An Algebraic Geometric Approach to Multidimensional Words

Jarkko Kari and Michal Szabados

Department of Mathematics and Statistics,
University of Turku, 20014 Turku, Finland
jkari@utu.fi

Abstract. We apply linear algebra and algebraic geometry to study infinite multidimensional words of low pattern complexity. By low complexity we mean that for some finite shape, the number of distinct subpatterns of that shape that occur in the word is not more than the size of the shape. We are interested in discovering global regularities and structures that are enforced by such low complexity assumption. We express the word as a multivariate formal power series over integers. We first observe that the low pattern complexity assumption implies that there is a non-zero polynomial whose formal product with the power series is zero. We call such polynomials the annihilators of the word. The annihilators form an ideal, and using Hilbert's Nullstellensatz we construct annihilators of simple form. In particular, we prove a decomposition of the word as a sum of finitely many periodic power series. We consider in more details a particular interesting example of a low complexity word whose periodic decomposition contains necessarily components with infinitely many distinct coefficients. We briefly discuss applications of our technique in the Nivat's conjecture and the periodic tiling problem. The results reported here have been first discussed in a paper that we presented at ICALP 2015.

1 Introduction

A *multidimensional infinite word*, or simply a *configuration*, $c \in A^{\mathbb{Z}^d}$ is a d-dimensional infinite array filled with symbols from a (usually finite) *alphabet* A. For each *cell* $\boldsymbol{v} \in \mathbb{Z}^d$, we denote by $c_{\boldsymbol{v}} \in A$ the symbol in position $\boldsymbol{v}$. Suppose that for some finite observation window $D \subseteq \mathbb{Z}^d$, the number of distinct patterns of shape D that exist in c is small, at most the cardinality $|D|$ of D. We investigate global regularities and structures in c that are enforced by such low local complexity assumption.

Suppose that the alphabet A is a subset of $\mathbb{Z}$. This can be established by renaming the symbols if A is finite. It is then possible to perform arithmetics on configurations; for example the sum of two configurations is defined cell wise. The main result that we report (Theorem 3) is that c can be expressed as a finite sum $c = c_1 + \cdots + c_m$ of *periodic* $c_1, \ldots, c_m \in \mathbb{Z}^{(\mathbb{Z}^d)}$. Recall that a configuration e is called *periodic* if it is invariant under some translation, so that there is a vector $\boldsymbol{u} \in \mathbb{Z}^d \setminus \{0\}$ such that $\forall \boldsymbol{v} \in \mathbb{Z}^d : e_{\boldsymbol{v}} = e_{\boldsymbol{v}+\boldsymbol{u}}$. Note that the periodic components c_i in the decomposition $c = c_1 + \cdots + c_m$ are not necessarily over any finite alphabet, but they are allowed to contain infinitely many distinct integer values. After the main result we present and analyze an example of a low local complexity configuration c over two letters, whose periodic decomposition uses *necessarily* an infinite alphabet. Finally, we briefly discuss applications of our results on two open problems: Nivat's conjecture [Niv97] and the periodic tiling problem [LW96].

To prove our main Theorem 3 we proceed in two steps.

(1) We show how the low complexity assumption on c implies that there is a non-trivial *filter* that annihilates c to the zero configuration. The filtering operation is the usual convolution of c with a finite mask, which we conveniently express in terms of multiplication by a multivariate polynomial. This step is based on basic linear algebra.
(2) We analyze configurations annihilated by non-trivial filtering, that is, by multiplying them with some non-zero polynomial. The set of annihilating polynomials is an ideal of the polynomial ring. Using Hilbert's Nullstellensatz we show that the annihilator ideal contains polynomials of simple form. In particular, we show that the configuration can be annihilated by a product of difference filters $(X^{\boldsymbol{v}} - 1)$ that subtract from a configuration its translated copy. This in turn implies a decomposition of the configuration into a sum of periodic components.

The result reported here have been presented in [KS15], except for the proofs related to the example in Section 5.

2 Preliminaries

Classically, configurations are just assignments $c : \mathbb{Z}^d \longrightarrow A$ of symbols of a (finite or infinite) alphabet A on an infinite grid. We use the subscript notation $c_{\boldsymbol{v}}$ for the symbol assigned in cell $\boldsymbol{v} \in \mathbb{Z}^d$. In order to apply algebra it is convenient to let the symbols in A be numbers, and to represent c as a *formal power series* over d variables $x_1, \ldots, x_d$ and with coefficients in A:

$$c(x_1, \ldots, x_d) = \sum_{v_1=-\infty}^{\infty} \cdots \sum_{v_d=-\infty}^{\infty} c_{v_1,\ldots,v_d} x_1^{v_1} \ldots x_d^{v_d}.$$

As usual, we abbreviate the vector $(x_1, \ldots, x_d)$ of variables as X, and write monomial $x_1^{v_1} \ldots x_d^{v_d}$ as $X^{\boldsymbol{v}}$ for $\boldsymbol{v} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$. Configuration c can now be expressed compactly as

$$c(X) = \sum_{\boldsymbol{v} \in \mathbb{Z}^d} c_{\boldsymbol{v}} X^{\boldsymbol{v}}. \tag{1}$$

Usually we let $A \subseteq \mathbb{Z}$ so that configurations are power series with integer coefficients, but to use Nullstellensatz we need an algebraically closed field, so that frequently we consider multivariate power series and polynomials over $\mathbb{C}$. Anyway, for $R = \mathbb{Z}$ or $R = \mathbb{C}$, we denote by $R[[X^{\pm 1}]]$ the set of formal power series as in (1) with coefficients $c_{\boldsymbol{v}}$ in domain R. Note that we include negative exponents in the series. We call power series (1) *integral* if all coefficients $c_{\boldsymbol{v}}$ are integers, and it is *finitary* if there are only finitely many distinct coefficients $c_{\boldsymbol{v}}$. In our usual setup $A \subseteq \mathbb{Z}$ is finite so that the corresponding power series is finitary and integral.

A *polynomial* over R is a formal sum $a(X) = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$ where $a_{\boldsymbol{v}} \in R$ and the sum is over a finite set of d-tuples $\boldsymbol{v} = (v_1, \ldots, v_d)$ with non-negative coordinates $v_i \geq 0$. If the coordinates are also allowed be negative we get a *Laurent polynomial* over R. We denote by $R[X]$ and $R[X^{\pm 1}]$ the sets of polynomials and Laurent polynomials over R. We sometimes use the term *proper* polynomial when we want to emphasize that $a(X)$ is a polynomial and not only a Laurent polynomial.

Here are some notational remarks: We use both notations $a(X)$ and a to denote (Laurent) polynomials and power series, that is, we may or may not explicitly write the formal variable in the notation. For any formal polynomial, Laurent polynomial or power series a we denote by $a_{\boldsymbol{v}}$ the value in cell $\boldsymbol{v}$, that is, the coefficient of monomial $X^{\boldsymbol{v}}$. Sometimes we may wish to write the coefficients explicitly differently, e.g., we may write $f(X) = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$.

The *support* of a polynomial or a Laurent polynomial $a(X)$ is the set

$$\text{supp}(a) = \{\boldsymbol{v} \in \mathbb{Z}^d \mid a_{\boldsymbol{v}} \neq 0\} \tag{2}$$

of cells with non-zero value.

The formal product between a power series and a (Laurent) polynomial is defined the usual way, as a convolution. This is a filtering operation, and the result is again a power series. Note that multiplying a power series with monomial $X^{\boldsymbol{v}}$ is equivalent to translating it by the vector $\boldsymbol{v}$. It follows that power series $c(X)$ is *periodic* with period $\boldsymbol{v}$ if and only if $(X^{\boldsymbol{v}} - 1)c = 0$. We say that $(X^{\boldsymbol{v}} - 1)$ annihilates $c(X)$.

3 Step 1: From Low Local Complexity to an Annihilating Filter

We are studying configurations in which the number of distinct patterns of some finite shape D is at most the size $|D|$ of the shape. More precisely, for any finite $D \subseteq \mathbb{Z}^d$ we denote by π_D the *projection* operator on $R[[X^{\pm 1}]]$ defined by

$$\pi_D(c) = \sum_{\boldsymbol{v} \in D} c_{\boldsymbol{v}} X^{\boldsymbol{v}},$$

and define the D-patterns of c to be the elements of

$$\text{Patt}_D(c) = \{\pi_D(X^{\boldsymbol{u}} c) \mid \boldsymbol{u} \in \mathbb{Z}^d\ \}.$$

Configuration c has *low complexity* with respect to a finite $D \subseteq \mathbb{Z}^d$ if

$$|\operatorname{Patt}_D(c)| \leq |D|, \tag{3}$$

and we say that c has low complexity if (3) is satisfied for some finite D.

We say that a Laurent polynomial $f(X)$ *annihilates* configuration $c(X)$ if $f(X)c(X) = 0$. The following lemma guarantees that each low complexity configuration is annihilated by some non-zero Laurent polynomial, and hence also by a non-zero proper polynomial.

Lemma 1. *Let R be a field or $R = \mathbb{Z}$. Let $c(X) \in R[[X^{\pm 1}]]$ be a configuration and $D \subset \mathbb{Z}^d$ a finite set such that $|\operatorname{Patt}_D(c)| \leq |D|$. Then there exists a non-zero polynomial $f(X) \in R[X]$ such that $f(X)c(X) = 0$.*

Proof. Let R be a field. We use elementary linear algebra. Let $D = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_n}\}$. By the low complexity assumption, the set

$$\{(1, c_{\boldsymbol{u_1}+\boldsymbol{v}}, \ldots, c_{\boldsymbol{u_n}+\boldsymbol{v}}) \mid \boldsymbol{v} \in \mathbb{Z}^d\}$$

of vectors in R^{n+1} contains at most $n = |D|$ elements. There exists hence a non-zero vector $(a_0, a_1, \ldots, a_n)$ orthogonal to the set. Consider the product of $c(X)$ and the Laurent polynomial $g(X) = a_1 X^{-\boldsymbol{u_1}} + \cdots + a_n X^{-\boldsymbol{u_n}}$. In any position $\boldsymbol{v}$, the coefficient in the product $g(X)c(X)$ is

$$a_1 c_{\boldsymbol{u_1}+\boldsymbol{v}} + \cdots + a_n c_{\boldsymbol{u_n}+\boldsymbol{v}} = -a_0.$$

Hence the product is a constant configuration, so that $(X^{\boldsymbol{v}} - 1)g(X)c(X) = 0$ for any $\boldsymbol{v}$. We conclude that $c(X)$ is annihilated by all non-zero Laurent polynomials $h(X) = (X^{\boldsymbol{v}} - 1)g(X)$.

To obtain a non-zero proper polynomial that annihilates c, notice that if $h(X)$ is an annihilator of $c(X)$, so is $a(X)h(X)$ for any Laurent polynomial $a(X)$. In particular, by choosing $a(X) = X^{\boldsymbol{u}}$ for $\boldsymbol{u} \in \mathbb{Z}^d$ with sufficiently large coordinates, we have that $f(X) = X^{\boldsymbol{u}} h(X) \in R[X]$ is a polynomial.

Consider then the case $R = \mathbb{Z}$. By the proof above (for $R = \mathbb{Q}$) we see that there exists a non-zero polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(X)c(X) = 0$. There is a positive integer m such that $m \cdot f(X) \in \mathbb{Z}[X]$, so that $m \cdot f(X)$ satisfies the claim.

□

As a first application of this simple observation we infer the classical Morse-Hedlund theorem [MH38]. Consider the case $d = 1$, and hence a one-variable configuration $c(x) \in \mathbb{C}[[x^{\pm 1}]]$ that satisfies the low complexity assumption. By Lemma 1, there is a (one variable) polynomial $f(x)$ that annihilates $c(x)$. Multiplying by a suitable monomial, we can take an annihilating $f(x)$ with the constant term one:

$$f(x) = 1 + a_1 x + a_2 x^2 + \ldots a_n x^n.$$

Now $f(x)c(x) = 0$ means that, for all $i \in \mathbb{Z}$,

$$c_i = a_1 c_{i-1} + a_2 c_{i-2} + \cdots + a_n c_{i-n},$$

so that the symbol in position i is determined by the n symbols on its left. A deterministic process on a finite set is necessarily periodic, so clearly c has to be a periodic configuration. We have established

Theorem 1 (Morse, Hedlund 1938). *If a one-dimensional bi-infinite word contains at most n distinct subwords of length n then the word is periodic.*

4 Step 2: From an Annihilating Filter to a Periodic Decomposition

Let c be a configuration. We define

$$\mathrm{Ann}(c) = \left\{ f \in \mathbb{C}[X] \;\middle|\; fc = 0 \right\}$$

to be the set of polynomials that annihilate it. Note that $\mathrm{Ann}(c)$ contains proper polynomials only. Note also that we take complex polynomials so that we can apply Hilbert's Nullstellensatz that requires an algebraically closed field.

It is easy to see that $\mathrm{Ann}(c)$ is an ideal of the polynomial ring $\mathbb{C}[X]$, the *annihilator ideal* of configuration c. We always have $0 \in \mathrm{Ann}(c)$ where 0 is the zero polynomial with zero coefficients. If $\mathrm{Ann}(c) = \{0\}$ then the annihilator ideal is *trivial*; if $\mathrm{Ann}(c)$ contains also some non-zero polynomial then it is *non-trivial*. By Lemma 1, the annihilator ideal of a low complexity configuration is always non-trivial. It is also easy to see that if c is integral and $\mathrm{Ann}(c)$ is non-trivial then $\mathrm{Ann}(c)$ contains a non-zero polynomial from $\mathbb{Z}[X]$, that is, a polynomial with integer coefficients.

More generally, if $\mathcal{C}$ is a set of configurations (e.g., a subshift), we let

$$\mathrm{Ann}(\mathcal{C}) = \left\{ f \in \mathbb{C}[X] \;\middle|\; fc = 0 \text{ for all } c \in \mathcal{C} \right\}$$

be the set of common annihilators. Again, $\mathrm{Ann}(\mathcal{C})$ is an ideal of the polynomial ring.

If $Z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ is a complex vector then it can be plugged into a polynomial, producing a complex value. In particular, plugging into a monomial $X^{\boldsymbol{v}}$ results in $Z^{\boldsymbol{v}} = z_1^{v_1} \cdots z_d^{v_d}$.

In this section we use Hilbert's Nullstellensatz as a tool to infer other elements of the ideal $\mathrm{Ann}(c)$. Recall the statement of the Nullstellensatz: Suppose $g(X)$ is a polynomial such that $g(Z) = 0$ for all common roots Z of $\mathrm{Ann}(c)$, that is, for all $Z \in \mathbb{C}^d$ such that $f(Z) = 0$ for all $f \in \mathrm{Ann}(c)$. Then $g^n(X) \in \mathrm{Ann}(c)$ for some n.

First we show that annihilating integral polynomials can be spatially "blown-up":

Lemma 2. *Let $c(X)$ be a finitary integral configuration and $f(X) \in \mathrm{Ann}(c)$ a non-zero integral polynomial, that is, $f(X) \in \mathrm{Ann}(c) \cap \mathbb{Z}[X]$. Then there exists an integer r such that for every positive integer n relatively prime to r we have $f(X^n) \in \mathrm{Ann}(c)$.*

Proof. Denote $f(X) = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$. First we prove the claim for the case when n is a large enough prime.

Let p be a prime, then we have $f^p(X) \equiv f(X^p) \pmod{p}$. Because f annihilates c, multiplying both sides by $c(X)$ results in

$$0 \equiv f(X^p)c(X) \pmod{p}.$$

The coefficients in $f(X^p)c(X)$ are bounded in absolute value by

$$s = c_{max} \sum |a_{\boldsymbol{v}}|,$$

where c_{max} is the maximum absolute value of coefficients in c. Therefore if $p > s$ we have $f(X^p)c(X) = 0$.

For the general case, set $r = s!$. Now every n relatively prime to r is of the form $p_1 \cdots p_k$ where each p_i is a prime greater than s. Note that we can repeat the argument with the same bound s also for polynomials $f(X^m)$ for arbitrary m – the bound s depends only on c and the (multi)set of coefficients $a_{\boldsymbol{v}}$, which is the same for all $f(X^m)$. Thus we have $f(X^{p_1 \cdots p_k}) \in \mathrm{Ann}(c)$. □

The next lemma establishes a polynomial $g(X)$ of simple form that becomes zero at all common roots of $\mathrm{Ann}(c)$:

Lemma 3. *Let c be a finitary integral configuration and $f(X) = \sum a_{\boldsymbol{v}} X^{\boldsymbol{v}}$ a non-trivial integral polynomial annihilator. Let $S = \mathrm{supp}(f)$ be the support of $f(X)$. Define*

$$g(X) = x_1 \cdots x_d \prod_{\substack{\boldsymbol{v} \in S \\ \boldsymbol{v} \neq \boldsymbol{v_0}}} (X^{r\boldsymbol{v}} - X^{r\boldsymbol{v_0}})$$

where r is the integer from Lemma 2 and $\boldsymbol{v_0} \in S$ arbitrary. Then $g(Z) = 0$ for any common root $Z \in \mathbb{C}^d$ of $\mathrm{Ann}(c)$.

Proof. Fix Z such that $h(Z) = 0$ for all $h \in \mathrm{Ann}(c)$. If any of its complex coordinates is zero then clearly $g(Z) = 0$. For this reason we included $x_1 \cdots x_d$ as a factor of $g(X)$.

Assume then that all coordinates of Z are non-zero. Let us define for $\alpha \in \mathbb{C}$

$$S_\alpha = \{ \boldsymbol{v} \in S \mid Z^{r\boldsymbol{v}} = \alpha \},$$
$$f_\alpha(X) = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} X^{\boldsymbol{v}}.$$

Because S is finite, there are only finitely many non-empty sets $S_{\alpha_1}, \ldots, S_{\alpha_m}$ and they form a partitioning of S. In particular we have $f = f_{\alpha_1} + \cdots + f_{\alpha_m}$.

Numbers of the form $1 + ir$ are relatively prime to r for all non-negative integers i, therefore by Lemma 2, $f(X^{1+ir}) \in \mathrm{Ann}(c)$. Plugging in Z we obtain $f(Z^{1+ir}) = 0$. Now compute:

$$f_\alpha(Z^{1+ir}) = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} Z^{(1+ir)\boldsymbol{v}} = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} Z^{\boldsymbol{v}} \alpha^i = f_\alpha(Z)\alpha^i$$

Summing over $\alpha = \alpha_1, \dots, \alpha_m$ gives

$$0 = f(Z^{1+ir}) = f_{\alpha_1}(Z)\alpha_1^i + \cdots + f_{\alpha_m}(Z)\alpha_m^i.$$

Let us rewrite the last equation as a statement about orthogonality of two vectors in $\mathbb{C}^m$:

$$(f_{\alpha_1}(Z), \dots, f_{\alpha_m}(Z)) \perp (\alpha_1^i, \dots, \alpha_m^i)$$

By Vandermode determinant, for $i \in \{0, \dots, m-1\}$ the vectors on the right side span the whole $\mathbb{C}^m$. Therefore the left side must be the zero vector, and especially for α such that $\boldsymbol{v_0} \in S_\alpha$ we have

$$0 = f_\alpha(Z) = \sum_{\boldsymbol{v} \in S_\alpha} a_{\boldsymbol{v}} Z^{\boldsymbol{v}}.$$

Because Z does not have zero coordinates, each term on the right hand side is non-zero. But the sum is zero, therefore there are at least two vectors $\boldsymbol{v_0}, \boldsymbol{v} \in S_\alpha$. From the definition of S_α we have $Z^{r\boldsymbol{v}} = Z^{r\boldsymbol{v_0}} = \alpha$, so Z is a root of $X^{r\boldsymbol{v}} - X^{r\boldsymbol{v_0}}$. □

Now we are ready to apply the Nullstellensatz to obtain a simple annihilator:

Theorem 2. *Let c be a finitary integral configuration with a non-trivial annihilator. Then there are non-zero $\boldsymbol{v_1}, \dots, \boldsymbol{v_m} \in \mathbb{Z}^d$ such that the Laurent polynomial*

$$(X^{\boldsymbol{v_1}} - 1) \cdots (X^{\boldsymbol{v_m}} - 1)$$

annihilates c.

Proof. This is an easy corollary of Lemma 3. First notice that the non-trivial annihilator can be taken so that it has integer coefficients. The polynomial $g(X)$ provided by Lemma 3 vanishes on all common roots of $\mathrm{Ann}(c)$, therefore by Hilbert's Nullstellensatz there is n such that $g^n(X) \in \mathrm{Ann}(c)$. Note that any monomial multiple of an annihilator is again an annihilator. Therefore also

$$\frac{g^n(X)}{x_1^n \cdots x_d^n X^{nr\boldsymbol{v_0}(|S|-1)}}$$

is, and it is a Laurent polynomial of the desired form. □

Multiplying a configuration by $(X^{\boldsymbol{v}} - 1)$ is a "difference operator" on the configuration. Theorem 2 then says that there is a sequence of difference operators which annihilates the configuration. We can reverse the process: let us start by the zero configuration and step by step "integrate" until we obtain the original configuration. This idea gives the Decomposition theorem:

Theorem 3 (Decomposition theorem [KS15]). *Let c be a finitary integral configuration with a non-trivial annihilator. Then there exist periodic integral configurations $c_1, \dots, c_m$ such that $c = c_1 + \cdots + c_m$.*

5 An Example

In this section we illustrate how the theory applies to a concrete example. Its properties were briefly mentioned in [KS15], without proofs. Recall that configurations are not assumed to be finitary or integral unless explicitly stated so.

Fix $\alpha \in \mathbb{R}$ irrational and define two-dimensional configurations $c^{(1)}, c^{(2)}, c^{(3)}$ and s by

$$c^{(1)}_{ij} = -\lfloor i\alpha \rfloor, \qquad c^{(2)}_{ij} = -\lfloor j\alpha \rfloor, \qquad c^{(3)}_{ij} = \lfloor (i+j)\alpha \rfloor,$$

$$s = c^{(1)} + c^{(2)} + c^{(3)}.$$

Then s is a finitary integral configuration over the alphabet $\{0, 1\}$. Obviously, $c^{(1)}, c^{(2)}, c^{(3)}$ are periodic in directions $(0, 1)$, $(1, 0)$, $(-1, 1)$ respectively, but they are not finitary. In the following we prove that s cannot be expressed as a finite sum of finitary periodic configurations.

There is a certain symmetry in s which becomes apparent when the configuration is affinely transformed such that these three directions become symmetric. In that case, it is natural to show the coefficients in a hexagonal grid, see Figure 1.

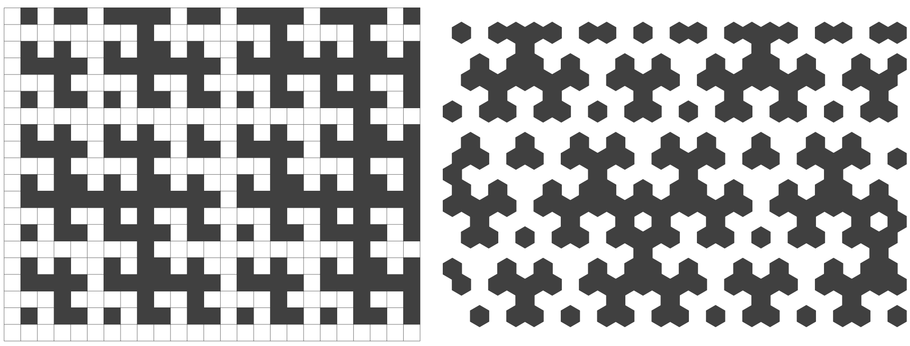

Fig. 1: The configuration s from Section 5 when α is the golden ratio is shown on the left. On the right the configuration is skewed such that the three directions $(0, 1)$, $(1, 0)$ and $(1, -1)$ became symmetrical, the bottom left corner is preserved.

For any Laurent polynomials $f_1, \dots, f_n \in \mathbb{C}[X^{\pm 1}]$, we let

$$\langle f_1, \dots, f_n \rangle = \{g_1 f_1 + \cdots + g_n f_n \mid g_1, \dots, g_n \in \mathbb{C}[X^{\pm 1}]\}$$

be the Laurent polynomial ideal they generate. Note that in this notation we let all involved polynomials be Laurent so that this is not a polynomial ideal. For Laurent polynomials $f(X)$ and $g(X)$, we denote $f \equiv g \bmod \langle f_1, \ldots, f_n \rangle$ if and only if $f(X) - g(X) \in \langle f_1, \ldots, f_n \rangle$.

A (Laurent) polynomial $a(X)$ is called a *line (Laurent) polynomial* if the support $\mathrm{supp}(a)$ defined by (2) contains at least two points and all the points of the support lie on a single line. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^d$ are such that $\{\boldsymbol{u} + t\boldsymbol{v} \mid t \in \mathbb{R}\}$ contains the support of a line (Laurent) polynomial $a(X)$ then we say that $\boldsymbol{v}$ is a *direction* of $a(X)$. By *rational directions* we mean elements of $\mathbb{Z}^d \setminus \{0\}$. We say that two line (Laurent) polynomials are *parallel* if they have the same directions.

Let configuration $c \in \mathbb{C}[[X^{\pm 1}]]$ be such that $\mathrm{Ann}(c)$ contains a line polynomial. We call such c *directed*. This terminology applies to both finitary and non-finitary configurations. Notice that for any line Laurent polynomial that annihilates c there is a parallel line polynomial in $\mathrm{Ann}(c)$, obtained by multiplying it with a monomial. If all line polynomials in $\mathrm{Ann}(c)$ are parallel to each other, we say that c is *one-directed*, and if c has non-parallel annihilating line polynomials we say that c is *multi-directed*. Non-finitary configurations can be directed without being periodic, but if c is finitary then it is one-directed if and only if it is periodic in one direction only, and it is multi-directed if and only if it has several directions of periodicity. In the two-dimensional setting $d = 2$, such configurations are sometimes called singly periodic and doubly periodic, respectively.

It is well known that a doubly periodic configuration is periodic in all rational directions. An analogous statement holds more generally for two-dimensional directed configurations:

Lemma 4. *If $a(X)$ and $b(X)$ are non-parallel two-dimensional line Laurent polynomials then $\langle a, b \rangle$ contains line Laurent polynomials in all rational directions. In particular, in two dimensions, if $\mathrm{Ann}(c)$ contains two non-parallel line polynomials then it contains a line polynomial in every rational direction.*

Proof. The proof is easy using simple algebraic geometry and zero dimensionality of $\langle a, b \rangle$. Here we give it as an elementary linear algebraic reasoning. It is easy to see that there is a finite domain $D \subseteq \mathbb{Z}^2$ (a parallelogram determined by the supports of a and b) such that for any Laurent polynomial f there is a Laurent polynomial $f' \equiv f \bmod \langle a, b \rangle$ with support $\mathrm{supp}(f') \subseteq D$.

Let $\boldsymbol{u} \in \mathbb{Z}^2 \setminus \{(0,0)\}$ be any rational direction. Consider the monomials $X^0, X^{\boldsymbol{u}}, X^{2\boldsymbol{u}}, \ldots$ and, for each $k = 0, 1, 2 \ldots$, let $f_k(X) \equiv X^{k\boldsymbol{u}} \bmod \langle a, b \rangle$ be the representative with $\mathrm{supp}(f_k) \subseteq D$. It follows from the finiteness of the support that $f_1, f_2, \ldots$ are linearly dependant, and hence there is a non-zero vector $(a_0, a_1, \ldots, a_n)$ of coefficients such that $a_0 f_0(X) + \cdots + a_n f_n(X) = 0$. But then $f(X) = a_0 X^0 + a_1 X^{\boldsymbol{u}} + \cdots + a_n X^{n\boldsymbol{u}}$ is in $\langle a, b \rangle$. If $f(X)$ is a monomial then $1 \in \langle a, b \rangle$ and hence $\langle a, b \rangle$ contains all Laurent polynomials. Otherwise $f(X)$ has at least two non-zero coefficients and it is then a line Laurent polynomial in direction $\boldsymbol{u}$. $\square$

The next lemma states that one-directed configurations in different directions are linearly independent.

Lemma 5. *Let $c_1(X), \ldots, c_n(X)$ be two-dimensional configurations that are one-directed and pairwise non-parallel. Then $a_1, \ldots, a_n \in \mathbb{C}$ satisfy $a_1c_1(X) + \cdots + a_nc_n(X) = 0$ if and only if $a_1 = \cdots = a_n = 0$.*

Proof. We prove the claim by induction on n. Case $n = 1$: since $c_1(X)$ is one-directed, it is not the zero power series. Hence $a_1c_1(X) = 0$ if and only if $a_1 = 0$.

Suppose then the claim has been proved for $n - 1$, and let $a_1, \ldots, a_n$ be such that

$$a_1c_1(X) + \cdots + a_nc_n(X) = 0. \tag{4}$$

Because $c_n(X)$ is one-directed it is annihilated by some line Laurent polynomial $a(X)$. We multiply (4) by $a(X)$.

Let $1 \leq i \leq n - 1$ and consider $a(X)c_i(X)$. It is annihilated by the same line Laurent polynomial that annihilates $c_i(X)$ so it is directed. If it were multi-directed then, by Lemma 4, it would be annihilated by some line Laurent polynomial $b(X)$ that is parallel to $a(X)$. Then $c_i(X)$ would be annihilated by the line Laurent polynomial $a(X)b(X)$ that is parallel to $a(X)$, a contradiction with the fact that $c_i(X)$ and $c_n(X)$ are one-directed in different directions. We conclude that $a(X)c_i(X)$ is one-directed in the same direction as $c_i(X)$.

Multiplying (4) by $a(X)$ implies that

$$a_1a(X)c_1(X) + \cdots + a_{n-1}a(X)c_{n-1}(X) = 0.$$

By the inductive hypothesis, $a_1 = \cdots = a_{n-1} = 0$. Case $n = 1$ applied to $a_nc_n(X) = 0$ shows that also $a_n = 0$. □

Now we are ready to analyze the configuration $s = c^{(1)} + c^{(2)} + c^{(3)}$ defined at the beginning of this section. We want to show that it is not a sum of finitely many periodic finitary configurations. Suppose the contrary: $c^{(1)} + c^{(2)} + c^{(3)} = f_1 + \cdots + f_n$ for some periodic finitary $f_i(X)$. By moving the terms on the same side, and combining terms that are directed in the same direction, we obtain that

$$(c^{(1)} + p_1) + (c^{(2)} + p_2) + (c^{(3)} + p_3) + p_4 + \cdots + p_m = 0, \tag{5}$$

for some directed finitary $p_i(X)$ with the following properties:

- Configurations $p_1(X), p_2(X)$ and $p_3(X)$ have line Laurent polynomial annihilators in the same directions $(0, 1)$, $(1, 0)$ and $(-1, 1)$ as $c^{(1)}(X), c^{(2)}(X)$ and $c^{(3)}(X)$, respectively. They may have line annihilators also in other directions so that any doubly periodic $f_i(X)$ in the original bounded periodic decomposition may be added in them.
- Configurations $p_4(X), \ldots, p_m(X)$ are one-directed in pairwise non-parallel directions. These directions are also not parallel to the directions $(0, 1)$, $(1, 0)$ and $(-1, 1)$ of the line annihilators of $c^{(1)}(X), c^{(2)}(X)$ and $c^{(3)}(X)$.

Lemma 6. *In* (5)*, configurations $c^{(k)} + p_k$ are one-directed, for $k = 1, 2, 3$.*

Proof. It is clear that $c^{(k)} + p_k$ is directed in the same direction as $c^{(k)}$, so it is enough to show that it is not multi-directed. For $k = 1$ or $k = 3$ let us read the coefficients of $c^{(k)} + p_k$ horizontally along cells $\ldots, (-1, 0), (0, 0), (1, 0), \ldots,$ and in the case $k = 2$ along the vertical line $\ldots, (0, -1), (0, 0), (0, 1), \ldots$. In each case we obtain a one-dimensional configuration $d(x) = c(x) + p(x)$ with $c_i = \lfloor i\alpha \rfloor$ for all $i \in \mathbb{Z}$, and with $p(x)$ finitary. (Note that in the cases $k = 1$ and $k = 2$ we negate the coefficients to get from $-\lfloor i\alpha \rfloor$ to $\lfloor i\alpha \rfloor$.)

If $c^{(k)} + p_k$ is multi-directed then by Lemma 4 it has an annihilating line Laurent polynomial in every direction and then, in particular, in the horizontal and vertical directions. This means that the one-dimensional configuration $d(x)$ has a non-trivial annihilator $b(x)$. Then also

$$(1 - x)d(x) = c'(x) + p'(x)$$

is annihilated by $b(x)$, where $c'(x) = (1 - x)c(x)$ has coefficient $c'_i = \lfloor i\alpha \rfloor - \lfloor (i - 1)\alpha \rfloor \in \{0, 1\}$ in cell i, and also $p'(x) = (1 - x)p(x)$ is finitary. A one-dimensional finitary configuration with a non-trivial annihilator is periodic by the determinism argument we used in the proof of Theorem 1, so that $d'(x) = (1 - x)d(x)$ is n-periodic for some $n > 0$. Let $h = d'_1 + \cdots + d'_n$ be the sum over one period. Notice that $d_i - d_0 = d'_1 + \cdots + d'_i$ for all $i > 0$, so that $d_{jn} = d_0 + jh$ for all $j > 0$. As $d(x) = c(x) + p(x)$ we have

$$p_{jn} = d_{jn} - c_{jn} = d_0 + jh - \lfloor jn\alpha \rfloor. \tag{6}$$

Because $p(x)$ is finitary, there are $j_1 < j_2$ such that $p_{j_1 n} = p_{j_2 n}$. By (6) this means $(j_2 - j_1)h = \lfloor j_2 n\alpha \rfloor - \lfloor j_1 n\alpha \rfloor$, so that h is a rational number and cannot hence be equal to irrational $n\alpha$. But then, using (6) again, $\lim_{j\to\infty} p_{jn} = \pm\infty$ so that $p(x)$ cannot be finitary, a contradiction. □

Now it is clear that (5) is a non-trivial linear dependency among one-directed configurations in pairwise non-parallel directions. This is impossible by Lemma 5 so (5) cannot hold. We have proved the following result:

Theorem 4. *Let $\alpha > 0$ be irrational. The two-dimensional configuration s over the binary alphabet $\{0, 1\}$ defined by*

$$s_{ij} = \lfloor (i + j)\alpha \rfloor - \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor$$

is a sum of three periodic integral configurations but not a sum of finitely many finitary periodic configurations.

6 Conclusions and Applications

We have proved that multidimensional configurations of low local complexity can be expressed as a sum of periodic configurations. We have also demonstrated that sometimes the periodic components are necessarily non-finitary. We believe that the periodic decomposition will be useful in tackling a number of questions in multidimensional symbolic dynamics and combinatorics of words. Here we present two open problems whose setup is amenable to our approach.

Nivat's conjecture

Nivat's conjecture (proposed by M. Nivat in his keynote address in ICALP 1997 [Niv97]) claims that in the two-dimensional case $d = 2$, the low complexity assumption (3) for a *rectangle* D implies that c is periodic. The conjecture is a natural generalization of the one-dimensional Morse-Hedlund theorem that we presented as Theorem 1. In the two-dimensional setting, for $m, n \in \mathbb{N}$, let us denote by $\mathrm{Patt}_{m\times n}(c)$ the set of $m \times n$ rectangles in configuration c.

Conjecture 1 (Nivat's conjecture). If for some m, n we have $|\mathrm{Patt}_{m\times n}(c)| \leq mn$ then c is periodic.

The conjecture has recently raised wide interest, but it remains unsolved. In [EKM03] it was shown $P_c(m, n) \leq mn/144$ is enough to guarantee the periodicity of c. This bound was improved to $P_c(m, n) \leq mn/16$ in [QZ04], and recently to $P_c(m, n) \leq mn/2$ in [CK13b]. Also the cases of narrow rectangles have been investigated: it was shown in [ST02] and recently in [CK13a] that $P_c(2, n) \leq 2n$ and $P_c(3, n) \leq 3n$, respectively, imply that c is periodic.

The analogous conjecture in the higher dimensional setups is false [ST00]. The following example recalls a simple counter example for $d = 3$.

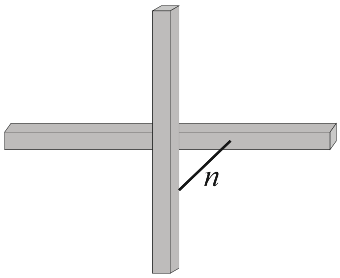

Fig. 2: A non-periodic three-dimensional configuration where two infinite stripes in orthogonal orientations are at distance n of each other. The number of distinct $n \times n \times n$ patterns in the configuration is $2n^2 + 1$.

Example 1. Fix $n \geq 3$, and consider the following $c \in \{0, 1\}^{\mathbb{Z}^3}$ consisting of two perpendicular lines of 1's on a 0-background, at distance n from each other: $c(i, 0, 0) = c(0, i, n) = 1$ for all $i \in \mathbb{Z}$, and $c(i, j, k) = 0$ otherwise. See Figure 2 for a picture of the configuration. For D equal to the $n \times n \times n$ cube we have $|\mathrm{Patt}_D(c)| = 2n^2 + 1$ since the D-patterns in c have at most a single 1-line piercing a face of the cube. Clearly c is not periodic although $2n^2+1 < n^3 = |D|$. Notice that c is the sum of two periodic configurations. Our results imply that any counter example must decompose into a sum of periodic components. □

In [KS15] we reported the following asymptotic result, using the approach discussed in the present paper. The detailed proof of the result will be published elsewhere.

Theorem 5 ([KS15]). *Let c be a two-dimensional non-periodic configuration. Then $|\mathrm{Patt}_{m\times n}(c)| > mn$ for all but finitely many pairs m, n.*

Periodic tiling problem

Another related open problem is the *periodic (cluster) tiling problem* by Lagarias and Wang [LW96]. A (cluster) tile is a finite $D \subset \mathbb{Z}^d$. Its co-tiler is any subset $C \subseteq \mathbb{Z}^d$ such that

$$D \oplus C = \mathbb{Z}^d. \tag{7}$$

The co-tiler can be interpreted as the set of positions where copies of D are placed so that they together cover the entire $\mathbb{Z}^d$ without overlaps. Note that the tile D does not need to be connected – hence the term "cluster tile" is sometimes used. The tiling is by translations of D only: the tiles may not be rotated.

It is natural to interpret any $C \subseteq \mathbb{Z}^d$ as the binary configuration $c \in \{0,1\}^{\mathbb{Z}^d}$ with $c_{\boldsymbol{v}} = 1$ if and only if $\boldsymbol{v} \in C$. Then the tiling condition (7) states that C is a co-tiler for D if and only if the $(-D)$-patterns in the corresponding configuration c contain exactly a single 1 in the background of 0's. In fact, as co-tilers of D and $-D$ coincide [Sze98], this is equivalent to all D-patterns having a single 1.

We see that the set $\mathcal{C}$ of all co-tiler configurations for D is a *subshift of finite type* [LM95]. We also see that the low local complexity assumption (3) is satisfied, even for the entire subshift of valid tilings so that $|\operatorname{Patt}_D(\mathcal{C})| \leq |D|$.

Conjecture 2 (Periodic Tiling Problem). If tile D has a co-tiler then it has a periodic co-tiler.

This conjecture was first formulated in [LW96]. In the one-dimensional case it is easily seen true, but already for $d = 2$ it is open. Interestingly, it is known that if $|D|$ is a prime number then *every* co-tiler of D is periodic [Sze98]. (See also [KS15] for an alternative proof that uses power series and polynomials.). The same is true if D is connected, that is, a polyomino [BN91].

References

[BN91] D. Beauquier and M. Nivat. "On Translating One Polyomino to Tile the Plane". In: *Discrete & Computational Geometry* 6 (1991).

[CK13a] V. Cyr and B. Kra. "Complexity of short rectangles and periodicity". In: *(submitted)* (2013). arXiv: `1307.0098 [math.DS]`.

[CK13b] V. Cyr and B. Kra. "Nonexpansive $\mathbb{Z}^2$-subdynamics and Nivat's conjecture". In: *Trans. Amer. Math. Soc. (to appear)* (2013).

[EKM03] C. Epifanio, M. Koskas, and F. Mignosi. "On a conjecture on bidimensional words". In: *Theor. Comput. Sci.* 1-3.299 (2003).

[KS15] J. Kari and M. Szabados. "An Algebraic Geometric Approach to Nivat's Conjecture". In: *ICALP*. Vol. 9135. Lecture Notes in Computer Science. Springer, 2015.

[LW96] J. C. Lagarias and Y. Wang. "Tiling the Line with Translates of One Tile". In: *Inventiones Mathematicae* 124 (1996), pp. 341–365.

[LM95] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

[MH38] M. Morse and G. A. Hedlund. "Symbolic Dynamics". In: *American Journal of Mathematics* 60.4 (1938), pp. 815–866.

[Niv97] M. Nivat. *Invited talk at ICALP, Bologna*. 1997.

[QZ04] A. Quas and L. Q. Zamboni. "Periodicity and local complexity". In: *Theor. Comput. Sci.* 319.1-3 (2004), pp. 229–240.

[ST00] J. W. Sander and R. Tijdeman. "The complexity of functions on lattices". In: *Theor. Comput. Sci.* 246.1-2 (2000), pp. 195–225.

[ST02] J. W. Sander and R. Tijdeman. "The rectangle complexity of functions on two-dimensional lattices". In: *Theor. Comput. Sci.* 270.1-2 (2002), pp. 857–863.

[Sze98] M. Szegedy. "Algorithms to Tile the Infinite Grid with Finite Clusters". In: *FOCS*. IEEE Computer Society, 1998, pp. 137–147.