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Strong stability measures for multicriteria quadratic integer programming problem of finding extremum solutions

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Abstract

We consider a wide class of quadratic optimization problems with integer and Boolean variables. In this paper, the lower and upper bounds on the strong stability radius of the set of extremum solutions are obtained in the situation where solution space and criterion space are endowed with various Hölder's norms.

Keywords: Multicriteria problem, extremum solutions, strong stability radius, Hölder's norms.

1 Problem formulation and basic definitions

Let $A = [a_{ijk}]$ be a $n \times n \times m$-matrix with corresponding cuts $A_k \in \mathbf{R}^{n\times n},\ k \in N_m = \{1, 2, \ldots, m\},\ m \geq 1$. Let also $X \subseteq \mathbf{Z}^n, 2 \leq |X| \leq \infty$, be a set of feasible solutions (integer vectors) $x = (x_1, x_2, \ldots, x_n)^T$, $n \geq 2$.

We define a vector criterion

$$f(x, A) = \big(f_1(x, A_1), f_2(x, A_2), \ldots, f_m(x, A_m)\big) \rightarrow \min_{x\in X},$$

with partial criteria being quadratic functions

$$f_k(x, A_k) = x^T A_k x,\ k \in N_m.$$

In decision making theory, along with the well-known Pareto optimality principle (see e.g. [1]), various choice functions are considered [2]–[5]. In this paper, under m-criteria quadratic problem $Z_m(A)$

we understand the problem of finding the set of extremum solutions defined in traditional way (see e.g. [2]–[4]):

$$C_m(A) = \{x \in X: \ \exists s \in N_m \ \ \forall x' \in X \ \ \big(g_s(x, x', A_s) \leq 0\big)\},$$

where

$$g_s(x, x', A_s) = f_s(x, A_s) - f_s(x', A_s) = (x - x')^T A_s (x - x').$$

Thus, the choice of extremum solutions can be interpreted as finding best solutions for each of m criteria, and then combining them into one set. In other words, the set of extremum solutions contains all the individual minimizers of each objective. Obviously, $C_1(A), \ A \in \mathbf{R}^{n\times n}$ is the set of optimal solutions for scalar problem $Z_1(A)$ with $A \in \mathbf{R}^{n\times n}$.

Taking into account that X is finite, the following formulae below are true:

$$C_m(A) = S_m(A)\backslash(P_m(A)\backslash L_m(A)) = L_m(A) \cup (S_m(A)\backslash P_m(A)),$$

$$C_m(A) \cap P_m(A) = L_m(A),$$

$$L_m(A) \subseteq P_m(A) \subseteq S_m(A),$$

$$L_m(A) \subseteq C_m(A) \subseteq S_m(A),$$

where $P_m(A)$ denotes the Pareto set [6], $S_m(A)$ denotes the Slater set [7], and $L_m(A)$ denotes the lexicographic set (see e.g. [1],[8]). Below we define all the three sets in a traditional way (see e.g. [9]–[11]):

$$P_m(A) = \Big\{x \in X: \ X(x, A) = \emptyset\Big\},$$

$$S_m(A) = \Big\{x \in X: \ \nexists x^0 \in X \ \ \forall k \in N_m \ \ \big(g_k(x, x^0, A_k) > 0\big)\Big\},$$

$$L_m(A) = \bigcup_{\pi \in \Pi_m} L(A, \pi),$$

$$L(A, \pi) = \Big\{x \in X: \ \forall x' \in X \ \ \big(g(x, x', A) \leq_\pi 0_{(m)}\big)\Big\},$$

$$X(x, A) = \Big\{x' \in X: \ g(x, x', A) \geq 0_{(m)} \ \& \ g(x, x', A) \neq 0_{(m)}\Big\},$$

$$g(x, x', A) = \big(g_1(x, x', A_1), g_2(x, x', A_2), \dots, g_m(x, x', A_m)\big),$$

$$0_{(m)} = (0, 0, \dots, 0) \in \mathbf{R}^m.$$

Here Π_m is the set of all $m!$ permutations of numbers $1, 2, ..., m$; $\pi = (\pi_1, \pi_2, ..., \pi_m) \in \Pi_m$; and the binary relation of lexicographic order between two vectors $y = (y_1, y_2, \dots, y_m) \in \mathbf{R}^m$ and $y' = (y'_1, y'_2, \dots, y'_m) \in \mathbf{R}^m$ is defined as follows

$$y \leq_\pi y' \Longleftrightarrow$$

$$\Big(y = y'\Big) \vee \Big(\exists k \in N_m \ \ \forall i \in N_{k-1} \ \ (y_{\pi_k} < y'_{\pi_k} \ \& \ y_{\pi_i} = y'_{\pi_i})\Big),$$

where $N_0 = \emptyset$. Obviously all the sets, $P_m(A), S_m(A), L_m(A)$ and $C_m(A)$, are non-empty for any matrix $A = [a_{ijk}] \in \mathbf{R}^{n \times n \times m}$ due to the finite number of alternatives in X.

We will perturb the elements of matrix $A \in \mathbf{R}^{n \times n \times m}$ by adding elements of the perturbing matrix $A' \in \mathbf{R}^{n \times n \times m}$. Thus the perturbed problem $Z_m(A + A')$ of finding extremum solutions has the following form

$$f(x, A + A') \to \min_{x \in X}.$$

In the solution space $\mathbf{R}^n$, we define an arbitrary Hölder's norm $l_p,\ p \in [1, \infty]$, i.e. under norm of vector $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ we understand the number

$$\|a\|_p = \begin{cases} \Big(\sum\limits_{j \in N_n} |a_j|^p\Big)^{1/p} & \text{if } \ 1 \leq p < \infty, \\ \\ \max\{|a_j| : \ j \in N_n\} & \text{if } \ p = \infty. \end{cases}$$

Thus, for any matrix $A_k \in \mathbf{R}^{n \times n}$, the norm of the matrix is defined as a norm of vector composed of all the matrix elements.

In the criterion space $\mathbf{R}^m$, we define another Hölder's norm $l_q,\ q \in [1, \infty]$, i.e. under norm of matrix $A \in \mathbf{R}^{n \times n \times m}$ we understand the number

$$\|A\|_{pq} = \|(\|A_1\|_p, \|A_2\|_p, \dots, \|A_m\|_p)\|_q,$$

It is easy to see that

$$\|A_k\|_p \leq \|A\|_{pq}, \;\; k \in N_m. \tag{1}$$

Let ζ be either p or q. It is well-known that l_ζ norm, defined in $\mathbf{R}^n$, induces conjugated l_{ζ^*} norm in $(\mathbf{R}^n)^*$. For ζ and ζ^*, the following relations hold

$$\frac{1}{\zeta} + \frac{1}{\zeta^*} = 1, \qquad 1 < \zeta < \infty.$$

In addition, if $\zeta = 1$, then $\zeta^* = \infty$. Obviously, if $\zeta^* = 1$, then $\zeta = \infty$. Also notice that ζ and ζ^* belong to the same range $[1, \infty]$. We also set $\frac{1}{\zeta} = 0$ if $\zeta = \infty$.

For any two vectors a and b of the same dimension, the following Hölder's inequalities are well-known (see e.g. [12])

$$|a^T b| \leq \|a\|_\zeta \|b\|_{\zeta^*}. \tag{2}$$

To any vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{Z}^n$, we assign a vector $\tilde{x}$ composed of all the possible products $x_i x_j$, i.e.

$$\tilde{x} = (x_1 x_1, x_1 x_2, \ldots, x_n x_{n-1}, x_n x_n)^T \in \mathbf{Z}^{n^2}.$$

Taking into account Hölder's inequalities (2), we can see that for any $x, x' \in \mathbf{Z}^n$ and $k \in N_m$ the following inequalities hold

$$|f_k(x, A_k)| = |x^T A_k x| = |A_k x x^T| \leq \|A_k\|_p \|\tilde{x}\|_{p^*},$$

$$|g_k(x, x', A_k)| \leq \|A_k\|_p \|\tilde{x} - \tilde{x'}\|_{p^*}. \tag{3}$$

It is easy to see that for any vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ with condition $|a_j| = \alpha$, $j \in N_n$, and any matrix $A_k = [a_{ijk}] \in \mathbf{R}^{n \times n}$ with condition $|a_{ijk}| = \alpha$, $(i, j) \in N_n \times N_n$, the following inequalities are valid

$$\|a\|_p = \alpha n^{\frac{1}{p}}, \tag{4}$$

$$\|A_k\|_p = \alpha n^{\frac{2}{p}}. \tag{5}$$

Given $\varepsilon > 0$, let

$$\Omega_{pq}(\varepsilon) = \left\{ A' \in \mathbf{R}^{n \times n \times m} : \;\; \|A'\|_{pq} < \varepsilon \right\}$$

be the set of perturbing matrices A' with cuts $A'_k \in \mathbf{R}^{n\times n},\ k \in N_m$, and $\|A'\|_{pq}$ is the norm of $A' = [a'_{ijk}] \in \mathbf{R}^{n\times n\times m}$. Denote

$$\Xi_{pq} = \Big\{\varepsilon > 0: \quad \forall\, A' \in \Omega_{pq}(\varepsilon) \quad \big(C_m(A+A') \cap C_m(A) \neq \emptyset\big)\Big\}.$$

Following [10] and [13], the number

$$\rho_m(p,q) = \begin{cases} \sup\ \Xi_{pq} & \text{if } \ \Xi_{pq} \neq \emptyset, \\ 0 & \text{if } \ \Xi_{pq} = \emptyset \end{cases}$$

is called the *strong stability* (in terminology of [14] and [15] T_1-stability) radius of problem $Z_m(A)$, $m \in \mathbf{N}$, with Hölder's norms l_p and l_q in the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Thus, the strong stability radius of problem $Z_m(A)$ defines the extreme level of independent perturbations of the elements of matrix A in the space $\mathbf{R}^{n\times n\times m}$ not leading to the situation where new extremum solutions appear.

2 Main result

Given $p, q \in [1, \infty]$, for problem $Z_m(A)$, $m \in \mathbf{N}$, we set

$$\phi_m(p) = \min_{x \notin C_m(A)} \ \min_{k\in N_m} \ \max_{x' \in X\setminus\{x\}} \ \frac{g_k(x,x',A_k)}{\|\tilde{x} - \tilde{x'}\|_{p^*}},$$

$$\psi_m(p) = \max_{x' \in C_m(A)} \ \max_{k\in N_m} \ \min_{x \notin C_m(A)} \ \frac{g_k(x,x',A_k)}{\|\tilde{x} - \tilde{x'}\|_{p^*}},$$

$$\gamma_m(p,q) = n^{\frac{2}{p}} m^{\frac{1}{q}} \min_{x \notin C_m(A)} \ \max_{k\in N_m} \ \max_{x' \in C_m(A)} \ \frac{g_k(x,x',A_k)}{\|\tilde{x} - \tilde{x'}\|_{1}}.$$

It is evident that if $C_m(A) = X$, the inequality

$$C_m(A+A') \cap C_m(A) \neq \emptyset$$

holds for any perturbing matrix $A' \in \Omega_{pq}(\varepsilon)$ with $\varepsilon > 0$. So, the stability radius is infinite when $C_m(A) = X$. The problem $Z_m(A)$ that satisfies $C_m(A) \neq X$ is called *non-trivial*.

Theorem 1. *Given $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, for the strong stability radius $\rho_m(p,q)$ of non-trivial problem $Z_m(A)$, the following lower bound is valid*

$$\rho_m(p,q) \geq \max\{\phi_m(p), \psi_m(p)\} > 0.$$

In addition,

$$\gamma_m(p,q) \geq \rho_m(p,q) \geq \max\{\phi_m(p), \psi_m(p)\} > 0 \tag{6}$$

if $Z^m(A)$ is a problem with Boolean variables, i.e. if $X \subseteq \mathbf{E}^n$.

Proof. Since the formula

$$\forall x \notin C_m(A) \quad \forall k \in N_m \ \ \exists x^0 \in X \ \ \big(g_k(x, x^0, A_k) > 0\big),$$

is true, the inequality $\phi_m(p) > 0$ tells us that the lower bound on the strong stability radius as well as the strong stability radius itself are always positive.

First, we prove that $\rho_m(p,q) \geq \phi_m(p)$. Let $A' \in \Omega_{pq}(\phi_m(p))$ be a perturbing matrix with cuts $A'_k \in \mathbf{R}^{n\times n}$, $k \in N_m$. Then according to the definition of the number $\phi_m(p)$, for any index $k \in N_m$ and any solution $x \notin C_m(A)$ there exists a solution $x^0 \in X\backslash\{x\}$ such that

$$\frac{g_k(x, x^0, A_k)}{\|\tilde{x} - \tilde{x^0}\|_{p^*}} \geq \phi_m(p) > \|A'\|_{pq} \geq \|A'_k\|_p,$$

due to (1). Using (3) we conclude that for any $k \in N_m$ there exists $x^0 \neq x$ such that

$$g_k(x, x^0, A_k + A'_k) = g_k(x, x^0, A_k) + g_k(x, x^0, A'_k) \geq$$

$$g_k(x, x^0, A_k) - \|A'_k\|_p \|\tilde{x} - \tilde{x^0}\|_{p^*} > 0,$$

i.e. $x \notin C_m(A + A')$. Thus, any solution that is not extremum in the problem $Z_m(A)$, so stays in the problem $Z_m(A+A')$. Then we conclude that for any perturbing matrix $A' \in \Omega_{pq}(\phi_m(p))$ the inclusion holds $\emptyset \neq C_m(A + A') \subseteq C_m(A)$. It implies that $C_m(A + A') \cap C_m(A) \neq \emptyset$ for any $A' \in \Omega_{pq}(\phi_m(p))$, and hence $\rho_m(p,q) \geq \phi_m(p)$.

Further, we prove that $\rho_m(p,q) \geq \psi_m(p)$. Since the formula

$$\exists x' \in C_m(A) \quad \exists k \in N_m \ \ \forall x \notin C_m(A) \ \ (C_k(x - x') > 0)$$

is true, the inequality $\psi_m(p) > 0$ is also evident.

Let $A' \in \Omega_{pq}(\psi_m(p))$ be a perturbing matrix with cuts $A'_k \in \mathbf{R}^{n\times n}$, $k \in N_m$. Then according to the definition of the number $\psi_m(p)$, there exist index $s \in N_m$ and solution $x^0 \in C_m(A)$ such that for any solution $x \notin C_m(A)$ we have

$$\frac{g_s(x, x^0, A_s)}{\|\tilde{x} - \tilde{x^0}\|_{p^*}} \geq \psi_m(p) > \|A'\|_{pq} \geq \|A'_s\|_p,$$

due to (1). Using (3), we conclude that for any $x \notin C_m(A)$ and any $A' \in \Omega_{pq}(\psi_m(p))$ the following inequalities hold

$$g_s(x, x^0, A_s + A'_s) = g_s(x, x^0, A_s) + g_s(x, x^0, A'_s) \geq$$

$$g_s(x, x^0, A_k) - \|A'_s\|_p \|\tilde{x} - \tilde{x^0}\|_{p^*} > 0.$$

Therefore,

$$\Big(X \backslash C_m(A)\Big) \cap C_s(x^0, A_s + A'_s) = \emptyset,$$

where

$$C_s(x^0, A_s + A'_s) = \{x \in X : \ g_s(x^0, x, A_s + A'_s) > 0\}.$$

Thus, any solution that is not extremum in the problem $Z_m(A)$ so stays in the problem $Z_m(A + A')$. Then we conclude that for any perturbing matrix $A' \in \Omega_{pq}(\psi_m(p))$ the following inequality holds

$$C_m(A + A') \cap C_m(A) \neq \emptyset,$$

and hence $\rho_m(p,q) \geq \psi_m(p)$.

Further we will consider the problem $Z_m(A)$ with Boolean variables ($X \subseteq \mathbf{E}^n$). And we demonstrate that $\gamma_m(p,q) \geq \rho_m(p,q)$. According to the definition of number $\gamma_m(p,q)$, there exists a Boolean solution $x^0 =$

$(x_1^0, x_2^0, ..., x_n^0) \notin C_m(A) \subseteq \mathbf{E}^n$ such that for any extremum solution $x \in C_m(A)$ and any index $k \in N_m$ we get

$$\gamma_m(p,q)\|\tilde{x^0} - \tilde{x}\|_1 \geq n^{\frac{2}{p}} m^{\frac{1}{q}} g_k(x^0, x, A_k). \tag{7}$$

Setting $\varepsilon > \gamma_m(p,q)$, we define the elements a_{ijk}^0 of any cut A_k^0, $k \in N_m$, of the perturbing matrix A^0 according to the formula

$$a_{ijk}^0 = \begin{cases} -\delta & \text{if } x_i^0 x_j^0 = 1, \\ \delta & \text{if } x_i^0 x_j^0 = 0, \end{cases}$$

where

$$\gamma_m(p,q) < \delta n^{\frac{2}{p}} m^{\frac{1}{q}} < \varepsilon. \tag{8}$$

Then according to (4) and (5), we get

$$\|A_s^0\|_p = \delta n^{\frac{2}{p}},$$

$$\|A^0\|_{pq} = \delta n^{\frac{2}{p}} m^{\frac{1}{q}},$$

$$A^0 \in \Omega_{pq}(\varepsilon).$$

In addition, due to the construction of matrix A_k^0, for any solution $x \neq x^0$ we have

$$g_k(x^0, x, A_k^0) = (x^0 - x)^T A_k^0 (x^0 - x) =$$

$$\sum_{i \in N_n} \sum_{j \in N_n} a_{ijk}^0 (x_i^0 x_j^0 - x_i x_j) = -\delta \|\tilde{x^0} - \tilde{x}\|_1. \tag{9}$$

Using (7), (8) and (9), we continue

$$g_k(x^0, x, A_k + A_k^0) = g_k(x^0, x, A_k) + g_k(x^0, x, A_k^0) \leq$$

$$\Big(\gamma_m(p,q)(n^{\frac{2}{p}} m^{\frac{1}{q}})^{-1} - \delta\Big)\|\tilde{x^0} - \tilde{x}\|_1 < 0.$$

Thus, $x \notin C_m(A + A^0)$ when $x \in C_m(A)$. Summarizing, for any $\varepsilon > \gamma_m(p,q)$, we can guarantee the existence of the perturbing matrix $A^0 \in \Omega_{pq}(\varepsilon)$ such that

$$C_m(A + A^0) \cap C_m(A) = \emptyset,$$

i.e. $\rho_m(p.q) < \varepsilon$ for any number $\varepsilon > \gamma_m(p,q)$. So, inequality (6) holds. □

From the Theorem we get the following result.

Corollary 1 If $Z_m(A)$, $A \in \mathbf{R}^{m\times n}$, is a non-trivial problem with Boolean variables, i.e. if $C_m(A) \neq X \subseteq \mathbf{E}^n$, then for any $m \in \mathbf{N}$.

$$0 < \max\{\phi, \psi\} \leq \rho_m(\infty, \infty) \leq \gamma,$$

where

$$\phi = \min_{x \notin C_m(A)} \min_{k \in N_m} \max_{x' \in X \setminus \{x\}} \frac{g_k(x, x', A_k)}{\|\tilde{x} - \tilde{x'}\|_1},$$

$$\psi = \max_{x' \in C_m(A)} \max_{k \in N_m} \min_{x \notin C_m(A)} \frac{g_k(x, x', A_k)}{\|\tilde{x} - \tilde{x'}\|_1},$$

$$\gamma = \min_{x \notin C_m(A)} \max_{k \in N_m} \max_{x' \in C_m(A)} \frac{g_k(x, x', A_k)}{\|\tilde{x} - \tilde{x'}\|_1}.$$

Corollary 2 If $Z_1(A)$, $A \in \mathbf{R}^n$, is a scalar non-trivial problem with Boolean variables ($X \subseteq \mathbf{E}^n$), then the following formula holds

$$\rho_1(\infty, q) = \min_{x \notin C_1(A)} \max_{x' \in X \setminus \{x\}} \frac{g(x, x', A)}{\|\tilde{x} - \tilde{x'}\|_1}.$$

Finally, we notice that Corollary 2 proves the attainability of $\phi_m(\rho)$ and $\gamma_m(p,q)$ when $m = 1$ and $p = \infty$.

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