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UNIVERSITY OF TURKU

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Harto Saarinen

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ABSTRACT

The classical setting in optimal stopping and optimal control theory assumes that the agent controlling the system can operate continuously in time. In optimal stopping this setting is highly stylized for many applications, for example, in mathematical finance due to illiquid markets. In optimal stochastic control this setting often leads to optimal strategies being singular with respect to the Lebesgue measure, and thus the strategies are not feasible in practice. Hence, it is of importance to study these problems from such a perspective that their solutions are practically more implementable.

In this thesis we alter the classical setting by introducing an exogenous constraint, in the form of a signal process, for the control opportunities of the agent. In order to keep the problems more tractable, especially time-homogeneous and Markovian, the signal process is assumed to be a Poisson process with constant intensity. Consequently, the agent can only have influence on the system at discrete times. We call these control problems Poisson constrained control problems and study them when the dynamics are governed by linear diffusion processes.

Linear diffusions are particular enough to have a rich theory but still general enough to offer a class of interesting dynamics that are applicable in various situations. A key factor is also that many control problems with diffusions will lead to closed-form solutions. This thesis investigates to which extent the classical theory of diffusion can be applied in this class of control problems to form closed-form solutions.

KEYWORDS: Linear diffusions, Poisson process, Resolvent operator, Stochastic control

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TIIVISTELMÄ

Optimaalisen pysäyttämisen ja optimaalisen säätöteorian ongelmissa oletetaan usein, että systeemiä ohjaava agentti voi halutessaan toimia rajoituksitta. Tämän seurauksena perinteisen säätöteorian ongelmissa optimaalinen strategia on usein säätää systeemiä jatkuvasti. Useissa sovelluksissa tällaiset stretegiat eivät kuitenkaan ole toteutettavissa. Esimerkiksi matemaattisen rahoituksen portfolion optimointiongelmissa ei myyminen ja ostaminen ole mahdollista milloin tahansa likvidisyysrajoitteista johtuen.

Tässä väitöskirjassa perinteisen säätöteorian ongelmiin lisätään signaaliprosessi, joka antaa agentille ajanhetket, joina säätäminen on mahdollista. Jotta säätöongelma olisi matemaattisesti ratkaistavissa, oletamme, että signaaliprosessi on vakiointensiteettinen Poisson-prosessi. Täten agentti voi vaikuttaa systeemin kulkuun vain diskreetteinä ajanhetkinä. Kutsumme näitä säätöongelmia Poisson-rajoitteisiksi säätöongelmiksi ja tutkimme niitä, kun säädettävää systeemiä kuvaa lineaarinen diffuusioprosessi.

Lineaariset diffuusiot ovat kohtuullisen yleisiä dynamiikaltaan, mikä mahdollistaa niiden käytön useissa erilaisissa sovelluksissa. Lisäksi lineaaristen diffuusioiden teoria on hyvin tutkittua, mikä takaa niiden sujuvan matemaattisen käsittelyn. Usein säätöteorian ongelmat, joissa dynamiikkaa kuvaa lineaarinen diffuusio, ratkeavatkin suljetussa muodossa. Tässä väitöskirjassa sovelletaan klassisen diffuusioteorian tuloksia ja menetelmiä Poisson-rajoitettujen säätöongelmien ratkaisemiseksi.

ASIASANAT: Lineaariset diffuusiot, Poisson prosessi, resolventti, stokastinen kontrolli

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10.03.2023, *Harto Saarinen*

Table of Contents

List of Original Publications

This dissertation is based on the following original publications, which are referred to in the text by their Roman numerals:

I J. Lempa and H. Saarinen. Ergodic control of diffusions with random intervention times. *Journal of Applied Probability*, 2021; 58(1): 1-21.

II J. Lempa and H. Saarinen. A zero-sum Poisson stopping game with asymmetric signal rates. *Applied Mathematics and Optimization*, 87:35, 2023.

III J. Lempa and H. Saarinen. A note on asymptotics between singular and constrained control problems of one-dimensional diffusions. *Acta Applicandae Mathematicae*, 181:13, 2022.

IV H. Saarinen. Two-sided Poisson control of linear diffusions. 2022; Submitted.

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1 Linear diffusions

In this chapter we go through some of the basic definitions in the theory of stochastic processes and the classical theory of diffusions at a level that is required for the rest of this thesis. The reader should thus be able to follow this thesis by having basic knowledge about probability spaces and Brownian motion.

We will consider a class of one-dimensional stochastic processes called *regular linear diffusions* (diffusions for short). In order to define diffusions we present some basics of stochastic processes in the first section. Unless otherwise indicated we refer to [1] for the contents of this chapter.

1.1 Stochastic processes

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions (completion by $\mathbb{P}$ null sets and right continuity). A *one-dimensional stochastic process* is a family of random variables X_t defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ that take values on space $(I, \mathcal{B})$, where I is an interval called a state space and $\mathcal{B}$ is the Borel sigma algebra on I. We denote by $l \geq -\infty$ the left endpoint and by $r \leq \infty$ the right endpoint of I. The *natural filtration* with respect to X_t is the filtration generated by X_t, that is $\sigma\{X_s, s \leq t\}$.

***Definition* 1.1.1** (Markov process). *Let $f : I \to \mathbb{R}$ be a bounded Borel function. Then X_t is called a* Markov process *if it is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and*

$$\mathbb{E}_x[f(X_{t+h}) \mid \mathcal{F}_t] = \mathbb{E}_{X_t}[f(X_h)], \quad \textit{for all } h \geq 0. \qquad (1)$$

Intuitively speaking, the future behaviour of a Markov process does not depend on its behaviour in the past but only on the present state.

***Definition* 1.1.2** (Stopping time). *A function $\tau : \Omega \to \mathbb{R}_+ \cup \infty$ is called a* stopping time *if*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

In other words, the random variable τ is a stopping time if one is able to decide, based on the present information, if τ has occurred or not. Important stopping times are *lifetime* and *explosion time*. The lifetime ζ of a stochastic process X_t is the time until it is sent to a state $\Delta \notin I$ called a *cemetery state*, where it stays until the end.

The cemetery state should be understood as an extra state that augments the state space I such that $I_\Delta = I \cup \Delta$ and $\mathcal{B}_\Delta = \sigma\{\mathcal{B}, \Delta\}$. Let $\tau_I = \inf\{t \geq 0 \mid X_t = l \text{ or } X_t = r\}$ be the first hitting time to either of the endpoints of the state space. Then the explosion time is defined as the minimum of the lifetime ζ and the hitting time τ_I.

Strong Markov processes are similar to Markov processes but (1) is assumed to be satisfied for all finite stopping times τ.

***Definition* 1.1.3** (Strong Markov process). *Let $f : I \to \mathbb{R}$ be a bounded Borel function. Then X_t is called a* strong Markov process *if for all finite stopping times τ and all $h \geq 0$*

$$\mathbb{E}_x[f(X_{\tau+h}) \mid \mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[f(X_h)],$$

where $\mathcal{F}_\tau = \{N \in \mathcal{F} \mid N \cap \{\tau \leq t\} \in \mathcal{F}_t,\, t \geq 0\}$.

Let τ_y denote the first hitting time of X_t to a state y, or in other words

$$\tau_y = \inf\{t \geq 0 \mid X_t = y\}.$$

A stochastic process X_t is called *regular* if

$$\mathbb{P}_x(\tau_y < \infty) > 0 \text{ for all } x, y \in I. \tag{2}$$

In other words, a regular stochastic process X_t can reach any state starting from any other state with positive probability. If instead

$$\mathbb{P}_x(\tau_y < \infty) = 1 \text{ for all } x, y \in I,$$

so that the process X_t reaches any state from any other state with probability one, X_t is called *recurrent*.

Another key concept in the theory of stochastic processes needed in this thesis is that of a *martingale*. A stochastic process is a martingale if the best approximation of the future is the present.

***Definition* 1.1.4** (Martingale). *A stochastic process X_t is called a* martingale *(with respect to a filtration $\mathcal{F}_t$) if X_t is $\mathcal{F}_t$-measurable, $\mathbb{E}_x[|X_t|] < \infty$ and*

$$\mathbb{E}_x[X_s \mid \mathcal{F}_t] = X_t, \quad \textit{for all } s \geq t. \tag{3}$$

If "$=$" in (3) is replaced by "$\leq$" or "$\geq$", X_t is called a *supermartingale* or *submartingale*, respectively. Consequently, if X_t is supermartingale and submartingale, then X_t is a martingale. A generalization of a martingale is given by a *local martingale*.

***Definition* 1.1.5** (Local martingale)**.** *Let X_t be $\mathcal{F}_t$-measurable. If there exists a sequence of stopping times τ_k such that it is almost surely increasing and divergent. Then X_t is called a* local martingale *if the stopped process $X_t^{\tau_k} = X_{\min\{t,\tau_k\}}$ is a martingale for all k.*

The name of a local martingale is descriptive as locally their paths are very similar to martingales and the difference occurs in the limit. Local supermartingales (submartingales) are defined in an analogous manner by saying that the stopped process is a supermartingale (submartingale). One can show that if a local supermartingale is bounded from below, it is a supermartingale.

A sufficient condition for a local martingale to be a martingale is also given by *uniform integrability*.

***Definition* 1.1.6** (Uniform integrability)**.** *A family $\{X_t\}_{t\geq 0}$ of random variables is said to be* uniformly integrable *if*

$$\sup_t \mathbb{E}[|X_t| \mathbb{1}_{\{|X_t|>c\ \}}] \to 0, \quad c \to \infty.$$

A useful necessary and sufficient condition for uniform integrability is given in section 2.6 of [2].

***Proposition* 1.1.7.** *A family $\{X_t\}_{t\geq 0}$ of random variables is uniformly integrable if and only if they are uniformly bounded*

$$\sup_t \mathbb{E}_x[|X_t|] < \infty$$

and uniformly absolutely continuous

$$\sup_t \mathbb{E}_x[|X_t| \mathbb{1}_A] \to 0, \text{ when } \mathbb{P}(A) \to 0.$$

We will need the uniform integrability of martingales in order to apply the following version of the optional stopping theorem that does not require the stopping time to be almost surely finite.

***Proposition* 1.1.8** (Optional stopping)**.** *Let X_t be a uniformly integrable martingale and τ a stopping time with respect to the filtration $\mathcal{F}_t$. Then*

$$\mathbb{E}_x[X_\tau] = \mathbb{E}_x[X_0].$$

1.2 Linear diffusions

In this subsection we define the class of stochastic processes called *linear diffusions* and study their main properties.

***Definition* 1.2.1** (Regular linear diffusion)**.** *A* regular linear diffusion *(or* regular one-dimensional diffusion*) X is a regular time-homogeneous strong Markov process that has $\mathbb{P}_x$ almost surely continuous paths and takes values in an interval I.*

Regularity is not usually included as part of the definition of a linear diffusion, but we will only be working with regular diffusions throughout the thesis.

***Definition* 1.2.2** (Infinitesimal generator)**.** *The operator $\mathcal{A}$ defined as*

$$\mathcal{A}f = \lim_{t \to 0+} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t} \tag{4}$$

is called the infinitesimal generator *of X_t. Here the function $f : I \to \mathbb{R}$ is such that the limit in (4) exists for all $x \in I$.*

Essentially, the infinitesimal generator describes the average movement of the diffusion X_t in infinitesimal time interval. Hence, it is not too surprising that in many cases it can be represented as a differential operator as we will shortly find out.

Let $S : I \to \mathbb{R}$ be a continuous increasing function and k, m non-negative measures that satisfy

$$m((x,y)) = \int_x^y m(dz) < \infty \quad \text{and} \quad k((x,y)) = \int_x^y k(dz) < \infty$$

for all $l < x < y < r$. Given such S, m and k, one can construct a diffusion whose behaviour is characterized by this triplet (see for example in section 5.6 of [3] and in section 7.2 of [4]). The functions S, m and k are the *basic characteristics* of the diffusion, and called the *scale function*, the *speed measure* and the *killing measure*, respectively. The names for the basic characteristics are rather descriptive: It can be shown that the speed measure describes the expected time that the process spends in small intervals, the killing measure is related to the distribution of the process at its lifetime ζ and the scale function has the property that it scales the state space in terms of the probabilities of hitting various states (see e.g. section 15.3 of [5]). Also, the scaled diffusion $S(X_t)$ turns out to be a local martingale.

We will consider the case where the basic characteristics are absolutely continuous with respect to the Lebesgue measure and that they have smooth derivatives. In this case

$$m(dx) = m(x)dx, \quad k(dx) = k(x)dx, \quad S(x) = \int^x S'(z)dz,$$

where $m > 0$, $S' > 0$ and $k \geq 0$ are continuous. Further, if S'' is continuous, then the infinitesimal generator $\mathcal{A}$ can be written for $f \in \mathcal{C}^2$ as

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x) + \mu(x)\frac{d}{dx}f(x) - c(x)f(x). \tag{5}$$

Here the *infinitesimal parameters* σ, μ, c of X_t are connected to the basic characteristics by

$$m(x) = 2\sigma^{-2}(x)e^{\mathcal{B}(x)}, \quad S'(x) = e^{-\mathcal{B}(x)}, \quad k(x) = 2\sigma^{-2}(x)c(x)e^{\mathcal{B}(x)}, \tag{6}$$

where $\mathcal{B}(x) = \int^x 2\sigma^{-2}(z)\mu(z)dz$. When $k(x) = 0$, the differential operator $\mathcal{A}$ can also be represented as a successive differentiations with respect to the scale function and the speed measure. Indeed,

$$\mathcal{A} = \frac{1}{2}\frac{d}{dm(x)}\frac{d}{dS(x)},$$

which is referred to as the *canonical representation*. Further, we note that the scale function S solves the ordinary differential equation $\mathcal{A}f(x) = 0$.

The basic characteristics can be shown to define the behaviour of the diffusion completely for any open interval in state space I. However, some boundary conditions on I are needed to have a full unique characterization. We will go through the characterization for the left endpoint l as similar definitions hold for the right endpoint. Define the functions

$$\begin{aligned} N(z) &= \int_l^z (m((a,z)) + k((a,z)))S(a)da, \\ \Sigma(z) &= \int_l^z (S(z) - S(a))(m(a) + k(a))da. \end{aligned}$$

The function $N(z)$ roughly describes the time it takes from the diffusion to reach a point $z \in (l, r)$ from the boundary l and $\Sigma(z)$ roughly describes the time it takes to reach l from $z \in (l, r)$ (see [5] section 15.6).

***Definition* 1.2.3.** *Let $z \in (l, r)$. The left endpoint of an interval I for a diffusion X_t is*

(i) exit, *if $N(z) < \infty$,*

(ii) entrance, *if $\Sigma(z) < \infty$,*

(iii) regular, *if it is exit and entrance,*

(iv) natural, *if it is neither exit nor entrance.*

The regular boundaries are further classified based on the values of $m(l)$ and $k(l)$. However, when the basic characteristics are absolutely continuous with respect to the Lebesgue measure the only two possibilities are *reflecting* boundary, when $m(l) = k(l) = 0$ and *killing* boundary, when $m(l) \neq \infty$ and $k(l) = \infty$.

We will from here on focus on the case where $k(x) = 0$ for all $x \in I$. Let $r > 0$ be a constant killing rate or discounting rate. Given the characterization of the infinitesimal generator 1.2.2 as a linear second order differential operator, we can study its properties from the perspective of ordinary differential equations. The two linearly independent functions $\psi_r(x)$ and $\varphi_r(x)$ of the ordinary differential equation

$$(\mathcal{A}-r)f(x)=0$$

are called the *fundamental solutions*. The *Wronskian* is

$$B_r=\frac{\psi_r'(x)}{S'(x)}\varphi_r(x)-\frac{\varphi_r'(x)}{S'(x)}\psi_r(x)>0,$$

and can be shown by direct differentiation to be independent of x. The fundamental solutions can be characterized as the unique (up to a multiplicative constant) positive solutions by demanding that ψ_r is increasing, φ_r decreasing, and giving boundary conditions on regular boundaries. When l is *reflecting*, we have $\frac{\psi_r'(l)}{S'(l)}=0$ and when l is killing, we have $\psi_r(l+)=0$. The functions ψ_r and φ_r also satisfy the following properties at the left endpoint l depending on the boundary behaviour of X_t:

(i) if l is entrance, $\psi_r(l+)>0$, $\frac{\psi_r'(l+)}{S'(l+)}=0$, $\varphi_r(l+)=\infty$, $\frac{\varphi_r'(l+)}{S'(l+)}>-\infty$,

(ii) if l is exit $\psi_r(l+)=0$, $\frac{\psi_r'(l+)}{S'(l+)}>0$, $\varphi_r(l+)<\infty$, $\frac{\varphi_r'(l+)}{S'(l+)}=-\infty$,

(iii) if l is natural $\psi_r(l+)=0$, $\frac{\psi_r'(l+)}{S'(l+)}=0$, $\varphi_r(l+)=\infty$, $\frac{\varphi_r'(l+)}{S'(l+)}=-\infty$.

As seen above the fundamental solutions carry information about the boundary behaviour of the diffusion X_t, and as we will see below, they also determine the distribution of the hitting times of the diffusion to various states. Recall that τ_z denotes the first hitting time of $z\in I$ by the diffusion X_t. Then using the fundamental solutions we can express the Laplace transform of τ_z as

$$\mathbb{E}_x[e^{-r\tau_z}]=\begin{cases}\frac{\psi_r(x)}{\psi_r(z)}, & z\le x,\\ \frac{\varphi_r(x)}{\varphi_r(z)}, & z>x.\end{cases} \tag{7}$$

The differential operator $(\mathcal{A}-r)$ has an inverse for any $r>0$, and the inverse can be given in terms of the diffusion that characterizes $\mathcal{A}$. Let $\mathcal{L}_r^1$ be the set of functions satisfying the integrability condition

$$\int_0^{\zeta}e^{-rt}|f(X_t)|dt<\infty,$$

where ζ is the lifetime of X_t. Then the inverse of $(\mathcal{A}-r)$ is the *resolvent* defined below.

***Definition* 1.2.4** (Resolvent)**.** *A* resolvent *of a function $f \in \mathcal{L}_r^1$ is*

$$(R_r f)(x) = \mathbb{E}_x\left[\int_0^\zeta e^{-rt} f(X_t)dt\right],$$

where $r > 0$ and ζ is the lifetime of X_t.

The following Proposition (see [1] pp. 4-5 and p. 11) highlights the main properties of the resolvent operator. Especially, the property (iv) is of interest as it connects the fundamental solutions ψ_r and φ_r to the resolvent operator in computationally useful way. It will be one of the main tools used throughout the thesis.

***Proposition* 1.2.5.** *Let $f \in \mathcal{L}_r^1$. Then the resolvent*

(i) *satisfies the* resolvent equation

$$(R_r - R_\lambda) = (\lambda - r)R_r R_\lambda, \quad \lambda > r > 0;$$

(ii) *is an inverse operator of the infinitesimal generator in the sense that for $f \in \mathcal{C}^2$*

$$(R_r(\mathcal{A} - r)f)(x) = -f(x);$$

(iii) *is a contraction map,*

$$||R_r|| \leq 1/r,$$

where $||R_r|| = \sup\{||R_r f|| \mid ||f|| = 1\}$ and $||f||$ is the usual sup-norm;

(iv) *can be represented using the fundamental solutions as*

$$(R_r f)(x) = B_r^{-1}\Big[\varphi_r(x)(\Psi_r f)(x) + \psi_r(z)(\Phi_r f)(x)\Big],$$

where

$$(\Psi_r f)(x) = \int_0^x f(z)\psi_r(z)m'(z)dz, \ (\Phi_r f)(x) = \int_x^\infty f(z)\varphi_r(z)m'(z)dz;$$

(v) *has a derivative*

$$(R_r f)'(x) = B_r^{-1}\Big[\varphi_r'(x)(\Psi_r f)(x) + \psi_r'(z)(\Phi_r f)(x)\Big].$$

The integral functionals $(\Psi_r f)$ and $(\Phi_r f)$ that appear in part (iv) and (v) of Proposition 1.2.5 will play a key role in the next chapters and one of their main properties is given in the next Proposition.

***Proposition* 1.2.6.** *(A) Assume that $f \in C^2$, $\lim_{x\to l+} |f(x)| < \infty$ and $(\mathcal{A} - r)f \in \mathcal{L}_r^1$. Then*

$$\frac{f'(x)}{S'(x)}\psi_r(x) - \frac{\psi_r'(x)}{S'(x)}f(x) = (\Psi_r(\mathcal{A} - r)f)(x) - \delta,$$

where $\delta = 0$ if l is natural or entrance and $\delta = B_r \frac{f(0)}{\varphi_r(0)}$ otherwise.

(B) Assume that $f \in C^2$, $\lim_{x\to r-} f(x)/\psi_r(x) = 0$ and $(\mathcal{A} - r)f \in \mathcal{L}_r^1$. Then

$$\frac{f'(x)}{S'(x)}\varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)}f(x) = (\Phi_r(\mathcal{A} - r)f)(x).$$

Using the above Proposition the fundamental solutions $\psi_r(x)$ and $\varphi_r(x)$ can be shown to satisfy the following useful inequalities for $z < x$ and $\lambda, r > 0$ (see lemma 4 in article IV)

$$\frac{\psi_{r+\lambda}(z)}{\psi_r(z)} \leq \frac{\psi_{r+\lambda}(x)}{\psi_r(x)} \leq \frac{\psi'_{r+\lambda}(x)}{\psi'_r(x)},$$
$$\frac{\varphi_{r+\lambda}(z)}{\varphi_r(z)} \leq \frac{\varphi_{r+\lambda}(x)}{\varphi_r(x)} \leq \frac{\varphi'_{r+\lambda}(x)}{\varphi'_r(x)}.$$

In order to have more tools at our disposal to solve the stochastic control problems in the following chapters, we further restrict ourselves to consider a class of diffusions called *Itô diffusions*. To define Itô diffusions, let W_t be a one-dimensional Brownian motion, $\mu : I \to \mathbb{R}$ be the drift function and $\sigma : I \to \mathbb{R}_+$ be the volatility function. Then stochastic process X_t given as the unique *weak solution* to a stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{8}$$

is called an *Itô diffusion*. Basically, a weak solution to (8) means that one can find a probability space, and a Brownian motion W_t defined on it, such that (8) holds for the stochastic process X_t. Hence, uniqueness should be understood as uniqueness in distribution. A set of sufficient conditions for a weak solution to the stochastic differential equation (8) to exist is given by

$$\sigma^2(x) > 0 \qquad \text{(non-degeneracy)}, \tag{9}$$
$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |\mu(z)|}{\sigma^2(z)} dz < \infty \qquad \text{(local integrability)}\,, \tag{10}$$

where the last condition means that for every $x \in I$ there exists a ε such that the given integral is finite.

The main tool that we will use related to stochastic differential equations is Itô's formula and its generalizations.

***Theorem* 1.2.7** (Itô's formula)**.** *Let f be a twice continuously differentiable function. Then*

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds.$$

The combination of tools from the classical theory of diffusions and stochastic differential equations gives us an ideal framework for the rest of this thesis. Finally, we give full statement connecting linear diffusions with Itô diffusions under our assumptions.

***Proposition* 1.2.8.** *Assume that σ and μ are continuous, $k \geq 0$, and the non-degeneracy (9) and local integrability (10) conditions hold. Then there exists a diffusion X_t such that it is given by a unique weak solution to the stochastic differential equation*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

and its infinitesimal generator up to an explosion time can be written for $f \in C^2$ as

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x) + \mu(x)\frac{d}{dx}f(x) - c(x)f(x).$$

1.3 Examples of diffusions

We give some examples of the most common diffusions defined through stochastic differential equations.

1.3.1 Brownian motion with drift

The diffusion X_t defined by

$$dX_t = \mu dt + dW_t, \quad X_0 = x, \tag{11}$$

where $\mu > 0$, is called a *Brownian motion with drift*. The state space of the process is $\mathbb{R}$ and its infinitesimal generator is given by

$$\mathcal{A}f = \frac{1}{2}\frac{d^2}{dx^2}f + \mu\frac{d}{dx}f.$$

The fundamental solutions are in this case known to be

$$\varphi_\lambda(x) = e^{-\left(\sqrt{\mu^2+2\lambda}+\mu\right)x}, \quad \psi_\lambda(x) = e^{\left(\sqrt{\mu^2+2\lambda}-\mu\right)x},$$

and the scale density and density of the speed measure read as

$$S'(x) = e^{-2\mu x}, \quad m'(x) = 2e^{2\mu x},$$

respectively. Both endpoints of the state space are in this case natural.

1.3.2 Geometric Brownian motion

The diffusion X_t defined by

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x, \tag{12}$$

where $\mu > 0$ and $\sigma > 0$, is called a *geometric Brownian motion*. The state space of the process is $\mathbb{R}_+$ and the infinitesimal generator is

$$\mathcal{A}f = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} f + \mu x \frac{d}{dx} f.$$

The scale density and the density of the speed measure read as

$$S'(x) = x^{-\frac{2\mu}{\sigma^2}}, \quad m'(x) = \frac{2}{\sigma^2} x^{\frac{2\mu}{\sigma^2}-2}.$$

Denote

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1,$$
$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0.$$

Then the fundamental solutions for X_t read as

$$\psi_r(x) = x^\beta, \quad \varphi_r(x) = x^\alpha.$$

Both endpoints of the state space are in this case natural.

1.3.3 Ornstein-Uhlenbeck process

Consider dynamics that are characterized by a stochastic differential equation

$$dX_t = -\delta X_t dt + dW_t, \quad X_0 = x,$$

where $\delta > 0$. This diffusion is often used to model continuous time systems that have mean reverting behaviour. The state space is $\mathbb{R}$ and the infinitesimal generator is

$$\mathcal{A}f = \frac{1}{2} \frac{d^2}{dx^2} f - \delta x \frac{d}{dx} f.$$

The scale density and the density of speed measure are

$$S'(x) = \exp(\delta x^2), \quad m'(x) = 2\exp(-\delta x^2),$$

and the fundamental solutions read as

$$\varphi_\lambda(x) = e^{\frac{\delta x^2}{2}} D_{-\lambda/\delta}(x\sqrt{2\delta}), \quad \psi_\lambda(x) = e^{\frac{\delta x^2}{2}} D_{-\lambda/\delta}(-x\sqrt{2\delta}),$$

where $D_\nu(x)$ is a parabolic cylinder function. We note that this process is positively recurrent and its stationary probability measure is given by

$$m(dx) = \frac{\sqrt{\delta}}{\sqrt{\pi}} e^{-\delta x^2} dx.$$

Both endpoints of the state space are in this case natural.

2 Some optimal control problems

In this chapter we go through some control problems of linear diffusions that can be solved using the tools presented in the first chapter. Thus, we assume for the rest of this thesis, without further notice, that the linear diffusion X_t satisfies the assumptions in Proposition 1.2.8, and that $k(x) = 0$ for all x. As the reader will notice, the choice of the framework to study the control problems allows the solutions to be presented rather explicitly (in a closed-form). The term explicit should be understood in this case to mean that the solution can be given in terms of the fundamental solutions ψ_r, φ_r and their functionals defined in the first chapter.

Since the focus of this thesis is in the Poisson constrained control problems (defined in chapter three), this chapter is mostly meant to highlight the aspects of these classical problems, which are relevant when comparing to the solutions of the Poisson constrained control problems.

2.1 Optimal stopping problems and optimal stopping games

2.1.1 Optimal stopping problems

Optimal stopping problems usually arise in applications where a decision maker (an agent) is faced with a problem of timing his/her decision to maximize a reward. Such problems appear, for instance, in stochastic analysis, sequential analysis, and mathematical finance. We refer to [6] for these applications, and also for the discussion in this subsection.

The optimal stopping problem is to find a value function V such that

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau} g(X_\tau)], \tag{13}$$

where the supremum is taken over the set of all stopping times with respect to the natural filtration of X_t, g is a non-negative, continuous function and the discounting factor $r \geq 0$ is such that

$$\mathbb{E}_x[\sup_{t\geq 0} e^{-rt} g(X_t)] < \infty. \tag{14}$$

The *optimal stopping time* τ^* is defined to be any stopping time for which the supremum in (13) is attained. The function g is usually called *exercise payoff* or *reward*.

The theory of optimal stopping is well developed and there are multiple ways of studying optimal stopping problems. One of the most celebrated results in optimal stopping is the characterization of the value using r-excessive functions.

***Definition* 2.1.1** (r-excessive function)**.** *Assume that $f : I \to \mathbb{R}_+$ is measurable. Then f is called* r-excessive *(with respect to X_t) if*

(i) $\mathbb{E}_x(e^{-rt} f(X_t)) \leq f(x)$ *for all* $x \in I$ *and*

(ii) $\lim_{t\to 0} \mathbb{E}_x(e^{-rt} f(X_t)) = f(x)$ *for all* $x \in I$.

***Definition* 2.1.2** (smallest r-excessive majorant)**.** *A function f is called a smallest r-excessive majorant of g if*

(i) f is r-excessive and $f(x) \geq g(x)$ for all x,

(ii) every other r-excessive majorant $\tilde{f}$ of g satisfies $\tilde{f}(x) \geq f(x)$ for all x.

***Proposition* 2.1.3.** *Let g be a continuous, non-negative function, which satisfies (14). Then the value function V in (13) is given by the smallest r-excessive majorant of the payoff g.*

From our perspective, two important approaches to optimal stopping are *Bellman's principle* (also called *dynamic programming principle*) and *free-boundary problems*. In our case, Bellman's principle can be written as the *Hamilton-Jacobi-Bellman variational inequality* (HJB)

$$V(x) = \max\{(\mathcal{A} - r)V(x) + V(x), g(x)\},$$

where the first component describes the option to gain a discounted profit by continuing over an infinitesimal period and the second component the option to stop and gain the payoff g. Bellman's principle is very general by itself and it reveals its usefulness for our approach after splitting the state space I in two regions. We denote by $C = \{x \mid V(x) > g(x)\}$ the *continuation region*, where the diffusion is allowed to evolve and is never terminated, and by $S = \{x \mid V(x) = g(x)\}$ the *stopping region*, where upon entering the diffusion X_t is immediately stopped. This split leads us to the free-boundary problem defined as

$$\begin{cases} (\mathcal{A} - r)V(x) = 0, & x \in C, \\ V(x) = g(x), & x \in S, \end{cases} \tag{15}$$

where ∂S is the boundary of the set S. However, to find the boundary ∂S additional conditions are needed. Usually, the so-called *principle of smooth fit* is adequate. This principle states that the value function V should be continuously differentiable across the boundary ∂S and that $V'(x) = g'(x)$ for $x \in \partial S$.

In practice it is often beneficial to start directly from (or derive using heuristic arguments) Bellman's principle or the free-boundary formulation and guess the shape of the optimal rule. For example, in many applications it turns out that a good guess for the shape of the continuation region is $C = (l, y^*)$, and consequently, the optimal stopping time is likely to be a hitting time to a state y^*, that is $\tau^* = \tau_{y^*} = \inf\{t \geq 0 \mid X_t = y^*\}$. These type of optimal policies are often referred to as *one-sided rules*, in contrary to *two-sided rules*, where $C = (x^*, y^*)$ for some $x^* < y^*$ and $x^*, y^* \in I$. For a one-sided stopping policy the free-boundary problem (15) takes the form

$$\begin{cases} (\mathcal{A} - r)V(x) = 0, & x < y^*, \\ V(x) > g(x), & x < y^*, \\ V(x) = g(x), & x = y^*, \\ V'(x) = g'(x), & x = y^*, \\ V(x) = g(x), & x > y^*, \end{cases} \tag{16}$$

After finding a candidate solution by solving (16), one needs to verify using a separate *verification theorem* (for example the Proposition 2.1.3) that the candidate is indeed the solution. This *guess and verify* approach is especially useful in time-homogeneous problems, where it is to be expected that the optimal stopping time is the first hitting time to some state or states of the state space.

We can see immediately from Proposition 2.1.3 that the fundamental solutions ψ_r and φ_r are related to solving the optimal stopping problem, because they are also identified as minimal excessive functions. The same is immediate from the free-boundary problem as it implies that $(\mathcal{A} - r)V(x) = 0$, when $x \in C$, and thus we have $V(x) = c_1\psi_r(x) + c_2\varphi_r(x)$, when $x \in C$. These observations are not, however, very surprising in the light of (7), as we notice that for one-sided rules, we have

$$\mathbb{E}_x[e^{-r\tau_z} g(X_{\tau_z})] = \begin{cases} \frac{\psi_r(x)}{\psi_r(z)} g(z), & z \leq x, \\ \frac{\varphi_r(x)}{\varphi_r(z)} g(z), & z > x. \end{cases}$$

2.1.2 Optimal stopping games

We consider optimal stopping games between two players, a *sup-player* and a *inf-player*. The sup-player attempts to maximize the expected present value of exercise payoff, whereas the inf-player's objective is to minimize the same quantity. This differs from a standard stopping problem in that the players have to take into account the strategy of the other player and possibly adjust their strategy. Some applications of stopping games in mathematical finance are for example cancellable options and convertible bonds. In what follows, we will focus on zero-sum stopping games called *Dynkin games*, and we follow [7] and [8].

The players are assumed to have their respective exercise payoff functions g_s and g_i, and are allowed to stop the process X_t at any stopping times defined with respect to the natural filtration of X_t. To define the value of the game we let

$$R(\tau,\sigma) = g_s(X_\tau)\mathbb{1}_{\{\tau<\sigma\}} + g_i(X_\sigma)\mathbb{1}_{\{\tau>\sigma\}} + g_p(X_\tau)\mathbb{1}_{\{\tau=\sigma\}},$$

where g_p is the payoff that corresponds to the simultaneous stopping of the players, and $g_i(x) \geq g_p(x) \geq g_s(x)$ for all $x \in I$. The lower and upper values of the game are defined as

$$\underline{V}(x) = \sup_\tau \inf_\sigma \mathbb{E}_x\left[e^{-r(\tau\wedge\sigma)}R(\tau,\sigma)\right], \quad \overline{V}(x) = \inf_\sigma \sup_\tau \mathbb{E}_x\left[e^{-r(\tau\wedge\sigma)}R(\tau,\sigma)\right].$$

We naturally have

$$g_s(x) \leq \underline{V}(x) \leq \overline{V}(x) \leq g_i(x).$$

When $\underline{V}(x) \geq \overline{V}(x)$, the equality

$$V(x) = \underline{V}(x) = \overline{V}(x) \tag{17}$$

holds and the zero-sum game is said to have a value V. The maximizing strategies in $\underline{V}$ and the minimizing strategies in $\overline{V}$ are called optimal and any pair of optimal strategies is a *Nash equilibrium*.

Dynkin games can be solved using methods similar to optimal stopping problems, even though they are usually mathematically more demanding. We refer to [9] for a similar approach to r-excessive functions in optimal stopping, [8; 10] using fluctuation theory, and [11] using variational inequalities.

2.2 Singular optimal control

Even though optimal stopping problems fit many applications it is usually necessary to model cases, where the decision maker (agent) is allowed to take action more than once instead of terminating the process completely once and for all. Applications in this direction are dividend payment problems [12] and rational harvesting of renewable resources [13].

Assume for now that the agent can control the path of the diffusion X_t downwards by applying a control D_t^s. We assume that the control D_t^s belongs to a set *admissible controls* $\mathcal{D}_s$, which is given by non-negative, non-decreasing, right-continuous, and $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes. Then, following [14], the controlled dynamics are given by the generalized stochastic differential equation

$$X_t^{D^s} = \mu(X_t^{D^s})dt + \sigma(X_t^{D^s})dW_t - \gamma dD_t^s, \quad X_0^{D^s} = x \in I,$$

where D_t^s denotes the applied control policy and γ is a positive constant called a *proportional transaction cost*. The following control problems are called *singular*, because the set of admissible controls allows for control policies that are singular with respect to the Lebesgue measure. These controls are usually also called *reflecting* controls, or *local time push* controls, as in many cases of interest the optimal policy is to prevent the process to cross some state of the state space by reflecting it by infinitesimal amount downwards. This behaviour will be in sharp contrast to the optimal policies in the Poisson constrained control problems defined in chapter three.

In this thesis we will investigate two different control criterion: *discounted criterion* and *ergodic criterion*. In discounted problems the agent is usually faced with the problem of minimizing (or maximizing)

$$\mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(X_t^{D^s})dt+\gamma dD_t^s)\right], \tag{18}$$

and in ergodic problems with minimizing (or maximizing)

$$\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}_x\left[\int_0^T(\pi(X_t^{D^s})dt+\gamma dD_t^s)\right], \tag{19}$$

where π is called a *running cost* and the minimization is done over the admissible controls $D_t^s\in\mathcal{D}_s$. In (18) and (19) the minimizers (or maximizers) are called *optimal controls* and the values attained with the optimal controls are called the *value function* $V_r(x)$ and the *minimum average cost* β. Even though these two criteria look quite different, their solutions are usually related. Indeed, in the literature the usual way to solve an ergodic problem is to first solve a discounted problem and then prove that the ergodic problem is solved by a so-called vanishing discount limit, where the discounting factor r goes to zero. More specifically, let

$$V(T,x)=\inf_{D^s\in\mathcal{D}_s}\mathbb{E}_x\left[\int_0^T(\pi(X_s^{D^s})ds+\gamma dD_t^s)\right]$$

denote the value of an associated finite time horizon problem. Due to sufficient ergodic properties of X_t, for large T we expect that we can separate the value as

$$V(T,x)\sim\beta T+W(x),$$

where β is the minimum average cost and W is called the *potential value function*. It is now expected that

$$rV_r(\bar{x})\to\beta\quad\text{and}\quad J_r(x)-J_r(\bar{x})\to W(x),$$

where $\bar{x}$ is a reference point and $V_r(x)$ is the value function of the discounted problem (see [15] and [16]).

Analogously to optimal stopping problems we can form a HJB variational inequality associated with discounted and ergodic control problems. In discounted problems it takes the form (see [17])

$$\max\{(\mathcal{A}-r)V_r(x)+\pi(x),\gamma-V_r'(x)\}=0,$$

and in ergodic problems (see [18])

$$\max\{(\mathcal{A}W(x)+\pi(x)-\beta,\gamma-W'(x)\}=0.$$

Using the variational inequalities we can proceed further by splitting the state space into a *continuation region* and an *action region*. These regions are very similar to a continuation and a stopping region in optimal stopping problems. Indeed, inside the continuation region the diffusion X_t is allowed to evolve without interventions, and if the diffusion is in the action region, a sufficient amount of control is immediately exerted to push the process to the boundary of the continuation region. This split allows us to write the free-boundary problem for each of the control problems. We will here directly consider the case where the split is assumed to be such that the continuation region is (l,y^*) and the action region $[y^*,\infty)$. Then in the discounted problem we have

$$\begin{cases}(\mathcal{A}-r)V_r(x)+\pi(x)=0, & x<y^*,\\ V_r'(x)=\gamma, & x=y^*,\\ V_r'(x)<\gamma, & x>y^*,\end{cases} \tag{20}$$

and in the ergodic problem

$$\begin{cases}\mathcal{A}W(x)+\pi(x)=\beta, & x<y^*,\\ V_r'(x)=\gamma, & x=y^*,\\ V_r'(x)<\gamma, & x>y^*.\end{cases} \tag{21}$$

In order to fully solve the free-boundary problems it is often further assumed that the value functions are twice continuously differentiable across the boundary points y^*. After finding the solution to the free-boundary problem a separate verification theorem can be used to prove that we have indeed found the solution to the problem.

The control problems defined above are termed as downward control problems because the path of the process is usually controlled downwards at some state y^*. In a similar manner we could define upward control problems, where only controlling upwards is allowed. The solutions to downward or upward problems are usually given by a *one-sided* control policy as highlighted in the free-boundary problems (20) and (21). An interesting generalization is to allow controlling both downwards and upwards (see e.g. [19] for a discounted problem and [20] for an ergodic problem), where in the discounted case the agent minimizes (or maximizes)

$$\mathbb{E}_x\left[\int_0^\infty e^{-rt}(\pi(X_t^{D^s})dt + \gamma_d dD_t^s - \gamma_u dU_t^s)\right] \tag{22}$$

over admissible control policies D_s^t and U_t^s. Then the optimal solution is often *two-sided* control, where the path of the diffusion is controlled upwards at some state x^* and downwards at state $y^* > x^*$. The solution methods are similar to one-sided problems but usually the assumptions are more demanding.

3 Poisson constrained optimal control

3.1 Restrictions on control times

The solutions to the optimal control problems in Chapter two are often unrealistic, or at least unfeasible in practice, as the optimal policies in the control problems include not only continuous monitoring of the underlying diffusion but also local time type optimal policies, where continuous controlling is required. There are various ways considered in the literature to make the setting more realistic. One possibility is to add a constant transaction cost to the control problems in section 2.2, so that a fixed amount K is paid every time that the agent applies the control. In this case the optimal policy is under similar assumptions an *impulse control*, where the agent chooses a set of stopping times $\{\tau_1, \tau_2, \ldots\}$, that describe when to exert the control, and corresponding impulse sizes $\{\zeta_1, \zeta_2, \ldots\}$, which describe how much the diffusion is controlled at each stopping time (see e.g. [21], [22] and references therein).

In this thesis we take another viewpoint and the main focus will be on the aspects of restricting the continuous controllability of the diffusion such that the diffusion is only controllable at discrete times given by an exogenous process. If the continuous observability is also restricted, the problems transform to those studied for example in [23]. Consequently, we introduce a *signal process*, or *constraint*, which we denote by N_t.

***Assumption* 3.1.1.** *The process N_t is assumed to be a Poisson process with parameter λ and it is assumed to be independent of X_t. Further, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is augmented such that it is rich enough to carry the process N_t. We denote by T_i the jump times of N_t and by $U = T_i - T_{i-1}$ the exponentially distributed inter arrival times.*

The reason for restricting the signal process to be a Poisson process is to have the inter arrival times be exponentially distributed, so that the memoryless property of the exponential distribution keeps the problem setting time-homogeneous and Markovian. One might expect that a restriction of control times to a given grid $\{nt \mid n \geq 0\}$ would make the problem more tractable, but this makes the problem time-inhomogenous and closed form solutions even in simple cases are not likely to exist. In this sense randomization of the control opportunities makes the problems more tractable. For these reasons, the introduction of restricted control times given by a Poisson process will not transform the problem to be very far from the standard problems introduced in Chapter two. This allows us to apply similar tools as in Chapter two to obtain solutions, that can be presented using similar functional forms, and consequently, enables rather direct comparisons between the solutions of the Poisson constrained and classical problems. One example of such comparison is to consider the case where the rate of the Poisson process λ tends to infinity. This corresponds to the case where more and more control opportunities are given to the agent, and hence we would expect that in the limit we would recover the related classical problem (see article III for results in this direction).

The literature on problems where the controlling is allowed only at exogenous times given by a Poisson (or more general) signal process has expanded quite rapidly in the past decade or so. Thus, we postpone most of the considerations of the literature from the coming sections to a separate section at the end of this Chapter.

3.2 Poisson constrained stopping problems and Poisson constrained stopping games

In this section we go trough similar control problems as in section 2.1 but the control times are restricted to jump times of a Poisson signal process as defined in section 3.1. As the reader will notice, the tools and solution methods coincide between the Poisson restricted problems and the classical problems. Thus, it is worthwhile to compare the problems defined in the sections below to Chapter two.

3.2.1 Poisson constrained stopping problems

We follow [24] and consider an optimal stopping problem, where stopping is only possible at the jump times T_i of a Poisson process N_t as given in assumption 3.1.1. We assume that $T_0 = 0$ and denote

$$\mathbb{N} = \{1, 2, \dots\},\ \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

The *Poisson constrained stopping problem* (*constrained stopping problem* or *Poisson stopping problem* for short) is to find the value function V_λ and a *Poisson stopping time* $\tau_\lambda \in \mathbf{T}^1$ such that

$$V_\lambda(x) = \sup_{\tau_\lambda \in \mathbf{T}^1} \mathbb{E}_x[e^{-r\tau} g(X_\tau)],$$

where

$$\mathbf{T}^1 = \{\tau \text{ is a } \mathcal{F}_t\text{-stopping time} \mid \text{for all } \omega, \tau(\omega) = T_n(\omega) \text{ for some } n \in \mathbb{N}\}$$

is the set of Poisson stopping times and $g : \mathbb{R} \to \mathbb{R}$ is sufficiently defined payoff. Notice that in this formulation exercising is not possible immediately as $T_0 = 0$ is not a possible Poisson stopping time. A related formulation is

$$V_\lambda^0(x) = \sup_{\tau_\lambda \in \mathbf{T}^0} \mathbb{E}_x[e^{-r\tau} g(X_\tau)],$$

where stopping immediately is admissible. In other words, the set of admissible Poisson stopping times is

$$\mathbf{T}^0 = \mathbf{T}^1 \cup \{0\} = \{\tau \text{ is a } \mathcal{F}_t\text{-stopping time} \mid \text{for all } \omega, \tau(\omega) = T_n \text{ for some } n \in \mathbb{N}_0\}.$$

It is worth noting that the first problem is time-homogeneous. However, the second is time-inhomogenous in the sense that at time $t = 0$ we can stop, but at every $t > 0$ we have to wait for a jump. This remark also yields the dynamic programming principle

$$V_\lambda^0(x) = \max\{g(x), V_\lambda(x)\}, \tag{23}$$

where the first component describes the option to stop at each Poisson jump to get the payoff g and the second component describes the option to wait until the next stopping opportunity. Conditioning on the first jump time of the Poisson process we further find that

$$V_\lambda(x) = \mathbb{E}_x[e^{-rU} V_\lambda^0(X_U)] = \lambda(R_{r+\lambda} V_\lambda^0)(x),$$

where $U \sim \text{Exp}(\lambda)$ and $R_{r+\lambda}$ is the resolvent operator (see definition 1.2.4). Hence, using (23) we get

$$V_\lambda(x) = \lambda \mathbb{E}_x\left[\int_0^\infty e^{-(r+\lambda)t} \max\{g(X_t), V_\lambda(X_t)\} dt\right].$$

Based on this identity we expect that V_λ will solve the variational inequality

$$(\mathcal{A} - r)V_\lambda(x) + \lambda(\max\{g(x), V_\lambda(x)\} - V_\lambda(x)) = 0.$$

Analogously to the classical stopping problem, the state space can be divided to a continuation region and a *Poisson stopping region*. The continuation region is defined in a similar manner, but the Poisson stopping region has the additional property that the agent is forced to wait there until a Poisson jump is observed (and the diffusion remains in that region). Based on the variational inequality this allows us to formulate the free-boundary problem

$$\begin{cases} (\mathcal{A} - r)V_\lambda(x) = 0, & x < y_\lambda^*, \\ V_\lambda(x) > g(x), & x < y_\lambda^*, \\ V_\lambda(y_\lambda^*) = g(y_\lambda^*), & x = y_\lambda^*, \\ (\mathcal{A} - (r+\lambda))V_\lambda(x) = -\lambda g(x), & x > y_\lambda^*, \\ V_\lambda(x) < g(x), & x > y_\lambda^*, \end{cases}$$

where we also assumed for simplicity that the optimal stopping rule is given by the one-sided *Poisson stopping rule* or the *Poisson entry time* to an interval $[y^*_\lambda, l)$

$$\tau^\lambda_{y^*_\lambda} = \min\{T_i, i \in \mathbb{N} \mid X_{T_i} \geq y^*_\lambda\}.$$

Since the considered Poisson stopping problem is essentially discrete (the possible stopping times form a discrete set), we can proceed by formulating the problem also in discrete time. The discrete time formulation is particularly useful in proving a verification theorem (see section 2.3 of [25] or section 3.2 of [24]). Thus, the guess and verify approach can be used to solve Poisson stopping problems in similar way as classical stopping problems.

3.2.2 Poisson constrained stopping games

Next, we define the Poisson constrained stopping game following [26] and article II. Similar to the stopping game considered in section 2.1.2, the players, *sup* and *inf*, have their respective exercise payoff functions g_s and g_i. The sup-player attempts to maximize the expected present value of exercise payoff, whereas the inf-players objective is to minimize the same quantity. However, contrary to the classical stopping game the players are allowed only to stop the diffusion at the arrivals of their respective signal Poisson processes N^s and N^i, which are assumed to be independent of each other and the diffusion X_t. We define

$$R(\tau,\sigma) = g_s(X_\tau)\mathbb{1}_{\{\tau<\sigma\}} + g_i(X_\sigma)\mathbb{1}_{\{\tau>\sigma\}}.$$

Then the lower and upper values of the game are

$$\underline{V}(x) = \sup_{\tau\in\mathbf{T}^s}\inf_{\sigma\in\mathbf{T}^i}\mathbb{E}_x\left[e^{-r(\tau\wedge\sigma)}R(\tau,\sigma)\right], \ \overline{V}(x) = \inf_{\sigma\in\mathbf{T}^i}\sup_{\tau\in\mathbf{T}^s}\mathbb{E}_x\left[e^{-r(\tau\wedge\sigma)}R(\tau,\sigma)\right],$$

where

$$\mathbf{T}^s = \{\ \tau \text{ is an } \mathcal{F}_t\text{-stopping time} \mid \text{for all } \omega : \tau(\omega) = T^s_n(\omega), \text{for some } n = 1,2,\dots\}$$
$$\mathbf{T}^i = \{\ \tau \text{ is an } \mathcal{F}_t\text{-stopping time} \mid \text{for all } \omega : \tau(\omega) = T^i_n(\omega), \text{for some } n = 1,2,\dots\},$$

and when the equality

$$V(x) = \underline{V}(x) = \overline{V}(x) \tag{24}$$

holds the zero-sum game is said to have a value V. There are two key differences compared to the classical stopping game. First, it is not necessary to include the possibility of simultaneous stopping as independent Poisson arrivals do not, almost surely, occur simultaneously. Second, the usual ordering of the payoffs $g_s \leq g_i$ is not directly needed to solve the problem.

To solve the problem (24), we could in principle proceed by heuristically forming a free-boundary problem to receive a candidate solution. However, it turns out it is easier to proceed by exploiting the discrete time aspects of the problem. Hence, we introduce two auxiliary problems I and S. Auxiliary problem I is defined via the lower and upper values

$$\underline{V}_0^i(x) = \sup_{\tau \in \mathbf{T}^s} \inf_{\sigma \in \mathbf{T}_0^i} \mathbb{E}_x \left[e^{-r(\tau \wedge \sigma)} R(\tau, \sigma) \right], \ \overline{V}_0^i(x) = \inf_{\sigma \in \mathbf{T}_0^i} \sup_{\tau \in \mathbf{T}^s} \mathbb{E}_x \left[e^{-r(\tau \wedge \sigma)} R(\tau, \sigma) \right],$$

where

$$\mathbf{T}_0^i = \{ \ \tau \text{ is a } \mathbb{F}\text{-stopping time} \mid \text{for all } \omega : \tau(\omega) = T_n^i(\omega), \text{for some } n = 0, 1, \dots \}.$$

Similarly, the auxiliary problem S is defined via the lower and upper values

$$\underline{V}_0^s(x) = \sup_{\tau \in \mathbf{T}_0^s} \inf_{\sigma \in \mathbf{T}^i} \mathbb{E}_x \left[e^{-r(\tau \wedge \sigma)} R(\tau, \sigma) \right], \ \overline{V}_0^s(x) = \inf_{\sigma \in \mathbf{T}_0^i} \sup_{\tau \in \mathbf{T}^s} \mathbb{E}_x \left[e^{-r(\tau \wedge \sigma)} R(\tau, \sigma) \right],$$

where

$$\mathbf{T}_0^s = \{ \ \tau \text{ is a } \mathbb{F}\text{-stopping time} \mid \text{for all } \omega : \tau(\omega) = T_n^s(\omega), \text{for some } n = 0, 1, \dots \}.$$

If conditions analogous to (17) hold, the values V_0^i and V_0^s are said to exist. We observe that the discrete time aspects of the problem is apparent in these auxiliary problems. In the problem I the inf-player is allowed to stop immediately, whereas the sup-player has to wait until his next stopping opportunity. Similar remarks hold for the problem S but from the perspective of the sup-player. These auxiliary problems suggest, for each of the players separately, a dynamic programming principle analogous to a Poisson stopping problem:

$$V_0^i(x) = \min\{g_i(x), V(x)\}, \tag{25}$$

$$V_0^s(x) = \max\{g_s(x), V(x)\}, \tag{26}$$

$$V(x) = \mathbb{E}_x \left[e^{-r(U^i \wedge U^s)} \left(V_0^i(X_{U^i}) \mathbb{1}_{\{U^i < U^s\}} + V_0^s(X_{U^s}) \mathbb{1}_{\{U^s < U^i\}} \right) \right], \tag{27}$$

where the exponential random variables $U^s \sim \text{Exp}(\lambda_s)$ and $U^i \sim \text{Exp}(\lambda_i)$ are independent. The first equation (25) reflects that when the inf-player receives a stopping opportunity, he/she either decides to receive the payoff g_i, or decides to wait. The second equation (26) has a similar interpretation for the sup-player. The last condition (27) is the expected present value of the next stopping opportunity, which will be for either of the players. This dynamic programming principle can be solved for example by relying on the properties of the resolvent operator, and then verified to solve the defined Poisson constrained Dynkin game using martingale techniques (see article II). We refer to [26; 27; 28] for an another approach using backward stochastic differential equations.

3.3 Poisson constrained singular control problems

In this section we consider downward Poisson constrained singular control problems, where the agent can act similarly as in section 2.2, but each action has to happen at a jump time of an independent Poisson process. This framework was first considered in [29]. Hence, we define the set of admissible control policies $\mathcal{D}$ as those non-negative, non-decreasing, right-continuous processes that can be represented as

$$\zeta_t = \int_0^{t-} \eta_s dN_s,$$

where η is an $\{F_t\}$-predictable process. Given the admissible controls $\mathcal{D}$, the controlled process X_t^ζ can be defined as the generalized Itô differential equation

$$X_t^\zeta = x + \int_0^t \mu(X_s^\zeta)ds + \int_0^t \sigma(X_s^\zeta)dW_s - \zeta_t.$$

As in section 2.2, the discounted control problem is to minimize over the admissible controls $\zeta \in \mathcal{D}$ the quantity

$$\mathbb{E}_x\left[\int_0^\infty e^{-rs}\big(\pi(X_s^\zeta)ds + \gamma d\zeta\big)\right]$$

and the ergodic problem to minimize

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\left[\int_0^T \big(\pi(X_s^\zeta)ds + \gamma d\zeta\big)\right],$$

where $\gamma > 0$ is the proportional transaction cost and π a running cost. The applied solution methods to Poisson constrained singular control problems are again connected to the classical counterparts. The HJB equation in the discounted Poisson constrained problem takes the form

$$(\mathcal{A} - r)V_{r,\lambda}(x) + \pi(x) - \lambda \max\{V'_{r,\lambda}(x) - \gamma, 0\} = 0 \tag{28}$$

and in the ergodic Poisson constrained problem

$$\mathcal{A}W_\lambda(x) + \pi(x) - \lambda \max\{W'_\lambda(x) - \gamma, 0\} = \beta_\lambda. \tag{29}$$

Analogous to (20) and (21), the split of the state space to continuation region and action region leads to free-boundary problems. In a Poisson constraint problem the action region has the property that the agent is, similar to constrained stopping, forced to wait for a Poisson event instead of immediately acting upon the entry to the action region. In a discounted problem the free-boundary problem takes the form (see [30])

$$\begin{cases} (\mathcal{A} - r)V_{r,\lambda}(x) + \pi(x) = 0, & x < y^*, \\ V'_{r,\lambda}(x) = \gamma, & x = y^*, \\ (\mathcal{A} - r)V_{r,\lambda}(x) + \pi(x) + \lambda(V'_{r,\lambda}(x) - \gamma) = 0, & x > y^*, \end{cases} \tag{30}$$

and in an ergodic problem (see article I)

$$\begin{cases} \mathcal{A}W_\lambda(x) + \pi(x) = \beta_\lambda, & x < y^*, \\ W'(x) = \gamma, & x = y^*, \\ \mathcal{A}W_\lambda(x) + \pi(x) - \lambda(W_\lambda'(x) - \gamma) = \beta_\lambda, & x > y^*. \end{cases} \tag{31}$$

Again, additional boundary conditions are needed to find the boundary point y^*, and usually a twice continuously differentiability requirement is sufficient as in the classical problems. These solutions to the free-boundary problems can then be verified to be optimal using a verification theorem (see [30] for a verification theorem for discounted Poisson constrained problem and article I for an ergodic Poisson constrained problem).

The optimal one-sided policies, characterized by y^*, implicit in the above free-boundary problems can be summarized as follows. When the process is below the threshold y^* we do not act, but if the process crosses the boundary, and the Poisson process jumps, we immediately push it down to y^* and start it anew. Hence, the optimal control is no longer singular reflecting barrier policy as in the singular control problems. However, the optimal policies in Poisson constrained problems and singular problems are similar in the sense that the optimal strategy is always to exert control at the 'maximum possible rate' when the process is at (or above) the corresponding boundary y^*, and otherwise leave it uncontrolled.

A generalization of the downward discounted Poisson constrained singular control to two-sided constrained control, similar to (22), is considered in article IV.

3.4 Literature on Poisson constrained control problems

The problems where independent Poisson processes alter the classical control problems in one way or another are rather new and mostly discussed in the literature during the past two decades. However, due to increasing interest, especially in applied probability, option pricing, portfolio optimization, credit risk modelling and actuarial risk models, it is of importance to see to what extent these problems are studied. In principle, there are multiple ways that an independent Poisson process can affect the classical control problems. These include admissible exercise times, the parameters of the underlying diffusion, the payoff structure, the time-horizon of the problem, or some combination of these.

The goal of this section is to give a short but rather extensive review of the literature associated with problems, where the exercise times are constrained to jump times of a Poisson process (or by some other more general signal process). Even though the problems in the aforementioned categories are very closely related, we try to keep them at least to some extent separated below.

3.4.1 Applied probability

The Poisson constrained stopping problem was first studied in [24], where stopping is only possible at Poisson events with constant arrival rate. The authors solve the problem for call-type payoff and geometric Brownian motion. This approach has been generalized and modified to cover variety of control problems. A more general payoff structure and linear diffusion structure is studied in [25], and the case where the underlying is a spectrally negative Lévy process in [31; 32]. The monotonicity and convexity of the value in Poisson constrained stopping for constant and state dependent rate of the Poisson process, is considered in [33]. In [34], the agent is also allowed to optimally choose the rate of the Poisson process to generate more stopping opportunities when needed, and in [35], the stopping opportunities given by the Poisson process are taken with a state dependent probability. Further, in [36] a regime-switching geometric Brownian motion is considered, where in addition to constrained stopping times, the stopping is possible only in one of the regimes. The authors in [37] study a variation of the problem, where the agent is stopping at the maximum of a geometric Brownian motion.

Multi-dimensional generalizations are considered in [38] and [39]. In [38] it is shown that the solution to the Poisson constrained stopping problem can be given as a solution to a penalized backward stochastic differential equation. In [39] the authors add a running cost to the model and note that a so-called "penalty method" is suitable for solving the Poisson constrained stopping problem (see also [40] for application of the penalty method for American options in this framework). Also, [39] notes that the Poisson constrained problem in multiple dimensions is often more mathematically and numerically tractable than the classical problem. A generalization for general Markov-Feller processes and more general signal processes is given in [41], and a similar generalization in a general Markovian framework is investigated in [42] using least squares Monte-Carlo methods.

The Poisson constrained control problems that restrict the classical singular problems to allow only controlling at Poisson events were first considered in [29], where the costs are assumed to be quadratic and the underlying process is a standard Brownian motion. The problem was solved under discounted and ergodic criteria. The discounted problem was generalized to a more general cost and underlying structure in [30], the ergodic problem in [43] (article I), and these problems were connected to each other and their classical counterparts in [44] (article III). The discounted problem was then considered in the case of two-sided controls in [45] (article IV). The case of general Markov-Feller processes and more general signal processes is treated in [46] for discounted criterion and in [47] for ergodic criterion.

Poisson constrained stopping games or constrained Dynkin games were first introduced in [26] and [27] relying on penalized backward stochastic differential equations. In [26] the authors consider a case, where a single Poisson process gives the stopping opportunities to both of the players simultaneously and in [27] (see also [28]) the case where each player has their own signal Poisson process. A similar game is also studied in article II and in [48]. In article II the perpetual game is solved relying on classical theory of diffusions and in [48] the authors consider Lévy dynamics.

In addition to the above problems, an optimal switching problem with constraint is studied in [49] and in [50]. In optimal switching the agent determines an optimal sequence of stopping times for a switching system, which is often modeled by a stochastic process with several regimes. Further, the studies [51; 52] are of interest as they study the fluctuation theory of Lévy processes observed at Poisson arrival times. Their results have been applied to Poisson constrained problems for example in [53; 54; 48; 55; 56; 57].

3.4.2 Mathematical finance

The Poisson constraint on the possible actions by the agent was first proposed in [58] as a simple model for liquidity effects (see also [59]). The authors consider the classical portfolio problem of Merton, but the portfolio can only be rebalanced at Poisson events with constant rate. The portfolio optimization problem under similar restrictions is further studied rather extensively. The model of [58] is studied in the case of finite time horizon in [60; 61; 62], and in [63] with proportional transaction costs. A related setting, where not only the trading times, but also the observations of the price process is allowed at the jump times of the Poisson process, leading to a coupled system of integro-differential equations, was first explored in [23; 64; 65; 66]. This framework is further elaborated in [67; 68; 69; 70]. The case of two assets, liquid (continuously tradeable) and illiquid or light stock (trading times modeled by Poisson constraint), is investigated in [69; 70; 71; 72; 73]. On the other hand, the authors in [67] consider finite time horizon with Lévy dynamics and in [68] the constant rate Poisson process is replaced by a Cox process. The extensions where the constraint is a Cox processes also appear in [74; 75]. A different direction compared to [23] is studied in [76] and [77], where the optimal strategy is not always to stop at the arrival of the Poisson events (in this case the Poisson events give information about the underlying process to the agent but do not directly restrict the intervention times). A portfolio insurance problem, where wealth is not allowed to fall under some fixed amount, appear in [78; 79]. Finally, further applications include credit risk, capital supply and default modelling [80; 81; 82; 83; 84; 85], optimal trade execution [86], real estate construction [87] and control of energy storage [88].

3.4.3 Actuarial risk models

Another application of constrained intervention times appears in actuarial risk models, where the surplus of a risky business is modelled by a stochastic process and the business pays dividends. In these models, the Poisson constrained dividend strategies are usually referred to as periodic strategies. The authors in [89; 90] study this setting when the surplus is given as a classical compound Poisson process, the dividend payments are done according to a Poisson constrained barrier strategy and ruin (negative surplus) can also only happen at Poisson times. These studies are now elaborated on in many aspects. In [91] the optimal barrier is derived when surplus of the company is modelled as Brownian motion and in [92] for Cramér-Lundberg risk process (see also [93]). The authors in [94] (see also [55]) investigate the situation where ruin can occur continuously in time but dividend decisions are still made at times given by a Poisson process.

Moreover, in [95] ruin is observed at Poisson events, but dividends can only be paid at some of the Poisson events. Generalizations of the above considerations to Lévy dynamics appear in [54; 56; 57; 96]. An extension to so-called hybrid strategies, where the dividends can be paid either periodically at Poisson events or continuously (with higher cost), are studied in [97; 98]. Other works include optimal dividends under regime switching models [99; 100], Markov additive models [100; 101], and bail-outs (restricting surplus to be nonnegative) [53].

4 Summaries of the included articles

In this section we give short summaries of the optimal control problems and their solutions in articles I-IV. We refer to earlier chapters for most of the notations used below, instead of repeating it here.

4.1 Article I: Ergodic control of diffusions with random intervention times

In the first paper, we consider the ergodic Poisson constrained problem, where the agent minimizes the ergodic cost criterion

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\left[\int_0^T (\pi(X_s^\zeta)ds + \gamma d\zeta_s)\right].$$

The controlled linear diffusion dynamics X_t^ζ are given by the Itô equation

$$X_t^\zeta = X_0 + \int_0^{\tau_0^\zeta\wedge t} \mu(X_s^\zeta)ds + \int_0^{\tau_0^\zeta\wedge t} \sigma(X_s^\zeta)dW_s - \zeta_t, \quad 0\le t\le \tau_0^\zeta,$$

where τ_0^ζ is the first exit time of X_t^ζ from the state space.

We first show, relying on heuristic arguments, that the solution (W,β) to this problem is expected to be given by the free-boundary problem

$$\begin{cases} W_\lambda \in C^2, & \\ \mathcal{A}W_\lambda(x) + \pi(x) = \beta_\lambda, & x < y^*, \\ W_\lambda'(x) = \gamma, & x = y^*, \\ \mathcal{A}W_\lambda(x) + \pi(x) - \lambda(\gamma(x-y^*) + W_\lambda(y^*)) = \beta_\lambda, & x > y^*. \end{cases} \tag{32}$$

We solve the free-boundary problem in closed-form and verify under mild additional assumptions compared to the classical version of the problem, that the we have indeed found the unique optimal solution (see [13] and [20] for the classical problem with similar structure). Further, we are able to show that the optimal strategy converges to its classical counterpart in the limit $\lambda\to\infty$.

4.2 Article II: A zero-sum Poisson stopping game with asymmetric signal rates

In the second paper, we solve a Poisson constrained Dynkin game. More specifically, we assume that the sup-player and the inf-player are only allowed to stop the diffusion at the arrivals of their respective signal processes Y^s and Y^i. The value of the game is defined trough the lower and upper values

$$\underline{V}(x) = \sup_{\tau \in \mathbf{T}^s} \inf_{\sigma \in \mathbf{T}^i} \mathbb{E}_x \left[e^{-r(\tau \wedge \sigma)} R(\tau, \sigma) \right], \quad \overline{V}(x) = \inf_{\sigma \in \mathbf{T}^i} \sup_{\tau \in \mathbf{T}^s} \mathbb{E}_x \left[e^{-r(\tau \wedge \sigma)} R(\tau, \sigma) \right],$$

where

$$\begin{aligned} R(\tau, \sigma) &= g_s(X_\tau) \mathbb{1}_{\{\tau < \sigma\}} + g_i(X_\sigma) \mathbb{1}_{\{\tau > \sigma\}}, \\ \mathbf{T}^s &= \{ \tau \text{ is a } \mathbb{F}\text{-stopping time} \mid \text{for all } \omega : \tau(\omega) = T_{n^s}(\omega), \text{for some } n^s = 1, 2, \dots \} \\ \mathbf{T}^i &= \{ \tau \text{ is a } \mathbb{F}\text{-stopping time} \mid \text{for all } \omega : \tau(\omega) = T_{n^i}(\omega), \text{for some } n^i = 1, 2, \dots \}. \end{aligned}$$

When the equality $V(x) = \underline{V}(x) = \overline{V}(x)$ holds, the game has a value V.

Using the resolvent operator we solve the dynamic programming principle (see section 3.2.2)

$$\begin{aligned} V_0^i(x) &= \min(g_i(x), V(x)), \\ V_0^s(x) &= \max(g_s(x), V(x)), \\ V(x) &= \mathbb{E}_x \left[e^{-r(U^i \wedge U^s)} \left(V_0^i(X_{U^i}) \mathbb{1}_{\{U^i < U^s\}} + V_0^s(X_{U^s}) \mathbb{1}_{\{U^s < U^i\}} \right) \right] \end{aligned}$$

in a closed-form and verify, relying on martingale techniques, that its solution is the value of the game. Further, we show that in the limit $\lambda \to \infty$ the solution coincedes with the classical Dynkin game with similar assumptions as ours (see [8]). We also demonstrate that the effects of changing the arrival rates of the Poisson processes on the strategies for the players.

4.3 Article III: A note on asymptotics between singular and constrained control problems of one-dimensional diffusions.

In the third paper, we show that the solutions to discounted and ergodic Poisson constrained control problems converge to the solutions of the corresponding classical singular control problems. Further, we also show that the two constrained problems, discounted and ergodic, are connected via a vanishing discounting factor limit $r \to 0$. These questions were left partly open in [30] and [43].

More precisely, we prove that the thresholds characterizing the optimal one-sided downward control policies (y_s^* and b_s^*) satisfy the following asymptotic results in terms of the intensity of the Poisson process

$$y^*(\lambda) \xrightarrow{\lambda\to\infty} y_s^*, \quad b^*(\lambda) \xrightarrow{\lambda\to\infty} b_s^*,$$

where $(y^*(\lambda)$ and $b^*(\lambda))$ are, respectively, the states characterizing the optimal policy in discounted and ergodic Poisson constrained problems. If the underlying diffusion is also sufficiently ergodic, we also show the vanishing discounting factor limits

$$y_s^*(r) \xrightarrow{r\to 0} b_s^*, \quad y^*(r) \xrightarrow{r\to 0} b^*.$$

Further, we show that similar results also hold for the corresponding value functions in terms of the intensity of the Poisson process

$$V_{r,\lambda}(x) \xrightarrow{\lambda\to\infty} V_r^s(x), \quad \beta_\lambda \xrightarrow{\lambda\to\infty} \beta^s,$$

where $V_{r,\lambda}(x)$ is the value of the discounted Poisson coinstrained problem, $V_r^s(x)$ the value of discounted singular problem and β_λ, β^s are the minimum average costs in Poisson constrained and classical ergodic problems, respectively. Analogously, we prove that the following Abelian limits hold for vanishing discounting factor

$$rV_{r,\lambda}(x) \xrightarrow{r\to 0} \beta_\lambda, \qquad rV_r^s(x) \xrightarrow{r\to 0} \beta^s.$$

These results are illustrated when the underlying diffusion is Brownian motion with drift and Ornstein-Uhlenbeck process.

4.4 Article IV: Two-sided Poisson control of linear diffusions

In the fourth paper, we consider a generalization of the Poisson constrained control problem defined in section 3.3, where controlling is allowed downwards and upwards. Consequently, we consider controlled dynamics given by the Itô equation

$$X_t^\zeta = x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s - \zeta_t^d + \zeta_t^u,$$

where

$$\zeta_t^d = \int_0^{t-} \eta_s^d dN_s, \quad \zeta_t^u = \int_0^{t-} \eta_s^u dN_s.$$

The problem is to find the value function

$$V(x) = \sup_{(\zeta_d,\zeta_u)} \mathbb{E}_x\left[\int_0^\infty e^{-rs}\big(\pi(X_s^\zeta)ds + \gamma_d d\zeta_d - \gamma_u d\zeta_u\big)\right],$$

where the supremum is taken over all admissible controls and γ_d and $-\gamma_u$ are constants, called the unit price and unit cost, respectively. The aim is also to characterize the optimal control policy (ζ_d^*, ζ_u^*) that realizes this supremum.

We expect that the value function V solves the free-boundary problem

$$
\begin{aligned}
&V \in \mathcal{C}^2 \\
&(\mathcal{A} - r)F(x) = -\pi(x), && a < x < b \\
&(\mathcal{A} - (r + \lambda))F(x) = -\pi(x) - \lambda(\gamma_d(x - b) + F(b)), && x \geq b \\
&(\mathcal{A} - (r + \lambda))F(x) = -\pi(x) - \lambda(\gamma_u(x - a) + F(a)), && x \leq a \\
&F'(b) = \gamma_d, \\
&F'(a) = \gamma_u.
\end{aligned}
$$

We solve the free-boundary problem in a closed-form and show that its unique solution is indeed the optimal solution. Our assumptions are only slightly altered compared to a singular version of the problem in a similar setting (see [19]). Further, the problem is connected to the one in [19] in the limit $\lambda \to \infty$, and results are illustrated when the underlying process is a geometric Brownian motion.

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