satisfies condition (1), then $F_{p,q}^t(\mathrm{Cyl}(w, -t)) \subseteq C_1$, and otherwise $F_{p,q}^t(\mathrm{Cyl}(w, -t)) \cap C_1 = \emptyset$. Also, if w satisfies condition (2), then $\mathrm{Cyl}(w, -t) \subseteq C_2$, and otherwise $\mathrm{Cyl}(w, -t) \cap C_2 = \emptyset$. Let $W_t \subseteq A_{pq}^{2t+l_1}$ be the collection of those words w that satisfy both conditions. It follows that

$$\mu(F_{p,q}^{-t}(C_1) \cap C_2) = \mu\left(\bigcup_{w \in W_t} \mathrm{Cyl}(w, -t)\right) = |W_t|(pq)^{-(2t+l_1)}.$$

Next, we estimate the number of words $w = w_1 w_2 w_3$ in W_t. In any case, to satisfy condition (2), w_2 must equal v_2. Then, for any of the $(pq)^{t+i}$ choices of w_1, the number of choices for w_3 that satisfy condition (1) is between $(pq)^{t+l_1-i-l_2}/(pq)^{l_1} - q^{2t}$ and $(pq)^{t+l_1-i-l_2}/(pq)^{l_1} + q^{2t}$ by Lemma 4.7 (and the paragraph following it). Thus,

$$\begin{aligned} &\left((pq)^{t-i-l_2} - q^{2t}\right)(pq)^{t+i}(pq)^{-(2t+l_1)} \leq \mu(F_{p,q}^{-t}(C_1) \cap C_2) \\ &\leq \left((pq)^{t-i-l_2} + q^{2t}\right)(pq)^{t+i}(pq)^{-(2t+l_1)}, \end{aligned}$$

and as t tends to infinity,

$$\lim_{t\to\infty} \mu(F_{p,q}^{-t}(C_1) \cap C_2) = (pq)^{-l_1-l_2} = \mu(C_1)\mu(C_2).$$

$\square$

Theorem 4.9. If $p > q > 1$ and $\epsilon > 0$, then there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1 - \epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$.

Proof. The previous theorem implies that $F_{p,q}$ is ergodic: if $S \in \Sigma(\mathcal{C})$ is such that $F_{p,q}^{-1}(S) = S$, then

$$\mu(S) = \lim_{t\to\infty} \mu(F_{p,q}^{-t}(S) \cap S) = \mu(S)\mu(S),$$

which means that $\mu(S) = 0$ or $\mu(S) = 1$.

Since $F_{p,q}$ is ergodic, by Lemma 4.4 there is a finite collection of cylinders $\{U_i\}_{i\in I}$ such that $\mu(\bigcup_{i\in I} U_i) < \epsilon$ and

$$\left\{ c \in A_{pq}^{\mathbb{Z}} \mid F_{p,q}^t(c) \in \bigcup_{i\in I} U_i \text{ for some } t \in \mathbb{N} \right\} = A_{pq}^{\mathbb{Z}}.$$

Without loss of generality we may assume that for every $i \in I$, $U_i = \mathrm{Cyl}(w_i, 1)$ and $w_i \in A_{pq}^k$ for a fixed $k > 0$. Consider the collection of words $W = A_{pq}^k \setminus \{w_i\}_{i\in I}$ and define

$$K_{p,q,\epsilon} = \bigcup_{v\in W} \left[\mathrm{real}_{pq}(v), \mathrm{real}_{pq}(v) + (pq)^{-k}\right).$$

The set $K_{p,q,\epsilon}$ has total length

$$\frac{|W|}{(pq)^k} = 1 - \frac{|I|}{(pq)^k} = 1 - \mu\left(\bigcup_{i\in I} U_i\right) \geq 1 - \epsilon.$$

Now let $\xi > 0$ be arbitrary and denote $c = \mathrm{config}_{pq}(\xi)$. There exists a $t \in \mathbb{N}$ such that $F_{p,q}^t(c) \in \bigcup_{i\in I} U_i$, and equivalently, $F_{p,q}^t(c) \notin \bigcup_{v\in W}(\mathrm{Cyl}(v, 1))$. This means that $\{\xi(p/q)^t\} \notin K_{p,q,\epsilon}$, and therefore $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$. $\square$

5 Conclusions

We have shown in Theorem 3.9 and Corollary 3.11 that for $p > q > 1$ and $\epsilon > 0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$. Moreover, by following the proof of this result, it is possible (at least in principle) to explicitly construct the set $J_{p,q,\epsilon}$ for any given ϵ. We have also shown in Theorem 4.9 that for $p > q > 1$ and $\epsilon > 0$ there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1-\epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$. The proof of this theorem is non-constructive.

Problem 5.1. Assume that $p > q > 1$. Is it possible to construct explicitly for every $\epsilon > 0$ a finite union of intervals S such that the total length of S is at least $1 - \epsilon$ and $Z_{p/q}(S) = \emptyset$?

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