Identification in $\mathbb{Z}^2$ using Euclidean balls*

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*Some of the results of this paper have been presented at the International Workshop on Coding and Cryptography 2009, WCC 2009.

[†]Research supported by the Academy of Finland under grant 210280.

Abstract

The concept of identifying codes was introduced by Karpovsky, Chakrabarty and Levitin. These codes find their application, for example, in sensor networks. The network is modelled by a graph. In this paper, the goal is to find good identifying codes in a natural setting, that is, in a graph $\mathcal{E}_r = (V, E)$ where $V = \mathbb{Z}^2$ is the set of vertices and each vertex (sensor) can check its neighbours within Euclidean distance r. We also consider a graph closely connected to a well-studied king grid, which provides optimal identifying codes for $\mathcal{E}_{\sqrt{5}}$ and $\mathcal{E}_{\sqrt{13}}$.

Keywords: Identifying code; optimal code; Euclidean distance; sensor network; fault diagnosis

1 Introduction

Let $G = (V, E)$ be a simple connected and undirected graph with V as the set of vertices and E as the set of edges. A nonempty subset of V is called a *code*, and its elements are called *codewords*. Let u and v be vertices of V. Then we say that u *covers* v if the vertices u and v are adjacent, i.e. there exists an edge between the vertices. The *ball centered at* u is defined as

$$B(G; u) = \{u\} \cup \{v \in V \mid u \text{ covers } v\}.$$

The ball $B(G; u)$ can also be written in short as $B(u)$ if the underlying graph G is known from the context. For a subset $U \subseteq V$, we denote

$$B(U) = B(G; U) = \bigcup_{u \in U} B(G; u).$$

If $U = \{u_1, u_2, \ldots, u_k\}$, then we can also write $B(G; U) = B(G; u_1, u_2, \ldots, u_k) = B(u_1, u_2, \ldots, u_k)$.

Let $C \subseteq V$ be a code and X be a subset of V. The size of the set X is denoted by $|X|$. The *I-set* of X with respect to the code C is

$$I(X) = I(C; X) = I(G, C; X) = B(G; X) \cap C.$$

Let also Y be a subset of V. The *symmetric difference* of X and Y is defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

Definition 1.1. Let ℓ be a positive integer. A code $C \subseteq V$ is said to be *ℓ-set-identifying* in G if for all $X, Y \subseteq V$ such that $|X| \leq \ell$, $|Y| \leq \ell$ and $X \neq Y$ we have

$$I(X) \neq I(Y).$$

If $\ell = 1$, then we simply say that C is *identifying*.

In other words, a code $C \subseteq V$ is ℓ-set-identifying in G if and only if for all $X, Y \subseteq V$ such that $|X| \leq \ell$, $|Y| \leq \ell$ and $X \neq Y$ we have

$$I(X) \triangle I(Y) \neq \emptyset.$$

The ℓ-set-identifying codes defined above are called $(1, \leq \ell)$-identifying codes in the terminology of, for example, [12].

Identifying codes were introduced in [15] for finding malfunctioning processors in a multiprocessor system. The topic forms an active field of research; see the numerous articles in the web-page [17] with various aspects considered; for a recent development we refer to [1, 7, 16]. Identifying codes find [18, 19] applications also in sensor networks. The network is modelled by a graph $G = (V, E)$. The sensors correspond to a code C ($\subseteq V$) and $B(u)$ is the set of vertices which the sensor u can check. The idea is that we determine the exact locations of objects (like faulty processor) $X \subseteq V$ using only the alarm signals (that is, the set $I(C; X)$) obtained from the sensors of C — this can be done provided that C is ℓ-set-identifying and $|X| \leq \ell$.

Assume now that the vertex set V is equal to $\mathbb{Z}^2$. Let then t be a positive integer and $\mathbf{u} = (x, y)$ be a vertex in $\mathbb{Z}^2$. The graph $\mathcal{S}_t$ with the ball

$$B(\mathcal{S}_t; \mathbf{u}) = \{(x', y') \in \mathbb{Z}^2 \mid |x - x'| + |y - y'| \leq t\}$$

is called the *square grid*. The graph $\mathcal{K}_t$ with the ball

$$B(\mathcal{K}_t; \mathbf{u}) = \{(x', y') \in \mathbb{Z}^2 \mid |x - x'| \leq t, |y - y'| \leq t\}$$

is called the *king grid*. The graphs $\mathcal{S}_t$ and $\mathcal{K}_t$ are illustrated in Figure 1. The ℓ-set-identification in $\mathcal{S}_t$ and $\mathcal{K}_t$ have been studied, for example, in [4, 10, 13] and [5, 8], respectively.

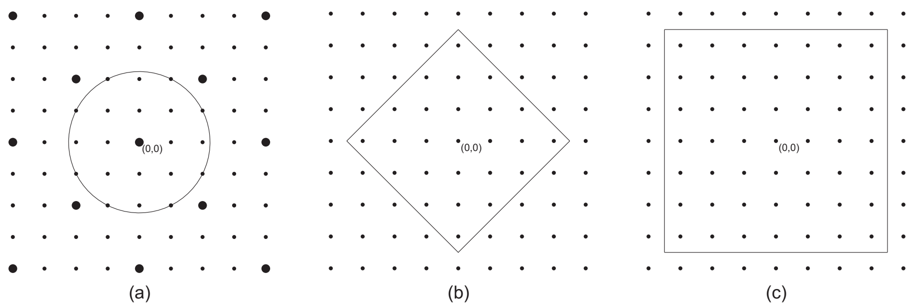

Figure 1: (a) The ball $B(\mathcal{E}_{\sqrt{5}}; (0, 0))$ and the code C_2 (defined in Section 2.2) illustrated. (b) The ball $B(\mathcal{S}_3; (0, 0))$ illustrated. (c) The ball $B(\mathcal{K}_3; (0, 0))$ illustrated.

Let now r be a positive real number. Let again $V = \mathbb{Z}^2$. The graph $\mathcal{E}_r = (V, E)$ is defined by the edge set E such that vertices $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{Z}^2$ are adjacent if the Euclidean distance of $\mathbf{u}$ and $\mathbf{v}$ is at most r. If $\mathbf{u} = (x, y) \in \mathbb{Z}^2$, then the ball

$$B(\mathcal{E}_r; \mathbf{u}) = \{(x', y') \in \mathbb{Z}^2 \mid (x - x')^2 + (y - y')^2 \leq r^2\}.$$

Obviously, $\mathcal{S}_1 = \mathcal{E}_1$, $\mathcal{K}_1 = \mathcal{E}_{\sqrt{2}}$, $\mathcal{S}_2 = \mathcal{E}_2$ and $\mathcal{K}_2 = \mathcal{E}_{2\sqrt{2}}$. The graph $\mathcal{E}_{\sqrt{5}}$ is illustrated in Figure 1. For larger values of t, the shape of the ball $B(u)$ in the graphs $\mathcal{K}_t$ and $\mathcal{S}_t$ is a square as can be seen in Figure 1. In this paper, we consider identification in the case when $B(u)$ is an Euclidean ball, which is a natural area for a sensor in $\mathbb{Z}^2$ to check. In other words, the aim is to find good ℓ-set-identifying codes in $\mathcal{E}_r$ for any real number $r \geq 1$. The motivation for considering different balls in $\mathbb{Z}^2$ also comes from [3] and [14, Section 5].

In order to measure codes in $\mathbb{Z}^2$, we define the notion of density of codes. For this, we first define

$$T_n = \{(x, y) \in \mathbb{Z}^2 \mid |x| \leq n, |y| \leq n\},$$

where n is a positive integer. Now the *density* of a code $C \subseteq \mathbb{Z}^2$ is defined as

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap T_n|}{|T_n|}.$$

Naturally, we seek identifying codes with density as small as possible. We say that an ℓ-set-identifying code is *optimal*, if there does not exist any identifying codes with lower density.

In the sequel we will need the following result from [4, Proposition 1].

Theorem 1.2 ([4])**.** *Let $G = (V, E)$ be a simple connected and undirected graph. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V$ be three vertices of G and C be an identifying code in G. Then the set $H(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = (B(\mathbf{u}_1) \triangle B(\mathbf{u}_2)) \cup (B(\mathbf{u}_1) \triangle B(\mathbf{u}_3)) \cup (B(\mathbf{u}_2) \triangle B(\mathbf{u}_3))$ contains at least two codewords.*

2 On ℓ-set-identifying codes with $\ell = 1$

In this section, we study 1-set-identifying codes in two families of graphs. We first start by considering identifying codes in $\mathcal{E}_r$. Then we examine identifying codes in a graph similar to the king grid. The identifying codes in this graph also provide optimal identifying codes for certain graphs $\mathcal{E}_r$.

2.1 Identifying codes in the graphs $\mathcal{E}_r$

In what follows, we construct a 1-set-identifying code for the graph $\mathcal{E}_r$, when $r \geq 1$ is an arbitrary real number, and also provide a lower bound on the density of such codes. For the considerations, we define the *horizontal line* as $L_i^{(h)} = \{(x', i) \mid x' \in \mathbb{Z}\}$ and the *vertical line* as $L_i^{(v)} = \{(i, y') \mid y' \in \mathbb{Z}\}$, where i is an integer. We also define the *diagonal with slope* -1 as $D_i^{(n)} = \{(x', y') \in \mathbb{Z}^2 \mid x' + y' = i\}$ and the *diagonal with slope* 1 as $D_i^{(p)} = \{(x', y') \in \mathbb{Z}^2 \mid x' - y' = i\}$. If $\mathbf{u}$ is a vertex in $\mathbb{Z}^2$ and X is a subset of $\mathbb{Z}^2$, then the *sum* of $\mathbf{u}$ and X is defined as $\mathbf{u} + X = \{\mathbf{u} + \mathbf{v} \mid \mathbf{v} \in X\}$. We first present the following technical lemma. The results (ii), (iii) and (iv) in the lemma are estimates (not always sharp), which are enough for our purposes in Section 3.

Lemma 2.1. *Let $\mathbf{u} = (x, y)$ be a vertex in $\mathbb{Z}^2$ and $r \geq 1$ be a real number.*

(i) In $B(x, y) \setminus B(x, y-1)$ there exist $2\lfloor r \rfloor + 1$ vertices, which lie on consecutive vertical lines $L_i^{(v)}$ with $i = x - \lfloor r \rfloor, \ldots, x + \lfloor r \rfloor$.

(ii) In $B(x, y) \setminus B(x-1, y-1))$ there exist $4\lfloor r/\sqrt{2} \rfloor + 1$ vertices, which lie on consecutive diagonals $D_i^{(p)}$ with $i = x - 2\lfloor r/\sqrt{2} \rfloor, \ldots, x + 2\lfloor r/\sqrt{2} \rfloor$.

(iii) In $B(x, y) \setminus B((x, y-1), (x+1, y))$ there exist $\lfloor r(1-1/\sqrt{2}) \rfloor$ vertices, which lie on consecutive vertical lines $L_i^{(v)}$ with $i = x - \lfloor r \rfloor, \ldots, x - \lfloor r \rfloor + \lfloor r(1 - 1/\sqrt{2}) \rfloor - 1$.

(iv) In $B(x,y)\backslash B((x-1,y-1),(x+1,y-1))$ there exist $2\lfloor r(1/\sqrt{2}-1/2)\rfloor-1$ vertices, which lie on consecutive diagonals $D_i^{(p)}$ with $i = x-2\lfloor r\sqrt{2}\rfloor, \ldots, x-2\lfloor r\sqrt{2}\rfloor+2\lfloor r(1/\sqrt{2}-1/2)\rfloor-2$.

Proof. (i) Moving the center $\mathbf{u} = (x, y)$ of a ball to $(x, y-1)$ means that $\mathbf{u}$ covers on $L_i^{(v)}$ $(i = x-\lfloor r\rfloor, \ldots, x+\lfloor r\rfloor)$ exactly one vertex of $\mathbb{Z}^2$ which is not covered by $(x, y-1)$, since the second coordinate decreases by one. The case (ii) is analogous. (iii) Suppose $r \geq 4$, otherwise the claim is trivial. Denote $Q_2^{(l)} = \{(-a, b) \in \mathbb{Z}^2 \mid 0 \leq a,\ 0 \leq b \leq a\}$. It is easy to check that the vertices of $\mathbf{u} + Q_2^{(l)}$ which are covered by $(x+1, y)$ belong to $B(x, y-1)$ also. Therefore, in $\mathbf{u} + Q_2^{(l)}$, it is enough to consider the vertices, that $\mathbf{u} = (x, y)$ covers but $(x, y-1)$ does not. We obtain the claim using (i) for the consecutive vertical lines $L_i^{(v)}$ for $x - r \leq i \leq x - r/\sqrt{2}$. The case (iv) is again similar (non-trivial for $r \geq 5$). □

Notice that analogous results to the previous lemma hold when the considered patterns are rotated by $\pi/2$, π and $3\pi/2$. For example, when the pattern in (i) is rotated anti-clockwise by $\pi/2$, we have that the set $B(x, y) \setminus B(x+1, y)$ contains vertices on $2\lfloor r\rfloor + 1$ consecutive horizontal lines.

For the construction of the identifying codes in $\mathcal{E}_r$, we first introduce the following sets of vertices

$$C^{(h)} = \{(j, 0) \in \mathbb{Z}^2 \mid j \equiv 0 \bmod 2\}$$

and

$$C^{(v)} = \{(0, j) \in \mathbb{Z}^2 \mid j \equiv 0 \bmod 2\}.$$

Define then a code C_k as follows:

$$\begin{aligned} C_k = &\bigcup_{i\in\mathbb{Z}} \Big((C^{(h)} + (0, i\cdot 2k)) \cup (C^{(h)} + (1, k + i\cdot 2k))\Big) \\ &\bigcup_{i\in\mathbb{Z}} \Big((C^{(v)} + (i\cdot 2k, 0)) \cup (C^{(v)} + (k + i\cdot 2k, 1))\Big), \end{aligned}$$

where $k \in \mathbb{Z}$ and $k \geq 1$. The following theorem shows that the previous code C_k provides a 1-set-identifying code for the graph $\mathcal{E}_r$.

Theorem 2.2. *Let $r \geq 1$ be a real number.*

(i) If $r^2 - \lfloor r\rfloor^2 \geq 1$, then the code $C_{2\lfloor r\rfloor+1}$ is identifying in $\mathcal{E}_r$.

(ii) If $r^2 - \lfloor r\rfloor^2 < 1$, then the code $C_{2\lfloor r\rfloor}$ is identifying in $\mathcal{E}_r$.

Proof. (i) Let $\mathbf{u} = (x, y)$ be a vertex in $\mathbb{Z}^2$. Assume first that $r^2 - \lfloor r\rfloor^2 \geq 1$. This assumption implies that the vertices $(x-\lfloor r\rfloor, y-1)$, $(x-\lfloor r\rfloor, y+1)$, $(x+\lfloor r\rfloor, y-1)$ and $(x+\lfloor r\rfloor, y+1)$ belong to $B(\mathbf{u})$. Therefore, the set $\{(i, j) \in \mathbb{Z}^2 \mid x - \lfloor r\rfloor \leq i \leq x + \lfloor r\rfloor, y-1 \leq j \leq y+1\}$ is a subset of $B(\mathbf{u})$. By the construction of $C_{2\lfloor r\rfloor+1}$, one of the $2\lfloor r\rfloor + 1$ consecutive vertical lines is such that every other vertex in the line is a codeword. Hence, the ball $B(\mathbf{u})$ contains a codeword. In other words, each vertex in $\mathbb{Z}^2$ is covered by a codeword.

Let $\mathbf{v} = (x + x', y + y')$ be a vertex in $\mathbb{Z}^2$ and $\mathbf{v} \neq \mathbf{u}$. Consider then the symmetric difference $B(\mathbf{u}) \triangle B(\mathbf{v})$. In order to prove that $C_{2\lfloor r\rfloor+1}$ is an identifying code in $\mathcal{E}_r$, we have to show that this symmetric difference always contains a codeword. Without loss of generality, we can assume that $x' \geq 0$ and $y' \geq 0$. (Other cases are analogous.) If $B(\mathbf{u}) \cap B(\mathbf{v}) = \emptyset$, then we are done. Hence, assume that $B(\mathbf{u}) \cap B(\mathbf{v}) \neq \emptyset$.

Assume first that $x' \geq 2$ or $y' \geq 2$. Let $y' \geq 2$ (the other case is analogous). Denote then $\mathbf{u}' = (x, y + y')$ and $\mathbf{v}' = (x + x', y')$. Using similar arguments as in the proof of Lemma 2.1 part (i), we conclude that each vertical line $L_i^{(v)}$ with $x - \lfloor r \rfloor \leq i \leq x + \lfloor x'/2 \rfloor$ contains two consecutive vertices in $B(\mathbf{u}) \setminus B(\mathbf{u}')$. (Recall that $r^2 - \lfloor r \rfloor^2 \geq 1$.) Clearly, these same points are also included in $B(\mathbf{u}) \setminus B(\mathbf{v})$. By symmetry, we can show that each vertical line $L_i^{(v)}$ with $x + \lceil x'/2 \rceil \leq i \leq x + x' + \lfloor r \rfloor$ contains two consecutive vertices in $B(\mathbf{v}) \setminus B(\mathbf{u})$. We have shown that each vertical line $L_i^{(v)}$ with $x - \lfloor r \rfloor \leq i \leq x + x' + \lfloor r \rfloor$ contains two consecutive vertices in $B(\mathbf{u}) \triangle B(\mathbf{v})$. Therefore, we conclude that there exists a codeword in $B(\mathbf{u}) \triangle B(\mathbf{v})$.

Assume now that $x' \leq 1$ and $y' \leq 1$. Then we have the following cases to consider:

1) Assume that $x' = 0$ and $y' = 1$. Let $L_k^{(v)}$ be a vertical line with $x - \lfloor r \rfloor \leq k \leq x + \lfloor r \rfloor$. By Lemma 2.1(i), the set $L_k^{(v)} \cap (B(\mathbf{v}) \setminus B(\mathbf{u}))$ is nonempty. Let $\mathbf{w} = (k, y+1+a) \in \mathbb{Z}^2$ be a vertex in $B(\mathbf{v}) \setminus B(\mathbf{u})$. Then, by symmetry, a vertex $\mathbf{w}' = (k, y - a) \in \mathbb{Z}^2$ belongs to $B(\mathbf{u}) \setminus B(\mathbf{v})$. Since the Euclidean distance between $\mathbf{w}$ and $\mathbf{w}'$ is equal to $2a + 1$, the parity of the second coordinates of the vertices $\mathbf{w}$ and $\mathbf{w}'$ are different. Therefore, since one of the vertical lines $L_i^{(v)}$ with $x - \lfloor r \rfloor \leq i \leq x + \lfloor r \rfloor$ is such that every other vertex in the line is a codeword, the symmetric difference $B(\mathbf{u}) \triangle B(\mathbf{v})$ contains a codeword.

2) If $x' = 1$ and $y' = 0$, then the proof goes exactly like in the case 1); just replace the vertical lines by horizontal ones.

3) Assume now that $x' = 1$ and $y' = 1$. Let $\mathbf{w} = (k, y + 1 + a) \in L_k^{(v)}$, where $x - \lfloor r \rfloor \leq k \leq x$, be a vertex such that $\mathbf{w} \in B(x, y + 1) \setminus B(x, y)$. By symmetry, the vertex $\mathbf{w}' = (k, y - a)$ belongs to $B(x, y) \setminus B(x, y + 1)$. Since $k \leq x$, the vertex $\mathbf{w}' \in B(x, y) \setminus B(x+1, y+1)$. If $\mathbf{w} \in B(x+1, y+1) \setminus B(x, y)$, then the vertical line $L_k^{(v)}$ contains two vertices ($\mathbf{w}$ and $\mathbf{w}'$) in $B(\mathbf{u}) \triangle B(\mathbf{v})$ such that the parity of their second coordinates are different. Assume then that $\mathbf{w} \notin B(x + 1, y + 1) \setminus B(x, y)$. Hence, by symmetry, the vertex $\mathbf{w}'' = (k, y + 1 - a) \in B(x, y) \setminus B(x + 1, y + 1)$. Clearly, the parity of the second coordinates of $\mathbf{w}'$ and $\mathbf{w}''$ are different. Analogous arguments also apply, when we are considering the vertical lines $L_k^{(v)}$ with $x+1 \leq k \leq x+1+\lfloor r \rfloor$. Hence, each line $L_i^{(v)}$ with $x - \lfloor r \rfloor \leq i \leq x+1+\lfloor r \rfloor$ contains two vertices in $B(\mathbf{u}) \triangle B(\mathbf{v})$ such that the parity of the second coordinates of the vertices are different. Thus, there exists a codeword in $B(\mathbf{u}) \triangle B(\mathbf{v})$.

In conclusion, we have shown that $C_{2\lfloor r \rfloor+1}$ is an identifying code in $\mathcal{E}_r$ when $r^2 - \lfloor r \rfloor^2 \geq 1$.

(ii) Let again $\mathbf{u} = (x, y)$ be a vertex in $\mathbb{Z}^2$. Assume then that $r^2 - \lfloor r \rfloor^2 < 1$. Define the set $A = \{(i, j) \in \mathbb{Z}^2 \mid x - \lfloor r \rfloor \leq i \leq x + \lfloor r \rfloor, y - 1 \leq j \leq y\} \setminus \{(x - \lfloor r \rfloor, y-1), (x+\lfloor r \rfloor, y-1)\}$. Let us then show that the set A contains a codeword of $C_{2\lfloor r\rfloor}$. If a vertical line $L_i^{(v)}$ with $x-\lfloor r\rfloor+1 \leq i \leq x+\lfloor r\rfloor-1$ is such that every other vertex in the line is a codeword, then we are clearly done. Otherwise, we know that the vertical lines $L_{x-\lfloor r\rfloor}^{(v)}$ and $L_{x+\lfloor r\rfloor}^{(v)}$ are such that every other vertex in the lines is a codeword. Hence, by the construction of $C_{2\lfloor r\rfloor}$, either the vertex $(x-\lfloor r\rfloor, y)$ or $(x+\lfloor r\rfloor, y)$ is a codeword. Since $A \subseteq B(\mathbf{u})$, the word $\mathbf{u}$ is covered by a codeword.

Let $\mathbf{v} = (x+x', y+y')$ be a vertex in $\mathbb{Z}^2$ and $\mathbf{v} \neq \mathbf{u}$. We need to show that the symmetric difference $B(\mathbf{u}) \triangle B(\mathbf{v})$ contains a codeword (when $B(\mathbf{u}) \cap B(\mathbf{v}) \neq \emptyset$). Without loss of generality, we can assume that $x' \geq 0$ and $y' \geq 0$. If ($x' = 0$ and $y' = 1$) or ($x' = 1$ and $y' = 0$), then the proof goes exactly as in the cases 1) and 2) of the part (i), respectively. Assume that $x' = 0$ and $y' \geq 2$. If now a vertical line $L_i^{(v)}$ with $x-\lfloor r\rfloor+1 \leq i \leq x+\lfloor r\rfloor-1$ is such that every other vertex in the line is a codeword, then we are done. Otherwise, either the vertex $(x-\lfloor r\rfloor, y)$ or $(x+\lfloor r\rfloor, y)$ in $B(\mathbf{u}) \triangle B(\mathbf{v})$ is a codeword. Therefore, $I(\mathbf{u}) \triangle I(\mathbf{v}) \neq \emptyset$. Similar arguments also apply when $x' \geq 2$ and $y' = 0$. If $x' = 1$ and $y' = 1$, then the proof goes exactly as in the previous case 3), but we just consider the $2\lfloor r\rfloor$ consecutive vertical lines $L_i^{(v)}$ with $x-\lfloor r\rfloor+1 \leq i \leq x+\lfloor r\rfloor$. If $x' \geq 1$ and $y' \geq 2$, then the proof is similar to the third paragraph of the proof of the part (i), but we just consider the vertical lines $L_i^{(v)}$ with $x-\lfloor r\rfloor+1 \leq i \leq x+x'+\lfloor r\rfloor-1$. The case with $x' \geq 2$ and $y' \geq 1$ goes the same way as the previous one. In conclusion, we have shown that $C_{2\lfloor r\rfloor}$ is an identifying code in $\mathcal{E}_r$ when $r^2 - \lfloor r\rfloor^2 < 1$. $\square$

It is easy to conclude that the density of the code C_k satisfies $D(C_k) \leq 1/k$. Therefore, by the previous theorem, we have shown that for any real number $r \geq 1$ there exists an identifying code C such that the density

$$D(C) \leq \frac{1}{2\lfloor r\rfloor}.$$

For small values of r, there exist identifying codes with smaller densities. Indeed, since $\mathcal{E}_{\sqrt{2}} = \mathcal{K}_1$ and $\mathcal{E}_{2\sqrt{2}} = \mathcal{K}_2$, we have optimal identifying codes in $\mathcal{E}_{\sqrt{2}}$ and $\mathcal{E}_{2\sqrt{2}}$ with densities 2/9 and 1/8, respectively (see [5]). Recall that $\mathcal{E}_1 = \mathcal{S}_1$ and $\mathcal{E}_2 = \mathcal{S}_2$. It has been shown in [6] that there exists an identifying code with density 7/20 in $\mathcal{S}_1$. Moreover, it was proved in [2] that there is no identifying codes in $\mathcal{S}_1$ with smaller density. There exists an identifying code in $\mathcal{S}_2$ with density 5/29 (see [13]). In [4], it has been shown that there does not exist an identifying code in $\mathcal{S}_2$ with density smaller than 3/20.

Consider then a lower bound on the density of an identifying code in $\mathcal{E}_r$. In order to provide a lower bound, we first need to present an auxiliary theorem. This theorem is a rephrased version of [11, Theorem 5]. For completeness, we have also included the proof.

Theorem 2.3. *Assume that $C \subseteq \mathbb{Z}^2$ is a code. Let $S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ be a subset containing k different points of $\mathbb{Z}^2$. For each $i = 1, 2, \ldots, k$ we choose a real number $w_i \geq 0$, which we call the weight of $\mathbf{s}_i$ and denote by $w(\mathbf{s}_i)$. For all subsets A of S we define*

$$w(A) = \sum_{\mathbf{a} \in A} w(\mathbf{a}).$$

If for all $\mathbf{v} \in \mathbb{Z}^2$ we have $w((\mathbf{v}+C) \cap S) \geq 1$, then the density of C satisfies

$$D(C) \geq \frac{1}{w_1 + w_2 + \cdots + w_k}.$$

Proof. Since S is finite, we can choose a constant h such that $S \subseteq T_h$. Consider then the sum $\sum_{\mathbf{v} \in T_{n-h}} w((\mathbf{v}+C) \cap S)$, where $n > h$. Now we have

$$|T_{n-h}| \leq \sum_{\mathbf{v} \in T_{n-h}} w((\mathbf{v}+C) \cap S) \leq \sum_{i=1}^{k} w_i f_i(n), \tag{1}$$

where $f_i(n)$ denotes the number of pairs $(\mathbf{c}, \mathbf{v})$ such that $\mathbf{c} \in C$, $\mathbf{v} \in T_{n-h}$ and $\mathbf{s}_i = \mathbf{v} + \mathbf{c}$. Since $\mathbf{v} \in T_{n-h}$ and $\mathbf{s}_i \in T_h$, we know that $\mathbf{c} = \mathbf{s}_i - \mathbf{v} \in T_n$. Hence, there at most $|C \cap T_n|$ choices for $\mathbf{c}$. Furthermore, for every $\mathbf{c}$ there is at most one possible choice for $\mathbf{v} \in T_{n-h}$ such that $\mathbf{s}_i = \mathbf{c} + \mathbf{v}$. Therefore, $f_i(n) \leq |C \cap T_n|$.

Combining this result with the equation (1), we have

$$|T_{n-h}| \leq (w_1 + w_2 + \cdots + w_k)|C \cap T_n|.$$

Thus,

$$\frac{|C \cap T_n|}{|T_n|} \geq \frac{|T_{n-h}|}{|T_n|} \cdot \frac{1}{w_1 + w_2 + \cdots + w_k}.$$

The claim immediately follows from this equation, since $|T_{n-h}|/|T_n| \to 1$ when $n \to \infty$. □

In what follows, we prove a lower bound on the density of an identifying code in $\mathcal{E}_r$. The lower bound is actually attained for some graphs $\mathcal{E}_r$ (see Theorem 2.7).

Theorem 2.4. *If $C \subseteq \mathbb{Z}^2$ is an identifying code in $\mathcal{E}_r$, then the density satisfies*

$$D(C) \geq \frac{3}{4\lfloor r \rfloor + 4\lfloor b \rfloor + 4\left\lfloor \sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \right\rfloor + 8},$$

where $b = -1/2 + 1/2 \cdot \sqrt{2r^2 - 1}$.

Proof. Let $C \subseteq \mathbb{Z}^2$ be an identifying code in $\mathcal{E}_r$. Denote $\mathbf{u}_1 = (0,0)$, $\mathbf{u}_2 = (-1,0)$, $\mathbf{u}_3 = (0,-1)$ and $\mathbf{u}_4 = (-1,-1)$. Define then the set

$$\begin{aligned} H = &(B(\mathbf{u}_1) \triangle B(\mathbf{u}_2)) \cup (B(\mathbf{u}_1) \triangle B(\mathbf{u}_3)) \cup (B(\mathbf{u}_1) \triangle B(\mathbf{u}_4)) \\ &\cup (B(\mathbf{u}_2) \triangle B(\mathbf{u}_3)) \cup (B(\mathbf{u}_2) \triangle B(\mathbf{u}_4)) \cup (B(\mathbf{u}_3) \triangle B(\mathbf{u}_4)) \end{aligned}$$

and H' as the set of vertices that belong to H and are covered by exactly two of the vertices $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$.

Notice that if $\mathbf{v} \in H \setminus H'$, then $\mathbf{v}$ is covered by exactly one or three of the vertices $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$. If a codeword $\mathbf{c} \in C$ belongs to $H \setminus H'$, then, by Theorem 1.2, there exist at least three codewords in H. On the other hand, if there does not exist any codewords in $H \setminus H'$, then there clearly exist at least two codewords in H'.

Using the notations of Theorem 2.3, we choose $S = H$. The weight of a vertex $\mathbf{s} \in H$ is now defined as follows: if $\mathbf{s} \in H'$, then $w(\mathbf{s}) = 1/2$, else $w(\mathbf{s}) = 1/3$. By the considerations in the previous paragraph, we conclude that for every $\mathbf{v} \in \mathbb{Z}^2$ we have $w((\mathbf{v}+C) \cap H) \geq 1$. By Theorem 2.3, we have

$$D(C) \geq \frac{1}{1/2 \cdot |H'| + 1/3 \cdot (|H| - |H'|)} = \frac{3}{|H| + 1/2 \cdot |H'|}.$$

For the lower bound, it is now enough to calculate the number of vertices in H and H'.

For the calculations, define the set $Q = \{(x, y) \in \mathbb{Z}^2 \mid x \geq 0, y \geq 0\}$. It is clear that a vertex $\mathbf{u} \in Q \cap H$ if and only if $\mathbf{u} \in B(0,0) \setminus B(-1,-1)$ and $\mathbf{u} \in Q$. Now, by straightforward computations, we have that the number of vertices in $Q \cap H$ is equal to

$$\sum_{i=0}^{\lfloor r \rfloor - 1} \left(\left\lfloor \sqrt{r^2 - i^2} \right\rfloor - \left\lfloor \sqrt{r^2 - (i+1)^2} - 1 \right\rfloor \right) + \left\lfloor \sqrt{r^2 - \lfloor r \rfloor^2} \right\rfloor + 1 = 2\lfloor r \rfloor + 1.$$

Therefore, by symmetry, the number of vertices in H is equal to $4(2\lfloor r \rfloor + 1) = 8\lfloor r \rfloor + 4$.

Consider then the number of vertices in H'. It is easy to see that the circles of radius r centered at the points $(-1, 0)$ and $(0, -1)$ intersect each other in the point (b, b), where $b = -1/2 + 1/2 \cdot \sqrt{2r^2 - 1}$. Define then the set $Q_b = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq b, y \geq 0\}$. It is clear that a vertex $\mathbf{u} \in Q_b \cap H'$ if and only if $\mathbf{u} \in (B(0,0) \cup B(-1,0)) \setminus (B(0,-1) \cup B(-1,-1))$ and $\mathbf{u} \in Q_b$. Hence, by straightforward computations, we have that the number of vertices in $Q_b \cap H'$ is equal to

$$\sum_{i=0}^{\lfloor b \rfloor} \left(\left\lfloor \sqrt{r^2 - (i+1)^2} \right\rfloor - \left\lfloor \sqrt{r^2 - i^2} - 1 \right\rfloor \right) = \left\lfloor \sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \right\rfloor + \lfloor b \rfloor - \lfloor r \rfloor + 1.$$

Therefore, by symmetry, the number of vertices in H' is equal to

$$8 \left(\left\lfloor \sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \right\rfloor + \lfloor b \rfloor - \lfloor r \rfloor + 1 \right).$$

Thus, we obtain the lower bound on the density

$$D(C) \geq \frac{3}{4\lfloor r \rfloor + 4\lfloor b \rfloor + 4 \left\lfloor \sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \right\rfloor + 8}.$$

□

Let us then consider more closely the lower bound given by the previous theorem. As in the theorem, let $C \subseteq \mathbb{Z}^2$ be an identifying code in $\mathcal{E}_r$ and denote $b = -1/2 + 1/2 \cdot \sqrt{2r^2 - 1}$. Denote further $\lfloor b \rfloor = k \in \mathbb{Z}$. Since now $b < k + 1$, we have that $r < \sqrt{1/2 \cdot (2k+3)^2 + 1/2}$. Therefore, we have

$$\sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \leq \sqrt{\left(\sqrt{1/2 \cdot (2k+3)^2 + 1/2} \right)^2 - (\lfloor b \rfloor + 1)^2} = k + 2.$$

Hence, we further obtain that $\left\lfloor \sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \right\rfloor \leq \lfloor b \rfloor + 1$. Thus, the denominator of the lower bound can be estimated as follows:

$$4\lfloor r \rfloor + 4\lfloor b \rfloor + 4 \left\lfloor \sqrt{r^2 - (\lfloor b \rfloor + 1)^2} \right\rfloor + 8 \leq 4\lfloor r \rfloor + 8\lfloor b \rfloor + 12 \leq 4(\sqrt{2} + 1)r + 12.$$

Therefore, we have the following approximation for the lower bound on the density of an identifying code C in $\mathcal{E}_r$:

$$D(C) \geq \frac{3}{4(\sqrt{2} + 1)r + 12} \geq \frac{1}{3,22r + 4}.$$

2.2 Identifying codes in the king grids without corners

In this section, we consider 1-set-identification in a graph closely related to the king grid. These considerations provide two optimal identifying codes in $\mathcal{E}_r$, as is shown in Theorem 2.7. The vertex set V is again equal to $\mathbb{Z}^2$. Let then t be a positive integer and $\mathbf{u} = (x, y)$ be a vertex in $\mathbb{Z}^2$. The edge set E of the considered graph $\mathcal{K}'_t$ is such that

$$B(\mathcal{K}'_t; \mathbf{u}) = B(\mathcal{K}_t; \mathbf{u}) \setminus \{(x+t, y+t), (x+t, y-t), (x-t, y+t), (x-t, y-t)\}.$$

The graph $\mathcal{K}'_t$ is called the *king grid without corners.* Notice that $\mathcal{K}'_1 = \mathcal{S}_1$. As was mentioned in Section 2.1, there exists an optimal identifying code in $\mathcal{S}_1$ with density 7/20.

Define a code

$$C_t = \bigcup_{i \in \mathbb{Z}} \{(2t \cdot i + \alpha, \alpha) \mid \alpha \in \mathbb{Z} \text{ and } \alpha \text{ is even}\}.$$

The code C_t is illustrated in Figure 1 when $t = 2$. Clearly, the density $D(C_t)$ is equal to $1/(4t)$. It has been shown in [5] that C_t is an optimal identifying code in $\mathcal{K}_t$. The following theorem shows that C_t is also an identifying code in $\mathcal{K}'_t$ — notice that now the ball in $\mathcal{K}'_t$ is smaller than the one in $\mathcal{K}_t$! In Theorem 2.6, we prove that there does not exist identifying codes in $\mathcal{K}'_t$ with lower density.

Theorem 2.5. *Let $t \geq 2$ be an integer. Then the code C_t is identifying in $\mathcal{K}'_t$.*

Proof. Let $\mathbf{w} = (x, y)$ be a vertex in $\mathbb{Z}^2$. Define then sets

$$A_h(\mathbf{w}) = \{(i, j) \in \mathbb{Z}^2 \mid x \leq i \leq x + 2t - 1, y \leq j \leq y + 1\}$$

and

$$A_v(\mathbf{w}) = \{(i, j) \in \mathbb{Z}^2 \mid x \leq i \leq x + 1, y \leq j \leq y + 2t - 1\}.$$

Let i be a integer. If i is even, then the horizontal line $L_i^{(h)}$ is such that one of the $2t$ consecutive vertices in the line is a codeword of C_t. The same also holds for the vertical lines. Thus, the sets $A_h(\mathbf{w})$ and $A_v(\mathbf{w})$ both contain at least one codeword.

Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be vertices in $\mathbb{Z}^2$. The I-set $I(\mathbf{u})$ is nonempty, since the ball $B(\mathbf{u})$ contains the set $A_h(\mathbf{w})$ with a suitable choice of $\mathbf{w}$, when $t \geq 2$. In order to prove the claim, we have to show that the symmetric difference $B(\mathbf{u}) \triangle B(\mathbf{v})$ always contains a codeword. Assume first that $|x_1 - x_2| \geq 3$ or $|y_1 - y_2| \geq 3$. Then the symmetric difference $B(\mathbf{u}) \triangle B(\mathbf{v})$ contains the set $A_v(\mathbf{w})$ or $A_h(\mathbf{w})$. Thus, $I(\mathbf{u}) \triangle I(\mathbf{v}) \neq \emptyset$.

Assume now that $|x_1 - x_2| \leq 2$ and $|y_1 - y_2| \leq 2$. Then we have the following cases to consider (other cases are analogous):

1) Assume that $\mathbf{v} = (x_1 + 1, y_1)$ or $\mathbf{v} = (x_1 + 2, y_1)$. Denote $X_1 = \{(x_1 - t, y_1 - t + 1), (x_1 - t, y_1 - t + 2), \ldots, (x_1 - t, y_1 + t - 1)\}$ and $X_2 = \{(x_1 + t + 1, y_1 - t + 1), (x_1 + t + 1, y_1 - t + 2), \ldots, (x_1 + t + 1, y_1 + t - 1)\}$. It is easy to see that $X_1, X_2 \subseteq B(\mathbf{u}) \triangle B(\mathbf{v})$ and $(x_1 - t + 1, y_1 - t), (x_1 + t, y_1 + t) \in B(\mathbf{u}) \triangle B(\mathbf{v})$. Assume first that $x_1 - t$ is even. Then, by the previous considerations, either X_1 contains a codeword or the vertex $(x_1 - t, y_1 - t)$ is a codeword. If X_1 contains a codeword, we are done. Otherwise, the

vertex $(x_1 - t, y_1 - t)$ is a codeword. Therefore, by the construction of C_t, the vertex $(x_1 - t + 2t, y_1 - t + 2t) = (x_1 + t, y_1 + t)$ is a codeword. Assume then that $x_1 - t$ is odd. Hence, $x_1 + t + 1$ is clearly even. The proof is now similar to the first case.

2) Assume that $\mathbf{v} = (x_1 + 1, y_1 + 1)$ or $\mathbf{v} = (x_1 + 2, y_1 + 2)$. Denote $Y_1 = X_1$ and $Y_2 = (0, 1) + X_2$. It is easy to see that $Y_1, Y_2 \subseteq B(\mathbf{u}) \triangle B(\mathbf{v})$ and $(x_1 - t + 1, y_1 - t + 1), (x_1 + t, y_1 + t) \in B(\mathbf{u}) \triangle B(\mathbf{v})$. Assume first that $x_1 - t$ is even. If Y_1 contains a codeword, we are done. Otherwise, the vertex $(x_1 - t, y_1 - t)$ is a codeword. Therefore, the vertex $(x_1 - t + 2t, y_1 - t + 2t) = (x_1 + t, y_1 + t)$ is a codeword. If $x_1 - t$ is odd, then $x_1 + t + 1$ is even and the proof is similar to the first case.

3) Assume that $\mathbf{v} = (x_1 + 2, y_1 + 1)$. The proof is now analogous to the case 2).

In conclusion, we have shown that the symmetric difference $I(\mathbf{u}) \triangle I(\mathbf{v})$ is always nonempty. Hence, the claim follows. □

The following theorem provides a lower bound on the density of an identifying code in $\mathcal{K}'_t$.

Theorem 2.6. *If C is an identifying code in $\mathcal{K}'_t$, then the density*

$$D(C) \geq \frac{1}{4t}.$$

Proof. Let C be an identifying code in $\mathcal{K}'_t$. Define the vertices $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbb{Z}^2$ and the sets $H, H' \subseteq \mathbb{Z}^2$ as in the proof of Theorem 2.4. Using similar arguments as in the proof of Theorem 2.4, we have

$$D(C) \geq \frac{3}{|H| + 1/2 \cdot |H'|}.$$

It is easy to calculate that $|H| = 8t + 4$ and $|H'| = 4(2t - 2)$. Therefore,

$$D(C) \geq \frac{3}{8t + 4 + 1/2 \cdot 4(2t - 2)} = \frac{1}{4t}.$$

□

In conclusion, we have shown that C_t is an optimal identifying code in $\mathcal{K}'_t$. Hence, we have the following theorem concerning identifying codes in $\mathcal{E}_r$, where $r = \sqrt{5}$ or $r = \sqrt{13}$.

Theorem 2.7. *The codes C_2 and C_3 are optimal identifying codes in $\mathcal{E}_{\sqrt{5}}$ and $\mathcal{E}_{\sqrt{13}}$, respectively.*

Proof. The claim immediately follows from the fact that $\mathcal{E}_{\sqrt{5}} = \mathcal{K}'_2$ and $\mathcal{E}_{\sqrt{13}} = \mathcal{K}'_3$. □

3 On ℓ-set-identifying codes with $\ell > 1$

Let $r \geq 1$ be a real number and let $\mathbb{Z}_+$ denote the set of positive integers. In what follows, we show that there exists a 2-set-identifying code C_r in $\mathcal{E}_r$ such that the density satisfies $D(C_r) = \Theta(1/r)$. We also prove that the density of a 2-set-identifying code in $\mathcal{E}_r$ is always at least $1/(2\lfloor r \rfloor + 1)$. In Theorem 3.2, we consider for which r a 3-set-identifying code can exist in $\mathcal{E}_r$. Theorem 3.3 shows that there does not exist a 4-set-identifying code in $\mathcal{E}_r$ for any r.

The following theorem considers 2-set-identifying codes in $\mathcal{E}_r$.

Theorem 3.1. *Let C_r be a 2-set-identifying code in $\mathcal{E}_r$, $r \geq 1$. Then C_r satisfies $D(C_r) \geq \frac{1}{2\lfloor r \rfloor + 1}$. Moreover, there exists a sequence of 2-set-identifying codes C_r such that $D(C_r) = \Theta(\frac{1}{r})$.*

Proof. Let C_r be any 2-set-identifying code in $\mathcal{E}_r$. The lower bound $D(C_r) \geq 1/(2\lfloor r \rfloor + 1)$ comes from comparing the sets $B(\mathbf{x})$ and $B(\mathbf{x}, \mathbf{x} + (1,0))$, where $\mathbf{x} \in \mathbb{Z}^2$. By Lemma 2.1(i) $|B(\mathbf{x}) \bigtriangleup B(\mathbf{x}, \mathbf{x} + (1,0))| = 2\lfloor r \rfloor + 1$ and there must be at least one codeword of C_r among these vertices. Applying Theorem 2.3 we obtain the result (choose $S = B(\mathbf{x}) \bigtriangleup B(\mathbf{x}, \mathbf{x} + (1,0))$ and all the weights equal to one).

Let $r \geq 14$, $P_1 = \lfloor r(1 - 1/\sqrt{2}) \rfloor$, $P_2 = 2\lfloor r(1/\sqrt{2} - 1/2) \rfloor - 1$ and

$$C_{1,r} = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv 0 \bmod P_1 \text{ or } y \equiv 0 \bmod P_1\}.$$

Let further

$$C_{2,r} = \{(x, y) \in \mathbb{Z}^2 \mid x + y \equiv 0 \bmod P_2 \text{ or } x - y \equiv 0 \bmod P_2\}.$$

We claim that the code

$$C_r = C_{1,r} \cup C_{2,r}$$

is 2-set-identifying in $\mathcal{E}_r$. Clearly, $|C| \leq 4/P_1$.

We need to show that for C_r we have $I(X) \neq I(Y)$ for any two sets $X, Y \subset \mathbb{Z}^2$, where $X \neq Y$ and $|X| \leq 2$ and $|Y| \leq 2$.

Suppose to the contrary that there exist distinct subsets X and Y of $\mathbb{Z}^2$ such that

$$I(X) = I(Y) \tag{2}$$

where $|X|, |Y| \leq 2$.

Clearly, if X or Y is the emptyset, we get $I(X) \neq I(Y)$. Therefore, assume that $|X| \geq 1$ and $|Y| \geq 1$.

Let L_1 (resp. L_2) be a horizontal line $L_i^{(h)}$ where i is such that $L_i^{(h)}$ contains at least one element of $X \cup Y$ but for any $j > i$ (resp. $j < i$) the line $L_j^{(h)}$ contains no elements of $X \cup Y$. Similarly, let L_3 (resp. L_4) be a vertical line $L_i^{(v)}$ where i is such that $L_i^{(v)}$ contains at least one element of $X \cup Y$ and for any $j < i$ (resp. $j > i$) the line $L_j^{(v)}$ contains no elements of $X \cup Y$. Denote by R the set of the vertices that belong to a rectangle or a line segment bordered by the the four lines L_1, L_2, L_3 and L_4. Clearly, all the vertices of $X \cup Y$ belong to R.

Of course, R is a line segment if (and only if) $L_1 = L_2$ or $L_3 = L_4$. Suppose first that this is the case; without loss of generality, let $L_1 = L_2$. We can also assume that on (at least) one end of the line segment R there is $\mathbf{x} \in$

$X \triangle Y$. Without loss of generality, let $\mathbf{x} \in X$ be on the left end of R. Now, by Lemma 2.1(i) (rotated anti-clockwise by $\pi/2$), we know that $B(\mathbf{x})$ contains $2\lfloor r \rfloor + 1$ vertices on consecutive horizontal lines, which $\mathbf{x} + (1, 0)$ does not cover. Since none of the vertices $\mathbf{x} + (a, 0)$, $a \in \mathbb{Z}_+$, covers them either, the elements of Y cannot cover them. By the definition of $C_{1,r}$, the set $I(C_r; \mathbf{x}) \triangle I(C_r; Y) \neq \emptyset$. Hence we get a contradiction with (2).

Consequently, we can assume that R is a rectangle.

1) Suppose first that (at least) one of the four corners of R contains $\mathbf{x} \in X \triangle Y$. Without loss of generality, we may assume that $\mathbf{x} \in X$ is in the north-west corner of R.

By Lemma 2.1(iii) there are at least P_1 vertices on consecutive vertical lines in $B(\mathbf{x}) \setminus B(\mathbf{u}, \mathbf{w})$ where $\mathbf{u} = \mathbf{x} + (0, -1)$ and $\mathbf{w} = \mathbf{x} + (1, 0)$. It is easy to verify that none of these P_1 vertices is covered by any vertex in $S = \{\mathbf{x} + (a, -b) \in \mathbb{Z}^2 \mid a \geq 0, b \geq 0, (a, b) \neq (0, 0)\}$. Since $Y \subseteq S$, these P_1 vertices belong to $B(\mathbf{x}) \setminus B(Y)$. Now the code $C_{2,r}$ guarantees that there is at least one codeword among these P_1 vertices, a contradiction with (2).

2) Suppose then that there is no vertices of $X \cup Y$ in any of the corners of R. Consequently, there must be an element of $X \cup Y$ on each line L_i, $i = 1, 2, 3, 4$. Therefore, $|X| = |Y| = 2$; denote $X = \{\mathbf{x}, \mathbf{y}\}$ and $Y = \{\mathbf{u}, \mathbf{w}\}$.

2.1) Assume first that the elements of X are on two non-intersecting lines, without loss of generality, let $\mathbf{x} \in L_1$ and $\mathbf{y} \in L_2$. Assume further $\mathbf{u} \in L_3$ and $\mathbf{w} \in L_4$.

By Lemma 2.1(iv), there are P_2 vertices of $B(\mathbf{x}) \cap \{\mathbf{x} + (-a, b) \mid a, b \in \mathbb{Z}_+, a \leq b\}$ on consecutive diagonals, which are neither in $B(\mathbf{x} + (1, -1))$ nor in $B(\mathbf{x} + (-1, -1))$. Again, none of the vertices $\mathbf{w} \in U = \{\mathbf{x} + (a, -b) \mid a, b \in \mathbb{Z}_+\}$ can cover these P_2 points. It is also easy to verify that none of the vertices $\mathbf{u} \in T = \{\mathbf{x} + (-a, -b) \mid a, b \in \mathbb{Z}_+, a \leq b\}$ can cover these P_2 points either. Consequently, if $\mathbf{u} \in T$, the code $C_{2,r}$ gives the codeword to the set $I(X) \triangle I(Y)$, which contradicts (2). Assume then that $\mathbf{u} \notin T$, that is, $\mathbf{u} = \mathbf{x} + (-a, -b)$ where $a, b \in \mathbb{Z}_+$ and $a > b$. In this case, we examine the vertices of $B(\mathbf{u}) \cap \{\mathbf{u} + (-c, d) \mid c, d \in \mathbb{Z}_+, c \geq d\}$ which are not covered by $\mathbf{u} + (1, 1)$ and $\mathbf{u} + (1, -1)$ — there are again P_2 of them by a result symmetrical to Lemma 2.1(iv). We observe that neither the vertex $\mathbf{y} = \mathbf{u} + (c, -d)$ for any $c, d \in \mathbb{Z}_+$ nor the vertex $\mathbf{x} = \mathbf{u} + (a, b)$ cannot cover these P_2 vertices in $B(\mathbf{u})$ (because the assumption $a > b$ now gives symmetric situation to the above case $\mathbf{u} \in T$). Therefore, there must be a codeword of $C_{2,r}$ in $I(X) \triangle I(Y)$ to give the contradiction.

2.2) Assume then that the elements of X are on two intersecting lines, and without loss of generality, let $\mathbf{x} \in L_1$, $\mathbf{y} \in L_3$, $\mathbf{u} \in L_2$ and $\mathbf{w} \in L_4$. Let $L_{\mathbf{x}} = \{\mathbf{x} + (0, -a) \mid a \in \mathbb{Z}_+\}$. Again $\mathbf{w} \in U$.

If $\mathbf{u} \in T \cup U \cup L_{\mathbf{x}}$, then the previous arguments give us the contradiction with (2). Indeed, if $\mathbf{u} \in U \cup L_{\mathbf{x}}$, the argument of 1) applies although $\mathbf{x}$ is not in a corner of R. If $\mathbf{u} \in T$, the case 2.1) yields the needed contradiction. Therefore, it suffices to assume that $\mathbf{w} \in U$ and $\mathbf{u} = \mathbf{x} + (-a, -b)$ where $a, b \in \mathbb{Z}_+$, $a > b$. Now consider $\mathbf{y}$ in the role of $\mathbf{x}$. In this case $\mathbf{u} \in U'$ and $\mathbf{w}$ belong to the area $T' \cup U' \cup L_{\mathbf{y}}$ where $T' = \{\mathbf{y} + (c, d) \mid c, d \in \mathbb{Z}_+, c \geq d\}$, $U' = \{\mathbf{y} + (c, -d) \mid c, d \in \mathbb{Z}_+\}$ and $L_{\mathbf{y}} = \{\mathbf{y} + (c, 0) \mid c \in \mathbb{Z}_+\}$. Now the area $T' \cup U' \cup L_{\mathbf{y}}$ for $\mathbf{y}$ is analogous to $T \cup U \cup L$ for $\mathbf{x}$. This gives the contradiction to (2).

3) Finally, it suffices to check the case where there is a vertex $\mathbf{y} \in X \cap Y$ in

one of the corners of R. By 1) we can assume that there is no other vertex of $X \cup Y$ in any corner. Consequently, we may assume that $\mathbf{y}$ is in the south-east corner and $\mathbf{x} \in L_1$ and $\mathbf{u} \in L_3$. This situation goes exactly like in 2.1). $\square$

The graphs $\mathcal{K}_t$ and $\mathcal{E}_r$ have balls of equal size, for example, when $r = 347$ and $t = 307$. In the king grid $\mathcal{K}_{307}$ the optimal density of a 2-set-identifying code equals 0.25 (see [8]) and by our previous construction we have a 2-set-identifying code in $\mathcal{E}_{347}$ with density at most 0.0396. Similarly, the square grid $\mathcal{S}_t$ and $\mathcal{E}_r$ have the same cardinality of vertices in a ball when, for instance, $r = 385$ and $t = 482$. The smallest possible density of a 2-set-identifying code in $\mathcal{S}_{482}$ is at least 0.125 (see [10]) and our construction gives a 2-set-identifying code of density at most 0.0357.

In general, an optimal 2-set-identifying code C_t in the king grid $\mathcal{K}_t$, $t \geq 3$, satisfies $D(C_t) = 1/4$ (see [8]). Similarly, in the square grid $\mathcal{S}_t$, we know (by [10]) that $D(C_t) \geq 1/8$ for any code C_t which is 2-set-identifying. In $\mathcal{E}_r$, however, the density of such codes can be arbitrarily small by the previous theorem. For the 2-set-identifying codes in $\mathcal{E}_1 = \mathcal{S}_1$, $\mathcal{E}_2 = \mathcal{S}_2$, $\mathcal{E}_{\sqrt{2}} = \mathcal{K}_1$ and $\mathcal{E}_{2\sqrt{2}} = \mathcal{K}_2$, we refer to [10].

Consider next the ℓ-set-identifying codes in $\mathcal{E}_r$ when $\ell = 3$. Since the sets $I((1,0),(-1,0))$ and $I((1,0),(-1,0),(0,0)$ must be distinct and also the sets $I((-1,-1),(1,1))$ and $I((-1,-1),(1,1),(0,0))$ must differ, we obtain the following statement for 3-set-identifying codes.

Theorem 3.2. *Let $r \in \mathbb{R}$, $r \geq 1$. If there exists a 3-set-identifying code in $\mathcal{E}_r$, then we must have*

$$\lfloor r \rfloor > \sqrt{r^2 - 1}$$

and, if $r \geq \sqrt{2}$, we also must have

$$\lfloor r/\sqrt{2} \rfloor > \sqrt{r^2/2 - 1}.$$

By the previous theorem we obtain that 1, 3, 17, 99, 577, 3363, 19601,... are the first values of an *integer* r such that the graphs $\mathcal{E}_r$ can have 3-set-identifying codes. By [9, Theorem 2], we know that there exists a 3-set-identifying code in $\mathcal{E}_1 = \mathcal{S}_1$. However, it remains open whether there exist a 3-set-identifying code in $\mathcal{E}_r$ for all possible values of r (listed above).

Moreover, if there exists a 3-set-identifying code C in $\mathcal{E}_r = (V, E)$ for $r = 3$ or $r = 17$, then necessarily $C = V$ and thus the density equals one due to the fact that

$$B((-1,-1),(1,2)) \triangle B((-1,-1),(1,2),(0,0)) = \{(2,-2)\}$$

(only one vertex!) for $r = 3$ and $B((-2,-1),(3,1)) \triangle B((-2,-1),(3,1),(0,0)) = \{(-8,15)\}$ for $r = 17$.

Theorem 3.3. *Let $r \geq 1$. There does not exist an ℓ-set-identifying code in $\mathcal{E}_r$ for any $\ell \geq 4$.*

Proof. The claim follows since

$$B((-1,0),(1,1),(1,-1)) = B((-1,0),(1,1),(1,-1),(0,0))$$

and thus these sets of three and four vertices cannot be distinguished. $\square$

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