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Stability kernel in finite games with perturbed payoffs*

by

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Abstract: The parametric concept of equilibrium in a finite cooperative game of several players in a normal form is introduced. This concept is defined by the partitioning of a set of players into coalitions. Two extreme cases of such partitioning correspond to Pareto optimal and Nash equilibrium outcomes, respectively. The game is characterized by its matrix, in which each element is a subject for independent perturbations., i.e. a set of perturbing matrices is formed by a set of additive matrices, with two arbitrary Hölder norms specified independently in the outcome and criterion spaces. We undertake post-optimal analysis for the so-called stability kernel. The analytical expression for supreme levels of such perturbations is found. Numerical examples illustrate some of the pertinent cases.

Keywords: post-optimal analysis, multiple criteria, kernel stability radius, independent perturbations, finite games, Pareto optimality, Nash equilibrium

1. Introduction

Rapid development of the various fields of information technology, economics and the social sphere, important part of which is related to integrity, high complexity and existence of uncertainty factors, requires an adequate development in the corresponding fields of system analysis, management and operations research. One of the main problem areas arising in this direction is related to

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multiobjective decision making in the presence of conflict, uncertainty and risk. An effective tool for modeling the respective decision-making processes is the apparatus of mathematical game theory.

Game-theoretic models target finding of classes of outcomes that are rationally coordinated in terms of possible actions and interests of participants (players) or groups of participants (coalitions). For each game, given in the so-called normal form, coalitional and non-coalitional equilibrium concepts (principles of optimality) are used, which usually lead to different game outcomes. In the theory of non-antagonistic games there is no single approach to the development of such concepts. The most famous one is the concept of the Nash equilibrium (Nash, 1950, 1951), as well as its various generalizations, related to the problems of group choice, which is understood as the aggregation of various individual preferences into a single collective preference.

This work implements a parameterization of the equilibrium concept of a finite game in normal form. The parameter of this parameterization corresponds to the method of dividing players into coalitions, in which the two extreme cases (a single coalition of players and the set of single-player coalitions) correspond to the Pareto optimal outcome and the Nash equilibrium outcome. Quantitative stability analysis for the set of all efficient (generalized equilibrium) outcomes from the point of view of invariance with respect to changes in the parameters of the game is carried out.

Usually, stability of a multicriteria discrete optimization problem is understood as a discrete analog of the Hausdorff upper semicontinuity property (see Aubin and Frankowska, 1990) of an optimal mapping that defines a choice function, i.e. in our case it is the existence of a neighborhood in the space of game parameters, inside which the appearance of new, different efficient outcomes is not possible. Modification of this requirement leads to the stability type, which is interpreted as the existence of a neighborhood of initial payoffs of the game, inside which there are stable efficient outcomes. That is, for any perturbation there exists at least one efficient outcome of the initial game that remains efficient under any admissible perturbations. Such type of stability is called kernel stability, and the stable outcomes constitute the kernel itself (see Sergienko and Shilo, 2003).

In the present paper the analytical expression for the stability kernel radius is found for the game with the given partition of players into coalitions under the assumption that arbitrary Hölder's norms are defined in the space of outcomes and in the criteria space.

Note that analogous quantitative characteristics of the various stability types of multicriteria parameterized problems of game theory and discrete linear programming problems with other principles of optimality, stability types and metrics defined in the space of parameters, were obtained in a series of works (see, e.g., Bukhtoyarov and Emelichev, 2006; Emelichev and Karelkina, 2009, 2021;

Emelichev et al., 2014; Emelichev and Kuzmin, 2006; Emelichev and Nikulin, 2019; or Emelichev, Nikulin and Mäkelä, 2013).

2. Basic definitions and notations

We consider the main object of study in the game theory, a finite game of n players in normal form (Osborne and Rubinstein, 1994), where each player $i \in N_n = \{1, 2, \ldots, n\}$, $n \geq 2$, has a set of outcomes $X_i \subset \mathbb{R}$, $2 \leq |X_i| < \infty$. The outcome of the game is a realization of the strategies chosen by all the players. This choice is made by the players independently. Let the linear payoff functions be given as follows:

$$f_i(x) = C_i x,\ i \in N_n,$$

where C_i is the i-th row of a square matrix $C = [c_{ij}] \in \mathbb{R}^{n\times n}$, and $x = (x_1, x_2, \ldots, x_n)^T \in X_j$ are defined on the set of all outcomes of the game

$$X = \prod_{j\in N_n} X_j \subset \mathbb{R}^n.$$

As a result of the game, which we call the game with matrix C, each player i gains the payoff $f_i(x)$, which a player tries to maximize using preference relationships.

We assume that all the players try to maximize their own payoffs simultaneously:

$$Cx = (C_1x, C_2x, ..., C_nx)^T \to \max_{x\in X}. \tag{1}$$

A non-empty subset $J \subseteq N_n$ is called a coalition of players. For a coalition J and game outcome $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T$ we introduce a set

$$V(x^0, J) = \prod_{j\in N_n} V_j(x^0, J)$$

where

$$V_j(x^0, J) = \begin{cases} X_j & \text{if } j \in J, \\ \{x_j^0\} & \text{if } j \in N_n \backslash J. \end{cases}$$

Thus, $V_j(x^0, J)$ is the set of outcomes that are reachable by the coalition J from the outcome x^0. It is clear that $V(x^0, N_n) = X$ and $V(x^0, \{k\}) = X_k$ for any $x^0, k \in N_n$.

Further on, we use a binary relation of preference $\prec$ by Pareto (Pareto, 1909) in the space $\mathbb{R}^k$ of arbitrary dimension $k \in \mathbb{N}$, assuming that for two different vectors $y = (y_1, y_2, \ldots, y_k)^T$ and $y' = (y'_1, y'_2, \ldots, y'_k)^T$ in the space $\mathbb{R}^k$ the following formula is valid

$$y \prec y' \;\Leftrightarrow\; y \leq y' \;\&\; y \neq y'.$$

The symbol $\overline{\prec}$, as usual, denotes the negation of $\prec$.

Let $s \in N_n$, and let $N_n = \bigcup_{k \in N_s} J_k$ be a partition of the set N_n into s nonempty sets (coalitions), i.e. $J_k \neq \emptyset$, $k \in N_s$, and $p \neq q \Rightarrow J_p \cap J_q = \emptyset$. A set of $(J_1, J_2, ..., J_s)$-efficient outcomes is introduced according to the formula:

$$\begin{aligned} &G(C, J_1, J_2, \ldots, J_s) = \\ &\quad \big\{x \in X \;:\; \forall k \in N_s \quad \forall x' \in V(x, J_k) \quad \big(C_{J_k} x \,\overline{\prec}\, C_{J_k} x'\big)\big\}, \end{aligned} \tag{2}$$

where C_{J_k} is a $|J_k| \times n$ submatrix of matrix C, consisting of rows that correspond to players in coalition J_k. For brevity, we denote this set by $G(C)$.

Thus, preference relations between players within each coalition are based on Pareto dominance. Therefore, the set of all N_n-efficient outcomes $G(C, N_n)$ ($s = 1$, i.e. all players are united in one coalition) is the Pareto set of the game (1) (set of efficient outcomes), see Pareto (1909):

$$P(C) = \big\{x \in X \;:\; X(x, C) = \emptyset\big\},$$

where

$$X(x, C) = \big\{x' \in X \;:\; Cx \prec Cx'\big\}.$$

Rationality of a cooperative-efficient outcome $x \in P(C)$ consists in that the increase of the payoff of any player is possible only by decreasing the payoff of at least one of the other players.

In the other extreme case, when $s = n$, $G(C, \{1\}, \{2\}, ..., \{n\})$ becomes the set of the Nash equilibria (Nash, 1950, 1951). This set is denoted by $NE(C)$ and is defined as follows:

$$\begin{aligned} &NE(C) = \Big\{x \in X \;:\; \nexists k \in N_n \quad \nexists x' \in X \\ &\Big(C_k x < C_k x' \;\&\; x_{N_n \setminus \{k\}} = x'_{N_n \setminus \{k\}}\Big)\Big\}, \end{aligned}$$

where $x_{N_n \setminus \{k\}}$ is a projection of vector $x \in X$ on the coordinate axis of space $\mathbb{R}^n$ with numbers from the set $N_n \setminus \{k\}$.

It is easy to see that the rationality of the Nash equilibrium consists in that no player can individually deviate from the own equilibrium strategy choice while others keep playing their equilibrium strategies and gain a positive change of payoff from such a deviation. Strict axioms regarding the perfect and common (shared) knowledge are assumed to be fulfilled (Osborne and Rubinstein, 1994).

Thus, we have just introduced a parameterization of the equilibrium concept for a finite game in normal form. The parameter s of this parameterization is the partitioning of all the players into coalitions $J = (J_1, J_2, ..., J_s)$, in which the two extreme cases (a single coalition of players and a set of n single-player coalitions) correspond to finding the Pareto optimal outcomes $P(C)$ and the Nash equilibrium outcomes $NE(C)$, respectively.

Denoted by $Z(C, J_1, J_2, \dots, J_s)$, the game consists in finding the set $G(C, J_1, J_2, \dots, J_s)$. Sometimes for brevity, we use the notation $Z(C)$ for this problem.

Without loss of generality, we assume that the elements of partitioning $N_n = \bigcup\limits_{k \in N_s} J_k$ are defined as follows:

$$
\begin{aligned}
&J_1 = \big\{1, 2, \dots, t_1\big\},\\
&J_2 = \big\{t_1 + 1, t_1 + 2, \dots, t_2\big\},\\
&\qquad \cdots\\
&J_s = \big\{t_{s-1} + 1, t_{s-1} + 2, \dots, n\big\}.
\end{aligned}
$$

For any $k \in N_s$, let C^k denote a square submatrix of size $|J_k| \times |J_k|$ with elements positioned at intersection of rows and column indexed with J_k. Let $P(C^k)$ be the Pareto set:

$$P(C^k) = \big\{z \in X_{J_k} \ : \ X(z, C^k) = \emptyset\big\},$$

where

$$X(z, C^k) = \big\{z' \in X_{J_k} \ : \ C^k z \prec C^k z'\big\},$$

of the $|J_k|$-criteria problem $Z(C^k)$.

$$C^k z \to \max_{z \in X_{J_k}},$$

where $z = (z_1, z_2, \dots, z_{|J_k|})^T$, and X_{J_k} is a projection of X onto J_k, i.e.

$$X_{J_k} = \prod_{j \in J_k} X_j \subset \mathbb{R}^{|J_k|}.$$

This problem is called the *partial problem* of the game $Z(C, J_1, J_2, \dots, J_s)$.

Due to the fact that the linear payoff functions $C_i x$, $i \in N_n$, are separable, according to (2), the following equality is valid:

$$G(C, J_1, J_2, \dots, J_s) = \prod_{k=1}^{s} P(C^k). \tag{3}$$

In the definition of $(J_1, J_2, ..., J_s)$-efficiency in the game with matrix $C \in \mathbb{R}^{n\times n}$, only block-diagonal elements $C^1, C^2, \dots, C^s$ matter. Thus, the set of $(J_1, J_2, ..., J_s)$-efficient outcomes of the game $Z(C, J_1, J_2, ..., J_s)$ will be denoted

$$G(\tilde{C}, J_1, J_2, \dots, J_s),$$

where $\tilde{C} = \{C^1, C^2, \dots, C^s\}$.

In the space of an arbitrary size, $\mathbb{R}^k$, we define Hölder's norm l_p, $p \in [1, \infty]$, i.e. by the norm of the vector $a = (a_1, a_2, ..., a_k)^T \in \mathbb{R}^k$, we mean the number

$$\|a\|_p = \begin{cases} \left(\sum\limits_{j\in N_k} |a_j|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\left\{|a_j| \ : \ j \in N_k\right\} & \text{if } p = \infty. \end{cases}$$

The norm of the matrix $C \in \mathbb{R}^{k\times k}$ with the rows $C_i,\ i \in N_k$, is defined as the norm of a vector, whose components are the norms of the rows of the matrix C. By that, we have

$$\|C\|_{pq} = \big\| \left(\|C_1\|_p, \|C_2\|_p, \dots, \|C_k\|_p\right) \big\|_q,$$

where $l_q,\ q \in [1, \infty]$, is another Hölder's norm, i.e. l_q may differ from l_p in general case.

It is easy to see that for any $p, q \in [1, \infty]$, and for any $i \in N_n$ we have

$$\|C_i\|_p \le \|C\|_{pq}. \tag{4}$$

The norm of the matrix bundle $\tilde{C} = \{C^1, C^2, \dots, C^s\},\ C^k \in \mathbb{R}^{|J_k|\times|J_k|},\ k \in N_s$ is defined as follows:

$$\|\tilde{C}\|_{\max} = \max\left\{\|C^k\|_{pq} \ : \ k \in N_s\right\}.$$

Perturbation of the elements of the matrix bundle $\tilde{C}$ is imposed by adding the *perturbing matrix bundle*

$$\tilde{B} = \{B^1, B^2, \dots, B^s\},$$

where $B^k \in \mathbf{R}^{|J_k|\times|J_k|}$ are the matrices with rows $B^k_i,\ i \in N_n,\ k \in N_s$. Thus, the set of the $(J_1, J_2, ..., J_s)$-efficient outcomes of the perturbed game will be denoted here and later on as $G(\tilde{C} + \tilde{B}, J_1, J_2, \dots, J_s)$.

For an arbitrary number $\varepsilon > 0$, we define a *bundle of perturbing matrices*

$$\Omega^{n\times n}(\varepsilon) = \Big\{\tilde{B} \in \prod_{k=1}^{s} \mathbf{R}^{|J_k|\times|J_k|} \ : \ \|\tilde{B}\|_{\max} < \varepsilon\Big\},$$

where

$$\|\tilde{B}\|_{\max} = \max\left\{\|B^k\|_{pq} \ : \ k \in N_s\right\}.$$

Following the terminology of Sergienko and Shilo (2003), the *stability kernel* of the game $Z(C, J_1, J_2, \ldots, J_s)$ is the set

$$[Ker(\tilde{C}) = Ker(\tilde{C}, J_1, J_2, \ldots, J_s) =$$
$$= \left\{x \in G(\tilde{C}) : \ \exists\varepsilon > 0 \ \forall\tilde{B} \in \Omega^{n\times n}(\varepsilon) \ \left(x \in G(\tilde{C} + \tilde{B})\right)\right\}.$$

Thus, the stability kernel represents the set of all $(J_1, J_2, \ldots, J_s)$-stable game outcomes.

The radius of stability kernel (in terminology of Sergienko and Shilo, 2003, T_2-stability, see also Lebedeva, Semenova and Sergienko, 2021) of the game $Z(C, J_1, J_2, \ldots, J_s)$ is a number defined as follows:

$$\rho{=}\rho^n_{pq}(J_1, J_2, \ldots, J_s){=}\begin{cases} \sup\Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \left\{\varepsilon > 0 : \ Ker(\tilde{C}, \varepsilon) \neq \emptyset\right\},$$

$$Ker(\tilde{C}, \varepsilon) = \left\{x \in G(\tilde{C}) : \ \forall\tilde{B} \in \Omega^{n\times n}(\varepsilon) \ \left(x \in G(\tilde{C} + \tilde{B})\right)\right\}.$$

We refer to the set $Ker(\tilde{C}, \varepsilon)$ as ε-*stability kernel* of the game $Z(C)$.

It is easy to see that

$$\Xi = \left\{\varepsilon > 0 : \ \exists x \in G(\tilde{C}) \ \forall\tilde{B} \in \Omega^{n\times n}(\varepsilon) \ \left(x \in G(\tilde{C} + \tilde{B})\right)\right\}.$$

Thus, the game $Z(C, J_1, J_2, \ldots, J_s)$ is T_2-stable (that is, $\rho_{pq}(J_1, J_2, \ldots, J_s) > 0$) if and only if the stability kernel of the game is nonempty.

So, we can easily conclude that the stability kernel radius of the game $Z(C, J_1, J_2, \ldots, J_s)$ is expressed by the formula:

$$\rho = \rho^n_{pq}(J_1, J_2, \ldots, J_s) =$$
$$\max\left\{\rho^n_{pq}(x) : \ x \in G(\tilde{C}, J_1, J_2, \ldots, J_s)\right\}, \tag{5}$$

where $\rho^n_{pq}(x)$ is the stability radius of the efficient outcome x, defined as follows:

$$\rho^n_{pq}(x) = \begin{cases} \sup\Theta & \text{if } \ \Theta \neq \emptyset, \\ 0 & \text{if } \ \Theta = \emptyset, \end{cases}$$

where

$$\Theta = \left\{\varepsilon > 0 : \ \forall\tilde{B} \in \Omega^{n\times n}(\varepsilon) \ \left(x \in G(\tilde{C} + \tilde{B})\right)\right\}.$$

3. Auxiliary statements and lemmas

In the outcome space $\mathbf{R}^n$, along with the norm $l_p,\ p \in [1, \infty]$, we will use the conjugate norm l_{p^*}, where the numbers p and p^* are connected, as usual, by the equality

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

assuming $p^* = 1$ if $p = \infty$, and $p^* = \infty$ if $p = 1$. Therefore, we further suppose that the range of variations of the numbers p and p^* is the closed interval $[1, \infty]$, and the numbers themselves are connected by the above conditions.

Further on, we shall use the well-known Hölder's inequality

$$|a^T b| \leq \|a\|_p \|b\|_{p^*} \tag{6}$$

that is true for any two vectors $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ and $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$.

Lemma 1 *For any $p \in [1, \infty]$, the following formula holds*

$$\forall b \in \mathbf{R}^n \quad \forall \sigma > 0 \quad \exists a \in \mathbf{R}^n$$

$$\big(|a^T b| = \sigma \|b\|_{p*} \ \&\ \|a\|_p = \sigma\big).$$

Proof It is well-known (see, e.g., Hardy, Littlewood and Polya, 1988) that Hölder's inequality becomes an equality for $1 < p < \infty$ if and only if

a) one of a or b is the zero vector;

b) the two vectors obtained from the non-zero vectors a and b by raising their components' absolute values to the powers of p and p^*, respectively, are linearly dependent (proportional), and sign $(a_i b_i)$ is independent of i.

When $p = 1$, inequality (6) is transformed into the following inequality:

$$|\sum_{i \in N_n} a_i b_i| \leq \max_{i \in N_n} |b_i| \sum_{i \in N_n} |a_i|.$$

This last inequality holds as equality if, for example, b is the zero vector or if $a_j \neq 0$ for some j such that $|b_j| = \|b\|_\infty \neq 0$, and $a_i = 0$ for all $i \in N_n \backslash \{j\}$.

When $p = \infty$, inequality (6) is transformed into the following inequality:

$$|\sum_{i \in N_n} a_i b_i| \leq \max_{i \in N_n} |a_i| \sum_{i \in N_n} |b_i|.$$

This last inequality holds as equality if, for example, b is the zero vector or if $a_i = \sigma \operatorname{sign}(b_i)$ for all $i \in N_n$ and $\sigma > 0$. □

Directly from (3), the following lemma follows.

LEMMA 2 *The outcome* $x \in X$ *is* $(J_1, J_2, \dots, J_s)$*-efficient, i.e.*

$$x \in G^n(\tilde{C}, J_1, J_2, \dots, J_s)$$

if and only if for any index $k \in N_s$

$$x_{J_k} \in P(C^k).$$

Hereinafter, x_{J_r} is a projection of the vector $x = (x_1, x_2, \dots, x_n)^T$ on the coordinate axes of $\mathbf{R}^n$ with coalition numbers J_r.

The norm $\|\cdot\|$, defined in space $\mathbf{R}^n$, is called monotone if for any vectors $y, y' \in \mathbf{R}^n_+$ inequality $y \leq y'$ implies $\|y\| \leq \|y'\|$. It is well-known (see, e.g., Hardy, Littlewood and Polya, 1988) that all Hölder's norms l_p, $p \in [1, \infty]$, are monotone.

Hereinafter, a^+ is a projection of a vector $a = (a_1, a_2, \dots, a_k) \in \mathbf{R}^k$ on the positive orthant, i.e.

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_k^+),$$

where $+$ implies the positive cut of vector a, i.e.

$$a_i^+ = [a_i]^+ = \max\{0, a_i\}.$$

LEMMA 3 *Given* $x \notin G^n(\tilde{C}+\tilde{B}, J_1, J_2, \dots, J_s)$*,* $\tilde{B} \in \Omega^{n \times n}(\psi)$*, and* $\psi > 0$*, there exist* $r \in N_s$ *and* $z^0 \in X_{J_r}$ *such that inequality*

$$\|[C^r(x_{J_r} - z^0)]^+\|_q < \psi\, \|x_{J_r} - z^0\|_{p^*} \tag{7}$$

holds.

PROOF Since $x \notin G^n(\tilde{C} + \tilde{B}, J_1.J_2, \dots, J_s)$, due to Lemma 2, there exists an index $r \in N_s$ such that

$$x_{J_r} \notin P(C^r + B^r).$$

Thus, due to the fact of external stability of the Pareto set (see, e.g., Noghin, 2018), there exists a vector $x^0 \in P^{|J_r|}(C^r + B^r)$ such that

$$(C^r + B^r)x_{J_r} \leq (C^r + B^r)z^0.$$

Then, we have

$$(C_i^r + B_i^r)(x_{J_r} - z^0) \leq 0, \; i \in J_r.$$

So, due to inequalities (6), we obtain

$$[C_i^r(x_{J_r} - z^0)]^+ \leq \|B_i^r\|_p \|x_{J_r} - z^0\|_{p*}, \; i \in J_r. \tag{8}$$

Let $J_r = \{i_1, i_2, \dots, i_v\}$, $1 \le i_1 \le i_2 \le \dots \le i_v \le n$. Taking into consideration (8), as well as the property of l_q-norm monotonicity, we deduce inequalities (7).

$$\|[C^r(x_{J_r} - z^0)]^+\|_q =$$
$$= \|[C^r_{i_1}(x_{J_r} - z^0)]^+, [C^r_{i_2}(x_{J_r} - z^0)]^+, \dots, [C^r_{i_v}(x_{J_r} - z^0)]^+\|_q \le$$
$$\le \|B^r\|_{pq}\, \|x_{J_r} - z^0\|_{p*} \le \|B\|_{pq}\, \|x_{J_r} - z^0\|_{p*} < \psi\, \|x_{J_r} - z^0\|_{p^*}.$$

□

Lemma 4 *Assume* $\emptyset \ne J_r \subseteq N_n$, $r \in N_s$, z^0, $z \in X_{J_r}$, $z^0 \ne z$. *Let a matrix* C^r *with rows* C^r_i, $i \in J_r$, *and vector* η *with positive elements* η_i, $i \in J_r$, *be such that inequality*

$$[C^r_i(z^0 - z)]^+ < \eta_i \|z^0 - z\|_{p*},\ i \in J_r \tag{9}$$

holds. Then, for any $\varepsilon > \|\eta\|_q$, *there exists a matrix*

$$B^r \in \mathbf{R}^{|J_r| \times |J_r|}$$

such that

$$z^0 \notin P^{|J_r|}(C^r + B^r),$$

$$\|B^r_i\|_p = \eta_i,\ i \in J_r,$$

$$\|B^r\|_{pq} < \varepsilon.$$

Proof Let $\varepsilon > \|\eta\|_q$. According to Hölder's inequality (6), for any matrix $D^r \in \mathbf{R}^{|J_r| \times |J_r|}$ with rows D^r_i, $i \in J_r$, the following inequalities are valid:

$$D^r_i(z^0 - z) \le \|D^r_i\|_p \|z^0 - z\|_{p*},\ i \in J_r.$$

Therefore, for any index $i \in J_r$, due to Lemma 1, there exists a matrix B^r with rows B^r_i, $i \in J_r$, such that

$$B^r_i(z^0 - z) = -\eta_i \|z^0 - z\|_{p*},$$

$$\|B^r_i\|_p = \eta_i,\ i \in J_r.$$

From the above expressions, taking into account (9), we deduce that

$$(C^r_i + B^r_i)(z^0 - z) \le [C^r_i(z^0 - z)]^+ - \eta_i \|z^0 - z\|_{p*} < 0,\ i \in J_r,$$

i.e. $z^0 \notin P^{|J_r|}(C^r + B^r)$, where $\|B^r\|_{pq} = \|\eta\|_p < \varepsilon$. □

4. Analytical expression for the stability kernel radius

For the game $Z(C, J_1, J_2, \dots, J_s)$, $C \in \mathbb{R}^{n\times n}$, $n \geq 2$, $s \in N_s$, and any $p, q \in [1, \infty]$, we define

$$\varphi = \varphi^n_{pq}(J_1, J_2, \dots, J_s) =$$
$$\max_{x\in G(\tilde{C})} \min_{k\in N_s} \min_{z\in X_{J_k}\setminus\{x_{J_k}\}} \frac{\|[C^k(x_{J_k} - z)]^+\|_q}{\|x_{J_k} - z\|_{p^*}}.$$

It is obvious that $\varphi \geq 0$.

Theorem 1 *For any $p, q \in [1, \infty]$, $C \in \mathbf{R}^{n\times n}$, $n \geq 2$ and any coalition partition $(J_1, J_2, \dots, J_s)$, $s \in N_n$, the stability kernel radius of the game $Z(C, J_1, J_2, \dots, J_s)$ is expressed by the formula:*

$$\rho^n_{pq}(J_1, J_2, \dots, J_s) = \varphi^n_{pq}(J_1, J_2, \dots, J_s). \tag{10}$$

Proof Taking into account (4), it is easy to see that in order to prove the correctness of (10) it suffices to prove the validity of

$$\rho^n_{pq}(x) = \min_{k\in N_s} \min_{z\in X_{J_k}\setminus\{x_{J_k}\}} \frac{\|[C^k(x_{J_k} - z)]^+\|_q}{\|x_{J_k} - z\|_{p^*}}$$

for any equilibrium outcome $x \in G(\tilde{C})$. Let ψ denote the right-hand side in the formula immediately above. It is obvious that $\psi \geq 0$.

First, we prove inequality $\rho^n_{pq}(x) \geq \psi$. If $\psi = 0$, there is nothing to prove. Let $\psi > 0$, then, according to the definition of ψ, for any vector $z \in X_{J_k}\setminus\{x_{J_k}\}$, the inequalities

$$\|[C^k(x_{J_k} - z)]^+\|_q \geq \psi\|x_{J_k} - z\|_{p^*} > 0,\ k \in N_s \tag{11}$$

hold. Assume that $\rho^n_{pq}(x) < \psi$. Therefore, according to the definition of $\rho^n_{pq}(x)$, there exists a matrix $\tilde{B} \in \Omega^{n\times n}(\psi)$, such that $x \notin G(\tilde{C} + \tilde{B})$. Due to Lemma 3, there exist index $r \in N_s$ and vector $z^0 \in X_{J_r}$ such that inequality (7) holds, and then we get a contradiction with (11). Thus, we have just shown that $\rho^n_{pq}(x) \geq \psi$.

Further, we prove $\rho^n_{pq}(x) \leq \psi$. Let the numbers $\varepsilon > \psi$ and $\Theta > 1$ be such that $\varepsilon > \Theta\psi > \psi$. According to the definition of ψ, the following formula holds

$$\exists r \in N_s\ \ \exists z \in X_{J_r}\setminus\{x_{J_r}\}\ \ \big(\|[C^r(x_{J_r} - z)]^+\|_q = \psi\|x_{J_r} - z\|_{p^*}\big).$$

Then, there exists a vector η with positive components η_i, $i \in J_r$, such that the following relations hold

$$[C^r_i(x_{J_r} - z)]^+ < \Theta[C^r_i(x_{J_r} - z)]^+ = \eta_i\|x_{J_r} - z\|_{p*},\ i \in J_r,$$

$$\|\eta\|_q = \Theta\psi < \varepsilon.$$

Therefore, using Lemma 4, we conclude that there exists a perturbing matrix B^r of size $|J_r| \times |J_r|$ with row $B_i^r,\ i \in J_r$, such that

$$x_{J_r} \notin P(C^r + B^r),$$

$$\|B_i^r\|_p = \eta_i,\ i \in J_r,$$

$$\|B^r\|_{pq} = \|\eta\|_q = \Theta\psi < \varepsilon.$$

Summarizing all the above and taking into consideration Lemma 2, we conclude that for any $x \in G(\tilde{C})$ the following formula holds

$$\forall \varepsilon > \psi \quad \exists \tilde{B} \in \Omega^{n\times n}(\varepsilon) \quad \big(x \notin G(\tilde{C} + \tilde{B}, J_1, J_2, \dots, J_s)\big).$$

Hence, for any $\varepsilon > \psi$ the inequality $\rho^n_{pq}(x) < \psi$ is valid, i.e. $\rho^n_{pq}(x) \le \psi$. Recalling the earlier proven $\rho^n_{pq}(x) \ge \psi$, we end the proof of the theorem. $\square$

5. Corollaries

Corollary 1 *For any $p, q \in [1,\infty]$, $C \in \mathbf{R}^{n\times n}$, $n \ge 2$, the stability kernel radius of the game $Z(C, N_n)$, consisting in finding the Pareto set $P(C)$, is expressed by the formula:*

$$\rho^n_{pq}(N_n) = \max_{x \in P(C)} \ \min_{x' \in X \setminus \{x\}} \frac{\|[C(x - x')]^+\|_q}{\|x - x'\|_{p^*}}.$$

We define the Smale set (Smale, 1974) as follows:

$$S(C) = \{x \in P(C):\ S(x, C) = \emptyset\},$$

where

$$S(x, C) = \{x' \in X \setminus \{x\}:\ Cx = Cx'\}.$$

Corollary 1 implies that the game $Z(C, N_n)$ is T_2-stable ($\rho^n_{pq}(N_n) > 0$) if and only if the Smale set $S(C)$ is non-empty.

From Theorem 1 it follows that

$$\rho^n_{pq}(\{1\}, \{2\}, \dots, \{n\}) = \max_{x \in NE(C)} \ \min_{i \in N_n} \ \min_{z \in X_i \setminus \{x_i\}} \frac{\|[c_{ii}(x_i - z)]^+\|_q}{\|x_i - z\|_{p^*}}.$$

From here, for any $x^0 \in NE(C)$ and $z \in X_i \setminus \{x_i^0\}$, the following equalities hold

$$\frac{\|[c_{ii}(x_i^0 - z)]^+\|_q}{\|x_i^0 - z\|_{p^*}} = \frac{\|c_{ii}(x_i^0 - z)\|_q}{\|x_i^0 - z\|_{p^*}} = |c_{ii}|.$$

So, we get the following result.

Corollary 2 *For any $p, q \in [1, \infty]$, $C \in \mathbf{R}^{n\times n}$, $n \geq 2$, the stability kernel radius of the game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$, consisting in finding the Nash set $NE(C)$ is expressed by the formula:*

$$\rho^n_{pq}(\{1\}, \{2\}, \ldots, \{n\}) = \max\{|c_{ii}| : \ i \in N_n\}.$$

Corollary 2 implies that the game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$ is kernel stable if and only if all the main diagonal elements of the matrix C are different from zero. Theorem 1 also implies the following result.

Corollary 3 *The outcome $x^0 = (x^0_1, x^0_2, \ldots, x^0_n)^T$ of the game with matrix $C \in \mathbf{R}^{n\times n}$, $n \geq 2$, is the Nash equilibrium, i.e. $x^0 \in NE(C)$ if and only if the equilibrium strategy for each player $i \in N_n$ is as follows:*

$$x^0_i = \begin{cases} \max\{x_i : \ x_i \in X_i\} & \text{if } c_{ii} > 0, \\ \min\{x_i : \ x_i \in X_i\} & \text{if } c_{ii} < 0, \\ x_i \in X_i & \text{if } c_{ii} = 0. \end{cases}$$

Corollary 3 implies that the game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$ is T_2-stable if and only if

$$|NE(C)| < \prod_{j\in N_n} |X_j|.$$

6. Numerical examples

Consider the following examples of bi-matrix games with two players. Let $C \in \mathbf{R}^{2\times 2}$ be a matrix with rows C_1 and C_2, and let $X_i \in \{0, 1\}$, $i \in N_2$, $x^{(1)} = (0, 0)^T$, $x^{(2)} = (0, 1)^T$, $x^{(3)} = (1, 0)^T$, $x^{(4)} = (1, 1)^T$. These examples illustrate the different interrelations between stability kernel radii for Nash and Pareto optimality principles. Set $p = q = \infty$. The payoff functions are written as

$$\begin{bmatrix} (C_1x^{(1)}, C_2x^{(1)}) & (C_1x^{(2)}, C_2x^{(2)}) \\ (C_1x^{(3)}, C_2x^{(3)}) & (C_1x^{(4)}, C_2x^{(4)}) \end{bmatrix}.$$

Additionally, set (see Corollaries 1 and 2)

$$\rho^2(P(C)) = \rho^2_{\infty\infty}(N_2) = \max_{x\in P(C)} \ \min_{x'\in X\setminus\{x\}} \max_{i\in N_2} \ \frac{\|C_i(x - x')\|_q}{\|x - x'\|_1}, \tag{12}$$

$$\rho^2(NE(C)) = \rho^2_{\infty\infty}(\{1\}, \{2\}) = \max\{|c_{ii}| : \ i \in N_2\}. \tag{13}$$

Example 1 *Let*

$$C = \begin{pmatrix} 2 & -6 \\ -2 & 1 \end{pmatrix}.$$

Then, we have a bi-matrix game $Z(C)$ with payoffs

$$\begin{bmatrix} (0,0) & (-6,1) \\ (2,-2) & (-4,-1) \end{bmatrix}.$$

Therefore,

$$P(C) = \{x^{(1)}, x^{(2)}, x^{(3)}\},$$

$$NE(C) = \{x^{(4)}\}.$$

It is evident that the Pareto optimal outcome $x^{(1)}$ is not the Nash equilibrium. This type of game is known as prisoner's dilemma, see, for instance, Osborne and Rubinstein (1994). According to formulae (12) and (13), we have

$$\rho^2(P(C)) = \rho^2(NE(C)) = 2.$$

So, we get a game, in which stability kernel radii are equal for both the Nash equilibrium and the Pareto optima sets. Now, consider the following game, where the Pareto and Nash sets have a non-empty intersection.

Example 2 *Let*

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then we have a bi-matrix game $Z(C)$ with payoffs

$$\begin{bmatrix} (0,0) & (-1,0) \\ (2,-1) & (1,-1) \end{bmatrix}.$$

Therefore,

$$P(C) = \{x^{(1)}, x^{(3)}\},$$
$$NE(C) = \{x^{(3)}, x^{(4)}\}.$$

According to formulae (12) and (13), we have

$$\rho^2(P(C)) = 1, \ \rho^2(NE(C)) = 2.$$

In this example a coalition formation, based on Pareto optimality, i.e. when all players are united in a single coalition, has a smaller stability kernel than in the situation when players do not form any coalitions and play independently.

7. Conclusion

As a result of parametric analysis performed, the formula for the stability kernel radius was obtained in a finite cooperative game of several players in a normal form with parametric optimality ranging from Pareto solutions to Nash equilibria in the case where criterion and solution spaces are endowed with various Hölder's norms.

One of the biggest challenges in this field is to construct efficient algorithms to calculate the analytical expression. To the best of our knowledge, there are not many results known in that area. Moreover, some of those results, which have been already known, put more questions than answers. As it was pointed out in Nikulin, Karelkina and Mäkelä (2013), calculating exact values of stability radii is an extremely difficult task in general, so that one could concentrate either on finding the easily computable classes of problems or on developing general metaheuristic approaches.

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