3-Abelian Cubes Are Avoidable on Binary Alphabets

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Abstract. A k-abelian cube is a word uvw, where u, v, w have the same factors of length at most k with the same multiplicities. Previously it has been known that k-abelian cubes are avoidable over a binary alphabet for $k \geq 5$. Here it is proved that this holds for $k \geq 3$.

Keywords: combinatorics on words, k-abelian equivalence, repetition-freeness

1 Introduction

The study of repetition-free infinite words (or even the whole area of combinatorics on words) was begun by Axel Thue [15, 16]. He proved that using three letters one can construct an infinite word that does not contain a square, that is a factor of the form uu where u is a non-empty word. Further, using two letters one can construct an infinite word that does not contain a cube, that is a factor of the form uuu where u is a non-empty word, or even an overlap, that is a factor of the form $auaua$ where u is a word and a is a letter. Due to their initial obscure publication, these results have been rediscovered several times.

The problem of repetition-freeness has been studied from many points of view. One is to consider fractional powers. This leads to the concept of repetition threshold and the famous Dejean's conjecture, which was proved in many parts. For example, an infinite number of cases were settled in [3], while the last remaining cases were settled independently in [4] and [14]. Another example is the repetition-freeness of partial words. It was shown that there exist infinite ternary words with an infinite number of holes whose factors are not matching any squares (overlaps) of words of length greater than one [12, 2]. For the abelian case an alphabet with as low as 5 letters is enough in order to construct an infinite word with an infinite number of holes with factors that do not match an abelian square of any word of length greater than two [1].

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In this paper abelian repetition-freeness is an important concept. An abelian square is a non-empty word uv, where u and v have the same number of occurrences of each symbol. Abelian cubes and nth powers are defined in a similar way. Erdős [6] raised the question whether abelian squares can be avoided, i.e., whether there exist infinite words over a given alphabet that do not contain two consecutive permutations of the same factor. It is easily seen that abelian squares cannot be avoided over a three-letter alphabet: Each word of length eight over three letters contains an abelian square. Dekking [5] proved that over a binary alphabet there exists a word that avoids abelian fourth powers, and over a ternary alphabet there exists a word that avoids abelian cubes. The problem of whether abelian squares can be avoided over a four-letter alphabet was open for a long time. In [11], using an interesting combination of computer checking and mathematical reasoning, Keränen proved that abelian squares are avoidable on four letters.

Recently, several questions have been studied from the point of view of k-abelian equivalence. For a positive integer k, two words are said to be k-abelian equivalent if they have the same number of occurrences of every factor of length at most k. These equivalence relations provide a bridge between abelian equivalence and equality, because 1-abelian equivalence is the same as abelian equivalence, and as k grows, k-abelian equivalence becomes more and more like equality. The topic of this paper is k-abelian repetition-freeness, but there has also been research on other topics related to k-abelian equivalence [9, 10].

In [9], the authors show that 2-abelian squares are avoidable only on a four letter alphabet. For $k \geq 3$, the question of avoiding k-abelian squares is open, the minimal alphabet size being either three or four. Computer experiments would suggest that there are 3-abelian square-free ternary words, but it is known that there are no pure morphic k-abelian square-free ternary words for any k [7].

It was conjectured in [9] that for avoiding k-abelian cubes a binary alphabet suffices whenever $k \geq 2$ since computer generated words of length 100000 having the property have been found. This was proved for $k \geq 8$ in [8] and for $k \geq 5$ in [13].

In this work it is proved that 3-abelian cubes can be avoided on a binary alphabet. The methods used are somewhat similar to those used in [8] and [13]: A word is constructed by mapping an abelian cube-free ternary word by a morphism. However, there are some crucial differences. Most importantly, the morphisms used in this paper are not uniform, and this makes many parts of the proofs different and more difficult. The method used in this article is fairly general, but using it requires an extensive case analysis, which can only be carried out with the help of a computer. The 2-abelian case remains open.

2 Preliminaries

We denote by Σ a finite set of symbols called *alphabet*. For $n \geq 0$, the n-letter alphabet $\{0, \ldots, n-1\}$ will be denoted by Σ_n. A *word* w represents a concatenation of letters from Σ. By ε we denote the *empty word*. We denote by $|w|$ the *length* of w and by $|w|_u$ the number of occurrences of u in w. For a factorization $w = uxv$, we say that x is a *factor* of w, and whenever u is empty x is a *prefix* and, respectively, when v is empty x is a *suffix* of w. The prefix of w of length k will be denoted by $\mathrm{pref}_k(w)$ and the suffix of length k by $\mathrm{suff}_k(w)$.

The *powers* of a word w are defined recursively, $w^0 = \varepsilon$ and $w^n = ww^{n-1}$ for $n > 0$. We say that w is an nth power if there exists a word u such that $w = u^n$. Second powers are called *squares* and third powers *cubes*.

Words u and v are *abelian equivalent* if $|u|_a = |v|_a$ for all letters $a \in \Sigma$. For a word $u \in \Sigma_n^*$, let $P_u = (|u|_0, \dots, |u|_{n-1})$ be the *Parikh vector* of u. Words $u, v \in \Sigma_n^*$ are abelian equivalent if and only if $P_u = P_v$.

Two words u and v are *k-abelian equivalent* if $|u|_t = |v|_t$ for every word t of length at most k. Obviously, 1-abelian equivalence is the same as abelian equivalence, and words of length less than $k - 1$ (or, in fact, words of length less than $2k$) are k-abelian equivalent only if they are equal. For words u and v of length at least $k - 1$, another equivalent definition can be given: u and v are k-abelian equivalent if $|u|_t = |v|_t$ for every word t of length k, $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$ and $\mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(v)$. This latter definition is actually the one used in the proofs of this article.

A *k-abelian nth power* is a word $u_1u_2 \dots u_n$, where $u_1, u_2 \dots, u_n$ are k-abelian equivalent. For $k = 1$ this gives the definition of an *abelian nth power*.

A mapping $f : A^* \to B^*$ is a *morphism* if $f(xy) = f(x)f(y)$ for any words $x, y \in A^*$, and is completely determined by the images $f(a)$ for all $a \in A$.

If no non-empty square is a factor of a word w, then it is said that w is *square-free*, or that w *avoids squares*. If there exists an infinite square-free word over an alphabet Σ, then it is said that *squares are avoidable on Σ*. Of course the only thing that matters here is the size of Σ. Similar definitions can be given for cubes and higher powers, as well as for k-abelian powers.

Unlike ordinary cubes, abelian cubes are not avoidable on a binary alphabet, and unlike ordinary squares, abelian squares are not avoidable on a ternary alphabet. However, Dekking showed in [5] that two letters are sufficient for avoiding abelian fourth powers, and three letters suffice for avoiding abelian cubes. An extension of the latter result is stated in the following theorem. It is proved that the word of Dekking avoids also many other factors in addition to abelian cubes.

Theorem 1. *Let $w = \sigma^\omega(0)$ be a fixed point of the morphism $\sigma : \Sigma_3^* \to \Sigma_3^*$ defined by*

$$\sigma(0) = 0012, \qquad \sigma(1) = 112, \qquad \sigma(2) = 022.$$

Then w is abelian cube-free and contains no factor $apbqcrd$ where a, b, c, d are letters and one of the following conditions is satisfied:

1. *$abcd = 0112$ and $P_p = P_q = P_r$,*
2. *$abcd = 0210$ and $P_p = P_q - (1, -1, 1) = P_r - (0, -1, 1)$,*
3. *$abcd = 0211$ and $P_p = P_q - (1, -1, 1) = P_r - (1, -2, 1)$,*
4. *$abcd = 0220$ and $P_p = P_q - (1, -1, 1) = P_r - (0, 0, 0)$,*
5. *$abcd = 0221$ and $P_p = P_q - (1, -1, 1) = P_r - (1, -1, 0)$,*

6. $abcd = 1001$ *and* $P_p = P_q = P_r$,
7. $abcd = 1002$ *and* $P_p = P_q = P_r$,

Proof. The word w was shown to be abelian cube-free in [5]. Similar ideas can be used to show that w avoids the factors $apbqcrd$. Case 1 was proved in [13]. Case 2 is proved here. Cases 3–6 are similar to the first two, so their proofs are omitted. Case 7 is more difficult, so it is proved here.

Let $f : \Sigma^* \to \mathbb{Z}_7$ be the morphism defined by

$$f(0) = 1, \qquad f(1) = 2, \qquad f(2) = 3$$

(here $\mathbb{Z}_7$ is the additive group of integers modulo 7). Then $f(\sigma(x)) = 0$ for all $x \in \Sigma$. If $apbqcrd$ is a factor of w, then there are u, s, t such that $\sigma(u) = sapbqcrdt$ and u is a factor of w. Consider the values

$$f(s), f(sa), f(sap), f(sapb), f(sapbq), f(sapbqc), f(sapbqcr), f(sapbqcrd). \quad (1)$$

These elements are of the form $f(\sigma(u')v') = f(v')$, where v' is a prefix of one of $0012, 112, 022$. The possible values for $f(v')$ are 0, 1, 2 and 4.

Consider Case 2. Let $abcd = 0210$. If $P_p = P_q - (1, -1, 1) = P_r - (0, -1, 1)$, then $f(p) = f(q) - 2 = f(r) - 1$. If we denote $i = f(s)$, $j = f(p)$, then the values for (1) are

$$i, i+1, i+j+1, i+j+4, i+2j+6, i+2j+1, i+3j+2, i+3j+3.$$

For all values of i and j, one is not 0, 1, 2 or 4. This is a contradiction.

Consider Case 7. Let $abcd = 1002$. Let $apbqcrd$ be the shortest factor of w satisfying the conditions of Case 7. Then $P_p = P_q = P_r$ and $f(p) = f(q) = f(r)$. If we denote $i = f(s)$, $j = f(p)$, then the values for (1) are

$$i, i+2, i+j+2, i+j+3, i+2j+3, i+2j+4, i+3j+4, i+3j.$$

It must be $i = 0$ and $j = 6$, because otherwise one of the values is not 0, 1, 2 or 4. There are letters a', b', c', d' and words $s', p', q', r', t', s_2, p_1, p_2, q_1, q_2, r_1, r_2, t_1$ such that

$$\begin{aligned}
u &= s'a'p'b'q'c'r'd't' & s &= \sigma(s')s_2 \\
s_2 1 p_1 &= \sigma(a') & p &= p_1\sigma(p')p_2 \\
p_2 0 q_1 &= \sigma(b') & q &= q_1\sigma(q')q_2 \\
q_2 0 r_1 &= \sigma(c') & r &= r_1\sigma(r')r_2 \\
r_2 2 t_1 &= \sigma(d') & t &= t_1\sigma(t'),
\end{aligned}$$

i.e. the situation is like in the following diagram:

s	1	p	0	q	0	r	2	t
	s_2 p_1		p_2 q_1		q_2 r_1		r_2 t_1	
$\sigma(s')$	$\sigma(a')$	$\sigma(p')$	$\sigma(b')$	$\sigma(q')$	$\sigma(c')$	$\sigma(r')$	$\sigma(d')$	$\sigma(t')$

Because $i = 0$, $s_2 = \varepsilon$. Then $\sigma(a')$ begins with 1, so $a' = 1$ and $p_1 = 12$. Thus $p = 12\sigma(p')p_2$. It must be $f(p_2) = f(p) - f(\sigma(p')) - f(12) = j - 0 - 5 = 1$, so $p_2 = 0$. Then $\sigma(b')$ begins with 00, so $b' = 0$ and $q_1 = 12$. Like above, it can be concluded that $q = 12\sigma(q')0$, and similarly also $r = 12\sigma(r')0$. But then $1p'0q'0r'2$ is a factor of w. If

$$M = \begin{pmatrix} |\sigma(0)|_0 & |\sigma(1)|_0 & |\sigma(2)|_0 \\ |\sigma(0)|_1 & |\sigma(1)|_1 & |\sigma(2)|_1 \\ |\sigma(0)|_2 & |\sigma(1)|_2 & |\sigma(2)|_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

and Parikh vectors are interpreted as column vectors, then

$$MP_{p'} = P_{\sigma(p')}, \qquad MP_{q'} = P_{\sigma(q')}, \qquad MP_{r'} = P_{\sigma(r')}.$$

Because M is invertible and $\sigma(p'), \sigma(q'), \sigma(r')$ are abelian equivalent, also p', q', r' are abelian equivalent. Because $1p'0q'0r'2$ is shorter than $1p0q0r2$, this contradicts the minimality of $1p0q0r2$. $\square$

If abelian cubes are avoidable on some alphabet, then so are k-abelian cubes. This means that k-abelian cubes are avoidable on a ternary alphabet for all k. But for which k are they avoidable on a binary alphabet? In [8] it was proved that this holds for $k \geq 8$, and conjectured that it holds for $k \geq 2$. In [13] it was proved that this holds for $k \geq 5$. In this article it is proved that this holds for $k \geq 3$. The case when $k = 2$ remains open.

3 3-abelian cube-freeness

Let $w \in \Sigma_m^\omega$. The following remarks will be used in the case where $m = 3$, $n = 2$, w is abelian cube-free and $k = 4$ or $k = 3$, but they hold also more generally.

For a word $v \in \Sigma_n^*$, let $Q_v = (|v|_{t_0}, \dots, |v|_{t_{N-1}})$, where $t_0, \dots, t_{N-1}$ are the words of Σ_n^k in lexicographic order. When doing matrix calculations, all vectors P_u and Q_v will be interpreted as column vectors.

Let $h : \Sigma_m^* \to \Sigma_n^*$ be a morphism. It needs to be assumed that h satisfies three conditions:

- There is a word $s \in \Sigma_m^{k-1}$ that is a prefix of $h(a)$ for every $a \in \Sigma_m$.
- The matrix M whose columns are $Q_{h(0)s}, \dots, Q_{h(m-1)s}$ has rank m.
- There are no k-abelian equivalent words v_1, v_2, v_3 of length less than

  $$2 \max \{h(a) \mid a \in \Sigma_m\}$$

  such that $v_1 v_2 v_3$ is a factor of $h(w)$.

Let M^+ be the Moore-Penrose pseudoinverse of M. The only properties of M^+ needed in this article are that it exists and can be efficiently computed, and that since the columns of M are linearly independent, M^+M is the $m \times m$ identity matrix. For any word $u \in \Sigma^*$, $Q_{h(u)s} = MP_u$.

Lemma 2. *If the word $h(w)$ has a factor $v_1v_2v_3$, where v_1, v_2, v_3 are k-abelian equivalent, then there are letters $a_0, a_1, a_2, a_3, b_2, b_3 \in \Sigma_m$, words $u_1, u_2, u_3 \in \Sigma_m^*$ and indices*

$$\begin{aligned} &i_0 \in \{0, \dots, |h(a_0)| - 1\}, \\ &i_1 \in \{k-1, \dots, |h(a_1)| + k - 2\}, \\ &i_2 \in \{k-1, \dots, |h(a_2)| + k - 2\}, \\ &i_3 \in \{k-1, \dots, |h(a_3)| + k - 2\} \end{aligned} \tag{2}$$

such that $a_0u_1a_1b_2u_2a_2b_3u_3a_3$ is a factor of w and $v_i = x_ih(u_i)y_i$ for $i \in \{1, 2, 3\}$, where

$$\begin{aligned} x_1 &= \mathrm{suff}_{|h(a_0)|-i_0}(h(a_0)) & y_1 &= \mathrm{pref}_{i_1}(h(a_1b_2)), \\ x_2 &= \mathrm{suff}_{|h(a_1b_2)|-i_1}(h(a_1b_2)) & y_2 &= \mathrm{pref}_{i_2}(h(a_2b_3)), \\ x_3 &= \mathrm{suff}_{|h(a_2b_3)|-i_2}(h(a_2b_3)) & y_3 &= \mathrm{pref}_{i_3}(h(a_3)s). \end{aligned} \tag{3}$$

Proof. It was assumed that $h(w)$ does not contain short k-abelian cubes, and a longer k-abelian cube $v_1v_2v_3$ must be of the form specified in the claim. □

Because s is a prefix of $h(u_i)$ and y_i, it follows that $Q_{v_i} = Q_{x_is} + Q_{h(u_i)s} + Q_{y_i}$.

The idea is to iterate over all values of $a_0, a_1, a_2, a_3, b_2, b_3$ and i_0, i_1, i_2, i_3 and in each case try to deduce that one of the following holds:

- There are no u_1, u_2, u_3 such that the words $v_i = x_ih(u_i)y_i$ are k-abelian equivalent.
- If $v_i = x_ih(u_i)y_i$ are k-abelian equivalent, then $a_0u_1a_1b_2u_2a_2b_3u_3a_3$ contains an abelian cube or a factor of the form mentioned in Theorem 1.

If we succeed, then there are words w such that $h(w)$ is k-abelian cube-free. The following lemmas will be useful.

Lemma 3. *Let $a_0, a_1, a_2, a_3, b_2, b_3 \in \Sigma_m$, indices i_0, i_1, i_2, i_3 be as in (2) and words x_1, x_2, x_3, y_1, y_2, y_3 be as in (3). Let the following condition be satisfied:*

$$\begin{aligned} &\mathrm{pref}_{k-1}(x_1s), \mathrm{pref}_{k-1}(x_2), \mathrm{pref}_{k-1}(x_3) \textit{ are not equal or} \\ &\mathrm{suff}_{k-1}(y_1), \mathrm{suff}_{k-1}(y_2), \mathrm{suff}_{k-1}(y_3) \textit{ are not equal.} \end{aligned} \tag{C1}$$

Then there are no u_1, u_2, u_3 such that the three words $v_i = x_ih(u_i)y_i$ would be k-abelian equivalent.

Proof. If the prefixes or suffixes of v_1, v_2, v_3 of length $k-1$ are not equal, then v_1, v_2, v_3 cannot be k-abelian equivalent. □

Lemma 4. *Let $a_0, a_1, a_2, a_3, b_2, b_3 \in \Sigma_m$, indices i_0, i_1, i_2, i_3 be as in (2) and words x_1, x_2, x_3, y_1, y_2, y_3 be as in (3). Let $R_i = Q_{x_is} + Q_{y_i}$ for $i \in \{1, 2, 3\}$. Let the following condition be satisfied:*

$$\begin{aligned} &M^+(R_1 - R_i) \textit{ is not an integer vector or} \\ &MM^+(R_1 - R_i) + R_i \textit{ are not equal for } i \in \{1, 2, 3\}. \end{aligned} \tag{C2}$$

Then there are no u_1, u_2, u_3 such that the three words $v_i = x_ih(u_i)y_i$ would be k-abelian equivalent.

Proof. If $v_i = x_i h(u_i) y_i$, then $Q_{v_i} = Q_{h(u_i)s} + R_i = MP_{u_i} + R_i$. If $Q_{v_1} = Q_{v_2} = Q_{v_3}$, then $P_{u_i} - P_{u_1} = M^+(R_1 - R_i)$. This must be an integer vector. The vectors $Q_{v_i} - MP_{u_1} = MM^+(R_1 - R_i) + R_i$ must be equal for $i \in \{1,2,3\}$. $\square$

Lemma 5. *Let $a_0, a_1, a_2, a_3, b_2, b_3 \in \Sigma_m$, indices i_0, i_1, i_2, i_3 be as in* (2) *and words x_1, x_2, x_3, y_1, y_2, y_3 be as in* (3). *Let $R_i = Q_{x_i s} + Q_{y_i}$ for $i \in \{1,2,3\}$. Let the following condition be satisfied:*

$$\begin{aligned} &\text{For } i \in \{0,1,2,3\} \text{ there are } c_i, d_i \in \{a_i, \varepsilon\} \text{ such that } c_i d_i = a_i \text{ and} \\ &\quad M^+(R_1 - R_1) + P_{d_0 c_1} \\ &= M^+(R_1 - R_2) + P_{d_1 b_2 c_2} \\ &= M^+(R_1 - R_3) + P_{d_2 b_3 c_3}. \end{aligned} \tag{C3}$$

If $a_0 u_1 a_1 b_2 u_2 a_2 b_3 u_3 a_3$ is abelian cube-free, then the three words $v_i = x_i h(u_i) y_i$ cannot be k-abelian equivalent.

Proof. Like in the proof of Lemma 4, the k-abelian equivalence of v_1, v_2, v_3 implies $P_{u_i} - P_{u_1} = M^+(R_1 - R_i)$. From this and (C3) it follows that

$$P_{u_1} + P_{d_0 c_1} = P_{u_2} + P_{d_1 b_2 c_2} = P_{u_3} + P_{d_2 b_3 c_3},$$

so $d_0 u_1 c_1, d_1 b_2 u_2 c_2, d_2 b_3 u_3 c_3$ are abelian equivalent. This contradicts the abelian cube-freeness of $a_0 u_1 a_1 b_2 u_2 a_2 b_3 u_3 a_3$. $\square$

Lemma 6. *Let $a_0, a_1, a_2, a_3, b_2, b_3 \in \Sigma_m$, indices i_0, i_1, i_2, i_3 be as in* (2) *and words x_1, x_2, x_3, y_1, y_2, y_3 be as in* (3). *Let $R_i = Q_{x_i s} + Q_{y_i}$ for $i \in \{1,2,3\}$ and $S_i = M^+(R_1 - R_i) + P_{b_i}$ for $i \in \{2,3\}$. Let the following condition be satisfied:*

$$\begin{array}{lll} (0 = S_2 = S_3 & \textit{and} & a_0a_1a_2a_3 = 0112) \textit{ or} \\ (0 = S_2 - (1,-1,1) = S_3 - (0,-1,1) & \textit{and} & a_0a_1a_2a_3 = 0210) \textit{ or} \\ (0 = S_2 - (1,-1,1) = S_3 - (1,-2,1) & \textit{and} & a_0a_1a_2a_3 = 0211) \textit{ or} \\ (0 = S_2 - (1,-1,1) = S_3 - (0,0,0) & \textit{and} & a_0a_1a_2a_3 = 0220) \textit{ or} \\ (0 = S_2 - (1,-1,1) = S_3 - (1,-1,0) & \textit{and} & a_0a_1a_2a_3 = 0221) \textit{ or} \\ (0 = S_2 = S_3 & \textit{and} & a_0a_1a_2a_3 = 1001) \textit{ or} \\ (0 = S_2 = S_3 & \textit{and} & a_0a_1a_2a_3 = 1002). \end{array} \tag{C4}$$

If $a_0 u_1 a_1 b_2 u_2 a_2 b_3 u_3 a_3$ is not of the form apbqcrd specified in Theorem 1, then the three words $v_i = x_i h(u_i) y_i$ cannot be k-abelian equivalent.

Proof. Like in the proof of Lemma 4, the k-abelian equivalence of v_1, v_2, v_3 implies $P_{u_i} - P_{u_1} = M^+(R_1 - R_i)$. From this and the first row of (C4) it follows that

$$P_{u_1} = P_{u_2} + P_{b_2} = P_{u_3} + P_{b_3},$$

so $u_1, b_2 u_2, b_3 u_3$ are abelian equivalent, which is a contradiction. The other rows lead to a contradiction in a similar way. $\square$

We can iterate over all values of $a_0, a_1, a_2, a_3, b_2, b_3$ and i_0, i_1, i_2, i_3. If in all cases one of Conditions C1, C2, C3 is true, then h maps all abelian cube-free words to k-abelian cube-free words. If in all cases one of Conditions C1, C2, C3, C4 is true, then h maps the word of Theorem 1 to a k-abelian cube-free word. In this way we obtain Theorems 7 and 8.

Concerning the actual implementation of the above algorithm, there are some optimizations that can be made. First, if i_1 and i_2 are such that b_1 and b_2 do not affect the definition of $x_1, x_2, x_3, y_1, y_2, y_3$ in (3), then instead of iterating over all values of b_1 and b_2, they can be combined with u_2 and u_3. Second, in most of the cases Condition C1 is true, and these cases can be handled easily. In the following theorems, there are a couple of thousand nontrivial cases, i.e. cases where Condition C1 is false. A Python file used for proving Theorems 7 and 8 is available on the Internet[3].

Theorem 7. *The morphism defined by*

$$0 \mapsto 10110100110, \qquad 1 \mapsto 101101001001, \qquad 2 \mapsto 1011001100100,$$

maps every abelian cube-free ternary word to a 4-abelian cube-free word.

Proof. The morphism satisfies all conditions stated at the beginning of this section:

- The images of 0, 1 and 2 have the common prefix 101.
- The rows of M corresponding to the factors 0010, 0101 and 1100 are $(0,1,2)$, $(1,0,1)$ and $(0,0,2)$, respectively. These are linearly independent, so the rank of M is 3.
- It can be checked that the image of any abelian cube-free word does not contain 4-abelian cubes of words shorter than 26.

Thus it suffices to check all cases as in the algorithm described above. Observe that here Condition C4 is not needed. □

Theorem 8. *The morphism defined by*

$$0 \mapsto 01010, \qquad 1 \mapsto 0110010, \qquad 2 \mapsto 0110110,$$

maps the word w of Theorem 1 to a 3-abelian cube-free word.

Proof. The morphism satisfies all conditions stated at the beginning of this section:

- The images of 0, 1 and 2 have the common prefix 01.
- The rows of M corresponding to the factors 010, 011 and 101 are $(2,1,0)$, $(0,1,2)$ and $(1,0,1)$, respectively. These are linearly independent, so the rank of M is 3.
- It can be checked that the image of w does not contain 3-abelian cubes of words shorter than 14.

[3] http://users.utu.fi/amsaar/en/code.htm

Thus it suffices to check all cases as in the algorithm described above. □

We end this work with some remarks regarding how the search of these morphisms was performed. A first observation is that in order to avoid short cubes and given the fact that we want the obtained images to have the same prefix of length $k-1$, we can only look at morphisms obtained by concatenation of elements from the set $\{ab, aab, abb, aabb\}$. Moreover, when investigating infinite words obtained by application of some morphism to the Dekking word, we note that not only all the images but also their concatenation with themselves must be k-abelian cube-free. Hence, one can generate all words up to some length, say 30, and check for which of these both them and their squares occur among factors. Next, using some backtracking one can check if any triple made of these words would in fact be fit for application on the Dekking word. One final observation is that in order to ensure that any of these triples constitute good candidates, one must check the k-abelian cube-freeness property for factors up to length 10,000, as it happened that the first occurrence of a 3-abelian cube of length over 1,000 started after position 7,000 of the generated word.

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