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Comparing traditional VaR with copula based VaR in extreme market volatility

Empirical Evidence from Finland between 2004 and 2009

Bachelor's thesis
in Accounting and Finance

Author:
Patrik Haverinen

Supervisor:
M.Sc. Matti Niinikoski

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This thesis examines the effectiveness of copula-based Value at Risk (VaR) compared to traditional VaR methods under extreme market volatility within the Finnish financial markets from 2004 to 2009. Traditional VaR has been widely used in risk management due to its simplicity; however, it often fails to capture the nonlinear dependencies, tail risks and stylized facts in financial time series. This inadequacy is critical during periods of significant market stress, such as the 2007-2009 financial crisis.

To address these limitations, this study employs copula-based VaR, which integrates copulas to model dependency structures from the marginal distributions, potentially offering enhanced accuracy in risk assessment during volatile periods. Utilizing historical data from Finnish stock and bond markets, the research applies ARMA-GARCH models to estimate the marginal distributions and Monte Carlo simulations to compute VaR. The VaR forecasts are done in rolling fashion manner, imitating a more practical approach. The performance of both copula-based and traditional VaR models is tested with two backtesting methods.

The results indicate that copula-based VaR, particularly the Student's t copula, provides a more accurate and reliable measure of risk under extreme market conditions compared to traditional VaR. These results not only underscore the potential limitations of traditional VaR in extreme market conditions but also highlight the advantages of copula-based approaches in enhancing risk assessment frameworks.

Key words: Risk Management, Value at Risk, copulas, GARCH, Monte Carlo, backtesting, normal distribution.

Kanditutkielma

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Tämä tutkielma tarkastelee kopulaan perustuvan Value at Riskin (VaR) tehokkuutta perinteisiin VaR-menetelmiin verrattuna, äärimmäisen volatiliteetin aikana, Suomen rahoitusmarkkinoilla vuosina 2004-2009. Perinteistä VaR:ia on laajasti käytetty riskienhallinnassa sen yksinkertaisuuden vuoksi se kuitenkin usein epäonnistuu epälineaaristen riippuvuuksien, häntäriskin ja poikkeavien tekijöiden huomioimisessa tuotto-sarjoissa. Tämä puutteellisuus on erityisesti kriittistä merkittävän markkinastressin aikana, kuten 2007-2009 finanssikriisin aikana.

Perinteiden VaR:in rajoitusten kiertämiseksi, tämä tutkimus käyttää kopulaan perustuvaa VaR:ia, joka yhdistää kopulat mallintamaan riippuvuusrakenteita marginaalijakaumista, mahdollisesti tarjoten enemmän tarkkuutta riskinarvioinnissa volatiileina aikoina. Käyttäen historiallista dataa Suomen osake- ja obligaatiomarkkinoilta, tutkimus soveltaa ARMA-GARCH -malleja marginaalijakaumien estimoimisessa ja Monte Carlo -simulaatioita VaR:in laskemiseksi. VaR-ennusteita tehdään rullaavan ikkunan menetelmällä, jäljitellen käytännönläheisempää lähestymistapaa. Sekä kopulaan perustuvan, että perinteisen VaR-mallien suorituskykyä testataan kahdella takaisintestauksen menetelmällä.

Tulokset osoittavat, että kopulaan perustuva VaR, erityisesti Studentin t-kopula, tarjoaa tarkemman ja luotettavamman riskimittarin äärimmäisissä markkinaolosuhteissa verrattuna perinteiseen VaR:iin. Nämä tulokset korostavat perinteisen VaR:n mahdollisia rajoituksia ja vahvistavat kattavien sekä monipuolisten lähestymistapojen merkitystä riskinarvioinnin kehittämisessä.

Avainsanat: Riskienhallinta, Value at Risk, kopulat, GARCH, Monte Carlo, takaisintestaus, normaalijakauma.

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1 Introduction

In the complex world of financial markets, the management of risk is crucial for the financial sector, including banking and insurance, and thus becomes essential for companies and the entire economy (McNeil, Frey and Embrechts 2005, 15–16). The 2007–2009 financial crisis serves as a stark example of the catastrophic consequences that can ensue when risks are not accurately assessed and managed. Despite the acknowledged importance of risk management, developing and implementing effective risk management strategies that also satisfy regulatory requirements remains a challenging task.

Value at Risk (VaR) has emerged as a cornerstone for risk management and is widely utilized as a risk metric due to its relative simplicity (Jorion, 2007). Although the traditional way of calculating VaR is widely used, it has been criticized for having several shortcomings; for example, it is noted in various academical studies that it fails to address the stylized facts of financial time series or the tail dependencies between assets (McNeil et al. 2005, 40–41). These problems arise from the traditional way of calculating VaR, which is based on the assumption that asset returns are elliptically distributed, especially normally.

To address the problem of inaccurate distributions, much research has been conducted on statistical models that can model nonlinear dependencies of the financial assets and therefore allow more accurate joint distributions to calculate risk metrics of. One of these being the statistical method known as copulas.

Copulas allow for the construction of multivariate distributions that are better at representing tail dependencies and nonlinear relationships between assets (Danielsson 2011, 25). The copula concept, originally introduced by Sklar (1959) and popularized in finance through the work of Embrechts, McNeil and Straumann (1999), has attracted much attention in the computation of the VaR of market portfolios Cherubini (2011). However, to the authors knowledge, there has been no research on how well the copula based VaR performs against its traditional counterpart in the Finnish stock and bond market, under extreme market volatility.

1.1 Research question

Amid growing concerns over traditional Value at Risk models' inability to accurately capture dynamics of asset returns and assess risks in extreme market conditions, this study aims to examine the performance of copula-based VaR against its traditional counterpart. By focusing specifically on the Finnish stock and bond market—an area yet unexplored in this context—the thesis aims to determine whether the copula-based model offers a more reliable and accurate risk measurement tool during periods of extreme volatility.

This study will adopt an empirical approach, utilizing historical data from two indexes that aim to mirror the Finnish economy. The data, taken from 2004 to 2008, will be used to estimate the market conditions of 2009. In addition to copulas, the study also employs GARCH-type models to estimate the marginal distributions of the assets to help also capture the stylized facts of asset returns. The main objective is to then find the best copula to be used modelling the joint distribution of the marginals and calculate the copula based VaR from it.

The study seeks to answer the question: *Does the Copula-based VaR perform better than traditional VaR under extreme market conditions*? This is studied with an approach more in line with the banking side of risk management, by calculating the 1-day 95% and 99% copula-based VaR and traditional VaR from the indexes. The results from these calculations will be backtested with two distinct methods, to answer the question.

1.2 Hypothesis

In the light of the theory behind the traditional VaR and copula based VaR infused with ARMA-GARCH model, we would assume that it performs better than traditional VaR, therefore the null hypothesis is as follows:

H0: Value at Risk based on copulas and ARMA-GARCH model gives better estimation than Value at Risk based on normal distribution assumption

1.3 Structure

The thesis is structured as follows: Chapter Two discusses the theoretical background of VaR and copulas, highlighting VaR's limitations and introducing copulas as a solution for improved risk estimation. Chapter Three details the methodology used to conduct this study, including the specification of the ARMA-GARCH model, identification of the copulas used, and the backtesting methods. Chapter Four presents the data used, analyzes the results of the empirical study, and compares the copula-based VaR model with the traditional one. The final chapter, Chapter Five, summarizes the findings and concludes the study also offering insights for future research.

2 Theory

2.1 Market Risk

Financial risk is commonly segmented into four main categories: market risk, credit risk, liquidity risk, and operational risk. In this thesis, we are focusing on market risk, which is often associated with movements in asset prices in liquid financial markets when new information enters the market (Miller 2018, 6). To measure market risk, many models have been developed. These models often assume that price movements follow a smooth and continuous path, which is an idealization. In reality, asset prices can exhibit sudden jumps or falls due to specific events as seen in financial crises. Market risk models can have high-frequency and they utilize large amounts of historical market data to evaluate risk. In classical finance, investors view risk as volatility, i.e., the standard deviations of returns, as noted in Markowitz (1952). The current risk metrics are often infused with other measures such as skewness, kurtosis, and quantiles. However, volatility has not been abandoned; it is reported alongside the current metrics and plays an important role in their calculation, such as in the case of Value at Risk (Miller 2018, 15–16).

The choice of risk metric to assess market risk depends on the application. On trading desks, modeling of risk is much more short-term than in senior management reports for the board. For this reason, the assessments of market risk have evolved quite separately and can be separated into three broad categories: banking, asset management, and large corporations (Alexander 2009, 7). In this thesis, we are focusing on the short-term aspects of market risk assessment used in the banking side of risk management.

Banks are under the Second Basel Accord (Basel II) regulatory framework. This regulation demands that banks and other Authorized Deposit-taking Institutions (ADIs) follow minimum capital requirements according to their riskiness (Hull 2018, 18–19, 359). Banks have adopted risk measures like Value at Risk to assess the risk of their operations and allocate their capital to match the minimum requirements (Luenberger 2014, 260–261).

2.2 Properties of financial time series

Studies have shown that financial time series possess certain characteristics that present challenges for researchers and risk management in general. These problems complicate the data generating process, since it is very important to identify good models for generating data that can capture the asset price dynamics and replicate them (Lu, Lai and Liang, 2011). Traditionally, models used to estimate the data-generating process are categorized into parametric and non-parametric approaches. Parametric models adopt a more rigid approach by assuming the data distribution is defined by a fixed set of parameters. In contrast, the non-parametric approach makes no assumptions about the probability distributions and relies solely on empirical data, enabling it to represent highly nonlinear functions effectively.

Cont (2001) identifies three main challenges affecting the application of statistics to financial time series, particularly in estimating asset returns. The first challenge is stationarity, which implies that the properties of a time series are independent of the time at that the series is observed. This means that no matter at what point we examine the series, its behavior and structure would be statistically similar. In finance literature the asset returns are assumed to be weakly stationary (Tsay 2010, 30). Furthermore, empirical studies have demonstrated that the assumption of stationarity does not hold for returns. For example, there are notable seasonal effects, such as the weekend and January effects.

The second issue concerns the ergodicity of asset returns, which requires that the empirical averages of asset returns converge to the values they aim to estimate, i.e., they must align with the values they are trying to predict. This theorem is often referred as the *ergodic theorem* (Tsay 2010, 180–181). Stationarity is essential for ensuring that the statistical properties of asset returns remain consistent over time; however, the ergodic property is crucial for guaranteeing convergence. This convergence is often assumed based on the asset returns being independent and identically distributed (i.i.d.), but empirical studies have challenged this assumption, showing that asset returns are not i.i.d. (Cont, 2001).

The third issue pertains to the properties of finite samples. The statistical estimator does not necessarily equal the quantity it aims to estimate. Although daily returns can be collected from thousands of data points, this amount is still significantly insufficient, leading to considerable estimation error. To mitigate this issue, confidence intervals are essential; however, their computation introduces new challenges. There is an underlying assumption that the residuals, or the noise terms of the returns process, are i.i.d. Furthermore, it is assumed that these residuals have well-defined higher-order moments. Yet, real-world data often exhibit heavy tails and nonlinear dependencies that challenge the i.i.d. assumption for residuals, thereby questioning the relevance and accuracy of traditional confidence intervals, which are designed under the premise of i.i.d. residuals (Artzner, Delbaen, Eber and Heath, 1999).

Cont (2001) also lists statistical properties that are common for asset returns across various markets.

(1) **Negligible autocorrelation**. Asset returns are generally not autocorrelated, with the exception over short intraday periods.

(2) **Heavy tails**. Unconditional distribution of asset returns exhibits a power law or Pareto-like tail behavior, characterized by pronounced tail risks which normal distribution models fail to accommodate.

(3) **Asymmetry of gain and losses.** Large losses are more likely to happen compared to equally large gains.

(4) **Aggregational Gaussianity.** When the time scale is increased the distribution of asset returns looks increasingly like Gaussian distribution.

(5) **Intermittency.** Asset returns experience substantial variability at all timescales, observed by irregular spikes of volatility.

(6) **Volatility clustering.** Although individual returns don't consistently show strong autocorrelation, the high volatility of returns does.

(7) **Conditional heavy tails.** Even after adjusting for volatility clustering, the time series of returns still exhibits heavy tails, though these tails are less pronounced than in the unconditional distribution

(8) **Slow decay of autocorrelation in absolute terms.** The autocorrelation of absolute returns diminishes slowly as a function of the time lag.

(9) **Leverage effect.** Returns of assets correlate negatively with the volatility of the asset.

(10) **Correlation of volatility and volume.** The trading volume of an asset is found to have a correlation with its volatility.

(11) **Asymmetry of volatility time scales.** When one uses coarser measures of volatility these tend to provide better predictions for the finer measures of volatility than the other way around.

Cont (2001) studies some stylized facts in more detail that are very important in this thesis point of view. These are the characteristics of return distributions and the dependent structure of the returns.

As early as Mandelbrot (1963) demonstrated that the asset returns can't be sufficiently modelled based on normal distribution. The normal distribution could not capture the kurtosis of realized returns. The empirical kurtosis was larger than the normal distribution, indicating heavier tails and sharper peak (Cont, 2001).

As previously mentioned, Cont (2001) highlighted volatility clustering and the absence of autocorrelations as key dependency properties of asset returns. Autocorrelation measures the strength of a time series' correlation with itself at different time points. If one however finds autocorrelation in the examined asset returns, it may be used to predict future returns. Volatility clustering, first also identified by Mandelbrot (1963), indicates that large changes in asset prices are likely to be followed by further large changes in returns, regardless of the direction. Similarly, small changes tend to be followed by small changes. These properties are linked to each other, as they suggest that past can be used to predict outcomes in the future. Tsay (2010, 131–134) suggests that GARCH models can be used to address the volatility clustering of asset returns, which will be further discussed in Chapter 3. These stylized facts are crucial for constructing models to estimate asset returns, even though simpler models often fail to capture these dynamics effectively.

2.3 Value at Risk

Most often, risk management focuses exclusively on the risk of loss rather than the possibility of gain. In mathematical language, this means concentrating on the lower tail of the probability density function (Luenberger 2014, 258–261). Value at Risk is the most common quantile risk metric that focuses on this lower tail of probability function, and it is the most used risk measure after volatility (Danielsson 2011, 76–77). VaR has it's downfalls, which can be lead to pretty crucial underestimations of risks, but when we consider it's simplicity and ease of backtesting, we can see why it is so used. VaR gives the best balance among the available risk measures and underpins most of the quantile risk models (Danielsson 2011, 76–77). As Hull (2018, 271) noted, VaR can be defined as the highest loss a portfolio endures in time horizon Δ with a given confidence level $1-\alpha$. Formally expressed as

$$\mathrm{VaR}_\alpha = \inf\{u : F(u) \geq \alpha\}, \tag{1}$$

where $F(\cdot)$ is some cumulative distribution function of portfolio returns in time horizon Δ. In probabilistic terms, VaR is thus simply a quantile of the return distribution. As McNeil et al. (2005, 38–39) noted VaR also can be defined from the loss distribution when it is the $1-\alpha$ quantile of the loss distribution. This way the VaR is a positive value.

So VaR has two basic parameters to be chosen, the significance level α (or confidence level $1-\alpha$) and the time horizon Δ, traditionally measured in trading days (Alexander 2009, 13–14). Most common values for α are $\alpha = 0.95$ or $\alpha = 0.99$ and the time horizon Δ is usually 1 or 10 trading days (McNeil et al. 2005, 38–39). In this thesis we are using both $\alpha = 0.99$ and 0.95, but Δ is 1 in both cases. It is also important to note that when calculating VaR we are assuming that the portfolio we are holding does not change during the time period Δ.

As mentioned before, VaR has its downfalls, and it is most commonly criticized for its properties. Artzner et al. (1999) created an objective framework with four properties that a risk measure should ideally possess to be considered a coherent risk measure. A risk measure should be invariant, subadditive, monotonic, and positively homogeneous; however, VaR does not satisfy the subadditive property. This failure means that VaR may contradict the diversification principle because it suggests that the total risk of a portfolio, as measured by VaR, could be greater than the sum of the VaR measures for the individual assets within the portfolio. In addition to that VaR has problems for measuring the extreme price losses, by definition VaR only measures the distribution quantile and disregards losses that exceed the VaR level (Yamai and Yoshiba, 2005). Although VaR's problems are well known it still remains the standard risk measure for financial organizations. While coherent risk measures like Conditional Value at Risk (CVaR), also known as expected shortfall, are available, many of these measures rely on VaR estimation (Alexander 2009, 34).

Estimating VaR accurately is an important and challenging problem, and many methods have been proposed. (Nadarajah and Chan 2016, 283-356) categorize these methods into two main types: parametric and nonparametric. The parametric method is based on an assumed distribution, while nonparametric methods do not make any assumptions about distributions. There are also combinations of both methods, which are referred to as semiparametric methods. In this thesis we are using the parametric Monte Carlo simulation for calculating VaR with copulas and also for traditional VaR calculations. The usage of Monte Carlo method will be further discussed in Chapter 3.

As the accuracy of VaR estimations depends heavily on the cumulative probability distribution of the portfolio returns, it is important to be able to estimate the distribution as precisely as possible. Danielsson (2011, 78) further notes that this is the most difficult and important aspect of risk modeling. As outlined in Chapter 2.2 the estimation of the probability distribution should satisfy the stylized facts of the time series. In the case of traditional VaR this inadequacy stems largely from the limitations of assuming normal distribution for asset returns, which fails to reflect the complexities of financial markets. More effective VaR calculations require the adoption of sophisticated probability distribution models that can accommodate these features. A key challenge is identifying suitable multivariate joint distributions that can model all the asset returns in the portfolio we are looking at. The reliance on linear correlation coefficients in multivariate normal distribution introduces further inaccuracies, as noted in Embrechts et al. (1999). Hu (2006) further underscores this issue, noting that in almost all cases where VaR was calculated based on these Gaussian assumptions significantly underestimated the downside risk. Copula models are thus introduced as a valuable tool for the multivariate modelling and examining the degree and structure of the dependence, since they do not require assumption of normal distribution.

2.4 Copulas

As mentioned before, asset returns don't usually follow a normal distribution, a fact that becomes particularly evident during financial crises when all assets tend to drop together. Therefore, if financial data were modeled using a multivariate normal distribution, correlations would decrease for extreme events, whereas, empirically, it has been shown that correlations actually increase (Danielsson 2011, 21). In other words, linear correlations overestimate the dependence of asset returns in non-crisis periods and underestimate them during crises. To address this problem, we need multivariate models that can also capture nonlinear dependence structures.

Copula functions provide us the tools to create multivariate distributions of random variables with a range of types of dependencies. As noted by McNeil et al. (2005, 184–185) every joint distribution for asset returns contains both a characterization of the marginal behavior of each individual asset as well as a description of their dependence structure. Copula functions offer a way to isolate and analyze the description of the dependence structure and do not require normality in marginal nor in joint distributions. In the context of risk management, and particularly for VaR, copulas are a natural fit because they express dependence on a quantile scale, which is useful for describing extreme outcomes (McNeil et al. 2005, 187).

Moreover, copulas support a bottom-up methodology for constructing multivariate models. This approach is especially beneficial in risk management, where our understanding of the marginal distributions of assets tends to be more comprehensive compared to our knowledge of their dependence structure (McNeil et al. 2005, 185). Furthermore, by employing the marginal distribution functions and any copula function, one can form a joint distribution (Patton, 2006). As copulas enable the representation of a joint distribution as a function derived from marginal distributions, they are capable of capturing all the dependence properties of the data generating process (Ibragimov, 2009; Danielsson, 2011).

2.5 Copula function

Copula can be thought as being a function that links together marginal distributions to form a joint distribution. As mentioned in the introduction, the concept of copula originates from the Skalar's theorem, by the work of Sklar (1959). McNeil et al. (2005, 185–186) define a d-dimensional copula as being multivariate distribution function $C : [0,1]^d \to [0,1]$ with standard uniform marginal distributions. Hence C is mapping from the unit hypercube $[0,1]^d$ into the unit interval $[0,1]$. McNeil et al. (2005, 185–186) also notes that a copula must have thee basic properties to be referred as copula,

(1) $C(u_1, \ldots, u_d)$ is increasing in each component u_i.

(2) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i,$ for all $i \in \{1, \ldots, d\}, u_i \in [0, 1]$.

(3) For all $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0, 1]^d$ with $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1i_1}, \ldots, u_{di_d}) \geq 0, \tag{2}$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \ldots, d\}$.

The first property asserts that if any argument u_i is increased, the value of the copula $C(u_i, \ldots, u_d)$ does not decrease, a requirement clearly essential for any multivariate distribution function. The second property, required for ensuring uniform marginal distributions, states that if all but one argument are set to 1, the copula returns the value of the remaining argument u_i. The third property, known as the rectangle inequality, also referred to as the 2-increasing or non-negativity condition (Cherubini 2011, 12), simply implies that the probability of a number falling within a given interval $[0, 1]$ cannot be negative.

As second property required the margins have to uniformly distributed. To achieve this requirement one can use the probability integral transformation, a result stemming from the research by Fisher (1932). Fisher demonstrated that if a random variable X has a continuous distribution F, then $U = F(X)$ will have a uniform distribution $U \sim U(0, 1)$ regardless of the original distribution F. Furthermore if F^{-1} is the generalized inverse of F, it follows that $F^{-1}(U) \sim X$. Given the that F is continuous and $F(X) \sim U(0, 1)$ we can express the joint distribution of F in d-dimensions as

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \tag{3}$$

where $u_1, \ldots, u_d$ are uniformly distributed marginals. Equation (3) shows how to express dependence on quantile scale since the value $C(u_i, \ldots, u_d)$ is the joint probability that a random variable X lies below its corresponding u-quantile McNeil et al. (2005, 187). Marginal distributions provide a complete overview of the behaviors of individual variables, while the joint distribution contains both the individual and combined behaviors of all variables, thus copulas must encapsulate the information on the dependence structure that exists among the variables (Patton 2009, 767-785). This is why sometimes copulas are known as 'dependence functions' (Galambos, 1978).

2.6 Sklar's theorem

Sklar's theorem is right at the core of the copula theory. Let F be a joint distribution function with margins $F_1, \ldots, F_d$. Then there exists a copula $C : [0,1]^d \rightarrow [0,1]$ such that, for all $x_1, \ldots, x_d \in \mathbb{R}$

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{4}$$

If the marginal distributions $F_1, \ldots, F_d$ are continuous then copula C is unique; otherwise C is uniquely determined on the Cartesian product of the ranges of the margins $RanF_1 \times RanF_2 \times \cdots \times RanF_d$ where $RanF_i$, is the set of all possible values that F_i can take (McNeil et al. 2005, 186).

Sklar's theorem further establishes that all multivariate distributions contain copulas, and conversely, all copulas can construct a multivariate distribution function with univariate ones. If one has a copula and marginal distributions, a valid multivariate distribution can be constructed (McNeil et al. 2005, 186–187). Complementing Sklar's theorem, Nelsen (2006) presents a corollary indicating that a copula can be isolated from any multivariate distribution. This allows for the construction of new joint distributions using any copula with any pair of marginal distributions and for the determination of the underlying copula and margins from a given joint distribution (Danielsson 2011, 27).

3 Methology

The model used in this thesis aims to capture the most crucial properties of asset returns and dependencies between two indexes by employing an ARMA-GARCH model, copulas, and Monte Carlo simulation, thus striving for accurate VaR estimations. The modeling process can be divided into four steps: first, modeling the marginal distributions of the indexes using the ARMA(1,1)-GARCH(1,1) model; second, using the residuals obtained from the marginals to fit four different copulas to the data and selecting the best fit for calculating VaR. Next the VaR is calculated using monte carlo method and lastly the results are evaluated with two backtesting methods and compared with the backtesting results of the traditional VaR. Next, the thesis will review the model used in VaR calculations without focusing on the methodology of the traditional model.

3.1 Marginal distribution modelling

When discussing the financial properties of time series in Chapter 2.2, it's noted that asset prices demonstrate asymmetries and volatility clustering, also known as conditional heteroskedasticity. This phenomenon means the conditional variance of the time series depends on time. Incorporating volatility clustering into the modeling of marginal distributions for variables is crucial. GARCH-type models, first introduced by Engle (1982), have been widely adopted to capture these characteristics of financial time series, as highlighted by Bollerslev, Engle and Nelson (1994); Lu et al. (2011). In this thesis we assume that marginal distributions are characterized by an ARMA(1,1)-GARCH(1,1) model. The model was first introduced by Bollerslev (1987) and subsequently utilized within a Copula framework by authors such as Patton (2006); Lu et al. (2011). The ARMA-GARCH method used in this thesis is defined as follows

$$r_t = \mu + \phi_1 r_{t-1} + \theta\epsilon_{t-1} + \epsilon \tag{5}$$

$$h_t = \omega + \alpha\epsilon_{t-1}^2 + \beta h_{t-1}$$

$$\epsilon_t = \eta_t\sqrt{h_t},$$

where μ, ϕ, θ are parameters of the ARMA model; ω, α, β are parameters of the GARCH model. The term h_t denotes the conditional variance, which is a positive, quantifiable and time-varying function of the information set at $t-1$ (Angelidis, Benos and Degiannakis, 2004). In the last equation ϵ_t is the residual and η_t is the standard residual with t distribution and v degrees of freedom. Lambert and Laurent (2001) proposed adopting the skewed extension of Student's t distribution by Fernández and Steel (1998) within GARCH frameworks. In this thesis, however the student's t will not have this extension for the sake of simplicity.

In practice, the most widely used approach to fitting GARCH models to data is maximum likelihood. While the Quasi-Maximum Likelihood (QML) method is a common preference for GARCH model estimation, alternative approaches like the Maximum Likelihood Estimate (MLE) are also viable, as Francq and Zakoian (2019) discuss. In this particular study, the method chosen for modeling the marginal distribution is the MLE approach, which aligns with the capabilities of the rugarch package in R utilized for this analysis. This approach is also used in similar studies that combine the ARMA-GARCH process with the copula framework, see, for example, Lu et al. (2011).

After estimating the ARMA-GARCH parameters for the two sets of return data, $r_{1,t}$ and $r_{2,t}$ where $t = 1, 2, \ldots, T$ represents time, we obtain the filtered standardized residuals $\eta_{1,t}$ and $\eta_{2,t}$ (Lu et al., 2011). Data can be perceived as being filtered, because ARMA-GARCH process can capture information from the variable itself. As such, this data holds limited relevance when the primary interest lies in examining the dependencies between different variables. These residuals will then be transformed to have uniform distributions through the probability integral transformation, as mentioned in Chapter 2.5.

3.2 Models for dependence

This part of the thesis will detail the copula models applied. Building on the foundations of copula theory and the established uniform distribution of the random variables, we will examine the dependency structure through copula functions. Four distinct bivariate copula functions from two copula families are being considered: Gaussian, Student's t, Gumbel, and Clayton copulas. These particular copulas were chosen due to their widespread use and their proven track record, both theoretically and empirically, in effectively estimating the dependence structure within financial time series.

3.2.1 Elliptical copulas

Elliptical copula functions are based on elliptical distributions, as the name suggests. These copulas are often referred to as implicit because they are not defined by closed-form expressions. However these copulas are fairly easy to use and parameters can be easily interpret (Embrechts, Lindskog and Mcneil, 2001). The Gaussian copula is very widely used copula in financial applications due its simplicity (Amengual and Sentana, 2020), although it often fails to fit in the real-world applications, as highlighted in (Embrechts, 2009). Gaussian Copula can be expressed using Equations (3) and (4) as follows

$$C_{\rho}^{Ga}(u) = \phi_{\rho}^{n}(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_n)), \tag{6}$$

where ϕ_R^n denotes the joint distribution function of the random vector X with correlation matrix ρ and ϕ^{-1} denotes the inverse function of the univariate standard normal distribution (Embrechts et al., 2001). As we can see from the equation above, the problem with Gaussian copulas is that they rule out any type of nonlinear dependence structures since the only parameter is linear correlation coefficient (Amengual and Sentana, 2020; Embrechts et al., 2001). Embrechts et al. (2001) further showcases that Gaussian copulas do not exhibit lower or upper tail dependence.

While Gaussian copula cannot model tail dependence, the student's t copula can. Like the Gaussian copula the t copula is based on the t-distribution, which has a additional parameter v, usually known as the degree of freedom (Amengual and Sentana, 2020). The degrees of freedom is responsible for so-called fat tail, which can be observed if v approaches ∞ making it the same as the normal distribution. This same principle applies to copulas as well; $\nu \to \infty$, $C_{\nu,\rho}^t = (u, v; \rho, \nu) \to C_\rho^{Ga}(u, v; \rho)$. The t Copula of X can be written as

$$C_{\nu,\rho}^t(u) = t_{\nu,\rho}^n(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)), \tag{7}$$

where $t_{\nu,\rho}(X)$ is the Student's t distribution and $t_\nu(x)$ is the univariate one. When the dependence of assets is moderately normally distributed but at the same time exhibits fat tails, i.e., has extreme co-movements, the t copula might be able to model the dependence quite accurately (Demarta and McNeil 2005; Cherubini 2011, 30). Empirical results suggested the same as demonstrated in (Lu et al., 2011).

3.2.2 Archimedean copulas

In the world of financial modeling the mentioned elliptical copulas have drawbacks. Embrechts et al. (2001) points out that elliptical copulas are restricted to have radial symmetry and therefore cannot capture asymmetries. As Author further mentions, in finance, it is reasonable to assume that the dependence is stronger between big losses than between big gains. As mentioned before, securities often fall together but do not necessarily rise together.

The Gumbel and Clayton copulas belong to the family of Archimedean copulas. Unlike elliptical copulas, they have an explicit closed-form expression. Archimedean copulas are also not derived from multivariate distributions using Sklar's theorem but rely on generator functions instead, which vary across different Archimedean copulas (Embrechts et al., 2001). If now φ is continuous and strictly decreasing function from $[0, \infty]$ to $[0, 1]$ such that $\varphi(1) = 0$, bivariate Archimedean copulas can be expressed as

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)), \tag{8}$$

where the function φ is called the generator of copula and $\varphi^{[-1]}$ is the pseudo-inverse of φ. The equation 8 can be referred to as a copula if and only if the generator function φ is convex.

The family of Archimedean copulas encompasses a variety of models designed for statistical analysis and is frequently applied within the realm of financial studies due to their effective alignment with financial data (Smith, 2003). Embrechts et al. (2001) further notes that The Archimedean family of copulas provides models that can only capture upper or lower tail dependence. The Gumbel copula is an Archimedean copula that is only able to capture the upper tail dependence and can be expressed as

$$C_\theta^{Gu}(u, v) = \exp(-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}), \tag{9}$$

where the θ has parameter space $1 \leq \theta < \infty$ (Smith, 2003). The generator function associated with Gumbel copula is

$$\varphi_\theta(t) = (-\ln t)^\theta \tag{10}$$

The parameter θ indicates the degree of dependency in case of Archimedean copulas. It defines the strength and type of dependency in the lower and upper tails of the distribution. Specifically, for the Gumbel and Clayton copulas, which are considered comprehensive copulas, the parameter θ allows for interpolation between a lower limit of countermonotonicity and an upper limit of comonotonicity as it traverses the boundaries of the parameter space (McNeil et al. 2005, 221). In the context of a Gumbel copula, θ value of 1 at the lower boundary denotes independence between the variables but dependence in the upper tail. In other words, the Gumbel copula captures the upper tail dependence, but since we are more concerned with lower tail dependence, the Gumbel copula may come up short. It is possible to rotate the Gumbel copula to capture lower tail dependence, but this will be ignored in this thesis. For more information, see, for example, Brechmann and Schepsmeier (2013).

The Clayton copula can be seen as opposite of Gumbel copula since it exhibits strong dependence in the lower tail. Therefore it is often used in modelling risk associated with losses (Danielsson 2011, 28). The Clayton copula can be defined as

$$C_\theta^{Cl}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/0}, \tag{11}$$

where the θ has parameter space $0 \leq \theta < \infty$ (Smith, 2003). The generator function associated with Clayton copula is,

$$\varphi_\theta(t) = \frac{1}{\theta}(t^\theta - 1) \tag{12}$$

As mentioned the Gumbel copula tends to the comonotonicity as $\theta \to \infty$. Using a similar technique, the lower tail coefficient λ_1 for Clayton copula is,

$$\lambda_1 = 2^{-1/\theta}, \tag{13}$$

given that $\theta > 0$ (McNeil et al. 2005, 209).

3.3 Estimation of parameters

There are several ways to estimate copula parameters. The most popular method is the maximum likelihood estimation (MLE), which according to Patton (2006) and (Cherubini 2011, 43) is the natural and the strongest way to do the estimation. In the MLE method, the parameters of both marginal distributions and the copulas are estimated simultaneously. This approach can be computationally intensive if dealing with a copula of higher dimension, which increases the number of parameters to be estimated (Lu et al., 2011). To address this issue, Joe and Xu (1996) proposed a method where the marginal parameters and copula parameters are estimated in two separate steps, known as the inference for margins (IFM). There is also canonical estimation method (CML) where the likelihood is estimated directly using non-parametric representations of the margins; i.e., making no assumptions about the marginals (Cherubini 2011, 44). In this thesis, we stick to the maximum likelihood estimation method and therefore estimate the margins for the standardized residuals $\eta_{1,t}$ and $\eta_{2,t}$ simultaneously with the copula parameters. The MLE method is combined with the log likelihood and can be written as

$$L(\theta) = \sum_{t=1}^{T} \log C(F_{\eta 1}(\eta_{1,t}), F_{\eta 2}(\eta_{2,t})) + \sum_{t=1}^{T} \log(f_{\eta 1}(\eta_{1,t}) + f_{\eta 2}(\eta_{2,t})), \tag{14}$$

where C is the copula, $F_{\eta 1}$ and $F_{\eta 2}$ are the Cumulative Distribution Functions and the $f_{\eta 1}$ and $f_{\eta 2}$ are the Probability Density Functions of each standardized residual.

3.4 Selecting criteria

The choice of copula that fits the data is very important to one just cannot just arbitrarily choose it as noted in Kole, Koedijk and Verbeek (2007); Durrleman, Nikeghbali and Roncalli (2000). Hence, the copula must be tested to see if it the copula can accurately describe the data. A natural way to test the goodness of fit for the copula is Akaike's information criterion (AIC) (Akaike 1998, 199-213) firstly introduced in 1973. The AIC is defined as

$$AIC = -2(LogL) + 2(p), \tag{15}$$

where $LogL$ is the log likelihood for each copula and p the number of parameters in each copula function. The copula that gets the lowest value in AIC test will the preferred one.

There are other methods like the Bayesian information criterion (BIC) and more, presented and compared in Genest, Rémillard and Beaudoin (2009). However, this study will focus only on the AIC, as Jordanger and Tjøstheim (2014) showed that it is sufficient to make a selection of copulas.

3.5 VaR estimation using Monte Carlo simulation

After we have obtained the marginal and bivariate distributions, we will estimate the VaR of portfolio returns. Given the diverse marginals and copulas, there is no straightforward analytic formula for calculating VaR (Fantazzini, 2008). However, Monte Carlo simulation, a method widely used in quantitative finance and VaR applications, can be utilized. Monte Carlo method enables the estimation of the VaR of portfolio returns r_1 and r_2 for time $T+1$. In this method first large number N of pseudo-observations $(\mu_1^1, \mu_2^1), (\mu_1^2, \mu_2^2), \dots, (\mu_1^N, \mu_2^N)$ is generated from the chosen copula function. In this study we are are using $N = 10,000$, which represents good compromise between accuracy and speed according to Lu et al. (2011). Next the pseudo-observations are transformed to standardized residuals $\eta_{1,t}$ and $\eta_{2,t}$ via inverse cumulative distribution functions $F_{\eta 1}^{-1}$ and $F_{\eta 2}^{-1}$. This step translates the copula's theoretical relationships into real-world scenarios of asset behavior. This transformation is given by

$$(r_{1,T+1}^j, r_{2,T+1}^j) = (\tilde{r}_{1,T+1} + \eta_1^j \cdot \sqrt{\tilde{h}_{1,T+1}}, \tilde{r}_{2,T+1} + \eta_2^j \cdot \sqrt{\tilde{h}_{2,T+1}}), \tag{16}$$

where $\tilde{r}_{1,T+1}$ and $\tilde{r}_{2,T+1}$ are the means and $\tilde{h}_{1,T+1}$, $\tilde{h}_{2,T+1}$ are the variances forecasted in time $T+1$ with ARMA(1,1)-GARCH(1,1) model (Lu et al., 2011). These equations adjust the standardized residuals by the forecasted mean and standard deviation to obtain potential future returns. By aggregating these simulated returns for each asset and aligning them with the portfolio weights, we derive a distribution of portfolio returns

$$r_{T+1}^j = w \cdot r_{1,T+1}^j + (1-w) \cdot r_{2,T+1}^j, \tag{17}$$

where w denotes the weight of each asset in the portfolio and $j = 1, 2, \dots, N$. Then we simply sort the values from the $10,000$ values of the portfolio r_{T+1}^j in increasing order, and for example the VaR with confidence level 1% is simply calculated as: 99% VaR is the absolute value of $10,000 \cdot (1-99\%) = 100$th ordered scenario of r^j_{T+1} (Lu et al., 2011).

After obtaining 1-day VaR, this process is repeated M times, where M is the number of shared trading days of the picked indexes over a 1-year period 2th January 2009 to 30th December 2009. These estimations are done using a rolling window technique in which the forecasts of VaR uses a segment of recent data, then moves forward a day at a time, constantly updating the dataset to include the newest information while dropping the oldest, ensuring the dataset always contains T observations. These results are then used for backtesting the VaR.

3.6 VaR evaluation: backtesting

After forecasting the daily Value at Risk using a rolling approach, we compare the predicted values with the actual observed results and assess the model's performance using two distinct backtesting methods. Backtesting is a crucial technique for determining the accuracy and reliability of VaR models. It stands as a central evaluative tool for financial institution supervisors and also for regulators, who provide guidelines on how institutions should interpret the results of backtesting (Alexander 2009, 335–336). VaR backtesting is generally divided into two categories frequency-based and size-based tests. Frequency-based tests evaluate how often the predicted VaR is breached On the other hand size-based tests assess the magnitude of the losses when breaches occur. This study focuses on frequency-based tests, which are more common in both practice and regulatory guidelines, where the primary concern is the number of times losses exceed the VaR predictions within a certain confidence level (Alexander 2009, 335–336).

Most backtest methods are based on the assumption that daily returns are generated by an i.i.d. *Bernoulli process.* A Bernoulli variable can take values of 1 or 0, success and failure. In our context, 'success' defined as an instance where the return exceeds the VaR, thereby taking the value of 1. Following Alexander (2009, 334) we can an define indicator function $I_{\alpha,T}$, which examines the exceedances as a binary sequence.

$$I_{t+1}(\alpha) = \begin{cases} 1, & \text{if } Y_{T+1} \leq -\mathrm{VaR}_{j,\alpha,T}, \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

where Y_{t+1} is the realized daily return at time $T+1$. According to Christoffersen (1998) the sequence I_{t+1} should exhibit the following characteristics to be considered accurate: the *Unconditional coverage property*, which dictates that the realized α of VaR should equal the probability of 'success' and the *Independence property*, which implies that occurrences of past exceedances should not influence or predict subsequent exceedances.

If both of these characteristics are satisfied simultaneously VaR forecast is said to have satisfactory conditional coverage and the sequence truly follows an i.i.d. Bernoulli process (Christoffersen 1998; Alexander 2009, 334–335).

3.6.1 Unconditional coverage test

The unconditional coverage test, also known as Kupiec's Proportion of Failures (POF), introduced by Kupiec in 1995, is a widely used VaR backtest. The POF test compares the observed number of exceedances against the number of expected, given the VaR confidence level. It uses a likelihood ratio test statistic that follows χ^2 (chi-squared) distribution with one degree of freedom. The formula for the POF test is given by:

$$POF = -2\ln\left(\frac{(1-\alpha)^{N-x}\cdot\alpha^x}{\left(1-\frac{x}{N}\right)^{N-x}\cdot\left(\frac{x}{N}\right)^x}\right), \tag{19}$$

where x is the number of exceedances, N is the total number of observations, and α is the predetermined confidence level of VaR. If the POF value exceeds the critical value from the chi-square distribution, the null hypothesis that the VaR model is accurate is rejected. The POF tests shortcoming is that it specifically focuses on the amount of exceedances without taking into account any possible dependency between them (Berkowitz, Christoffersen and Pelletier, 2011; Jorion, 2007).

3.6.2 Conditional coverage test

Conditional coverage ratio test was developed to asses shortfalls of Kupiec's test by Christoffersen (1998) by evaluating both the frequency and the independence of exceedances of VaR forecasts. It combines the POF (noted LR_{uc} in this context) test and adds separate statistic for the independence of exceedances. The independence test checks if successive exceedances are independent of each other. Conditional coverage ratio test can be defined as

$$LR_{cc} = LR_{uc} + LR_{ind}, \tag{20}$$

where LR_{cc} is the conditional coverage ratio with a χ^2 distribution and two degrees of freedom, reflecting the two statistics being tested. The independence test LR_{ind} can be defined as

$$LR_{ind} = -2\ln\left(\frac{(1-\alpha)^{n_{00}+n_{01}}\alpha^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\pi_0^{n_{01}}(1-\pi_1)^{n_{10}}\pi_1^{n_{11}}}\right), \tag{21}$$

where n_{ij} denotes the number of days a given condition j occurs following another condition i on the previous day. A value of 1 denotes exceedance as in the POF test, and 0 indicates no exceedance. This leads to the definition of transition probabilities

$$\pi_{ij} = \frac{n_{ij}}{\sum_i n_{ij}} \tag{22}$$

For the sequence of exceedances to be considered independent, the probability of an exceedance on any given day should remain unaffected by the previous day's outcome, more formally $\pi_{0,1} = \pi_{1,1} = \alpha$. he main disadvantage of Christoffersen's test is that it requires a more substantial sample size compared to Kupiec's test to ensure the validity of its results (Angelidis et al., 2004).

4 Data and results

This thesis analyzes data sourced from Bloomberg and LSEG Refinitiv Workspace, utilizing R and the R packages *rugarch*, as introduced by Galanos (2023), and *copula*, as detailed by Hofert, Kojadinovic, Maechler and Yan (2023). The dataset comprises the Bloomberg Finland Treasury Bond Index and OMXH GI closing prices, spanning from 7th April 2004 to 30th December 2009. Ideally, a larger dataset would have been preferable; however, data prior to 7th April 2004 were not available. These indexes were selected aiming to mirror Finnish stock and bond markets during one of Finland's most volatile historical periods. The portfolio of these two indexes was constructed with equal weights.

As the Bloomberg Finland Treasury Bond Index and the OMXH Index represent aggregations of their respective market data rather than tradeable assets, discrepancies arose due to missing values when the underlying assets of one index had updated prices while the other did not. To address this issue, adjustments were made by eliminating instances where both indexes did not have concurrently updated values. Returns were calculated before this elimination, thus avoiding discrepancies such as attributing a three-day return for one index against a one-day return for the other due to non-simultaneous update days. Subsequently, the returns were converted into log returns using the formula defined by (Tsay 2010, 5):

$$r_t = \ln \frac{P_t}{P_{t-1}} \tag{23}$$

This process resulted in a dataset consisting of 1422 log returns for each index. The dataset was divided into two parts: the first 1174 observations (from 7th April 2004 to 30th december 2008) were used to estimate the model, and the last 248 observations (from 2th January 2009 to 30th December 2009) were reserved for out-of-sample estimations and backtesting. This division is crucial for conducting out-of-sample estimations, as the model requires a substantial amount of data to accurately capture the dynamics of the returns. Because the data was not available before the April 2004 the study made a compromise, and used the whole 2008 period also for estimation of the model and left only the 2009 time period for the out-of-sample estimations. The estimation window T, remained constant of 1174 as the analysis was conducted in a rolling sample fashion.

Table 1: Descriptive Statistics for log returns

The table presents the descriptive statistics for the log returns of both indices, segmented into the initial estimation window T and the entire dataset. Values denoted with an asterisk (*) have been scaled by a factor of 100. The table also includes results from the Jarque-Bera test, with p-values provided in parentheses.

	OMXH GI		FI Treasury Index	
	1. estimation window	Whole data	1. estimation window	Whole data
Mean	-0.0011*	0.0116*	0.0166*	0.0170*
St. dev.	1.4407*	1.5265*	0.1850*	0.1890*
Min	-0.0792	-0.0792	-0.0099	-0.0099
Max	0.0885	0.0885	0.0096	0.0096
Skewness	-0.1102	-0.0396	-0.0074	-0.0552
Kurtosis	6.0030	4.3232	2.9300	2.3200
Jarque-Bera	1174	1113	423	322
p-value	(<0.001)	(<0.001)	(<0.001)	(<0.001)

Table 1 displays the descriptive statistics for the daily log returns. From the mean comparison, it is evident that the average return of the OMXH GI was negative during the first estimation window but turned positive in the subsequent backtesting window, reflecting the market's recovery from the financial crisis. Correspondingly, we observe an increase in volatility, aligning with the motivating factors of our study. The Finland Treasury Index exhibits similar patterns, though with less pronounced changes, which is expected given that government bonds typically exhibit less volatility than stocks.

Skewness measures the symmetry, or lack thereof, of the distribution of returns. A skewness greater than zero suggests a distribution with a long right tail (more extreme values on the positive side), while a negative value suggests a long left tail (more extreme values on the negative side). As discussed in Chapter 2.2, asset returns are generally negatively skewed, indicating that large losses are more probable than large gains. This characteristic is observable across both indices and time horizons.

Kurtosis measures how heavily the tails of a distribution differ from those of a normal distribution. The OMXH GI shows a leptokurtic distribution, as evidenced by a kurtosis value well above 3, suggesting 'fat tails'. In contrast, the FI Treasury Index, with a kurtosis below 3, suggests a platykurtic distribution, implying a reduced likelihood of outliers compared to a normal distribution.

The Jarque-Bera test evaluates the normality of a dataset by examining its skewness and kurtosis (McNeil et al. 2005, 69). The test conclusively rejects the normality assumption for both indices, as the p-values indicate strong evidence against a normal distribution.

This result implies that traditional Value at Risk calculations based on the presumption of normality may lead to incorrect estimations as stated before.

In Table 2 The presence of volatility clustering was examined via the volatility clustering teste created by by Nordhausen, Matilainen, Miettinen, Virta and Taskinen (2021). The test assesses autocorrelation in the squared returns of the time series, and evidence of such autocorrelation signals the presence of volatility clustering (Ljung and Box, 1978). Furthermore, Table 2 includes findings from the Dickey-Fuller test for stationarity, since stationarity is a fundamental requirement for employing GARCH models on the dataset (Francq and Zakoian 2019, 74).

Table 2: Tests for Volatility clustering and Stationarity

The table reports Volatility clustering test and Augmented Dickey-Fuller test (ADF) for the whole data of both indices.

	OMXH GI	FI Treasury Index
Volatility clustering	5414.9551	1506.3682
p-value	(<0.001)	(<0.001)
Dickey-Fuller	-10.8993	-10.9219
Lag order	11	11
p-value	(<0.01)	(<0.01)

Based on the volatility clustering test, the p-values indicate significant volatility clustering in the returns for both the OMXH GI and FI Treasury Index. With p-values well below the 0.01 threshold, we reject the null hypothesis that the time series has no squared autocorrelation. This result supports the findings of Cont (2001), reinforcing the suitability of employing a GARCH model to capture and model this characteristic. The Augmented Dickey-Fuller test, designed to test the null hypothesis of a non-stationarity in the time series, also leads us to reject this null hypothesis, as the p-values are smaller than 0.01. This affirms that the data is stationary and appropriate for GARCH modeling. Furthermore, Table 3 showcases the estimated parameters for the ARMA(1,1)-GARCH(1,1) model with t-distributed residuals for both indices' return series. The analysis was conducted in a rolling manner; hence, the table displays the model specifications for the initial and final fits, without documenting the day-to-day changes as they are minor.

Table 3: ARMA(1,1)-GARCH(1,1) parameters and tests for autocorrelation and ARCH effects

This table presents the first (30th of December 2008) and last (30th of December 2009) model fits' parameter estimates and p-values in parentheses. The table also reports results of the Ljung-Box test and Engle test for three different lags.

	OMXH GI		FI Treasury Index	
	First fit	Last fit	First fit	Last fit
μ	0.1098	0.1161	0.0127	0.0080
	(<0.001)	(<0.001)	(<0.01)	(0.0934)
ϕ	-25.5916	-32.8776	-7.5223	-2.6894
	(0.5752)	(0.5321)	(0.9682)	(0.9913)
θ	28.8362	35.0347	8.8086	3.9952
	(0.5228)	(0.5015)	(0.9631)	(0.9874)
ω	0.0000	0.0000	0.0000	0.0000
	(0.7604)	(0.7872)	(0.9934)	(0.9861)
α	7.8392	8.1247	3.2341	3.8699
	(0.2451)	(0.1708)	(<0.01)	(0.0161)
β	91.1969	91.5254	96.6159	95.8595
	(<0.001)	(<0.001)	(<0.001)	(<0.001)
v	621.3288	773.5615	2602.0211	1500.3391
	(<0.001)	(<0.001)	(0.0767)	(<0.01)
Ljung-Box Test				
Lags	First fit	last fit	fist fit	last fit
1	4.2641	5.3349	0.009	0.0019
	(0.0389)	(0.0209)	(0.9239)	(0.9612)
5	0.4298	1.4632	1.3071	3.5313
	(0.5124)	(0.2257)	(0.2527)	(0.0602)
10	5.4112	10.3868	0.9712	2.0961
	(0.0213)	(0.0013)	(0.3241)	(0.1478)
Engle Test				
Lags	First fit	last fit	fist fit	last fit
1	0.0246	0.0383	0.0208	0.0254
	(0.8753)	(0.8449)	(0.8852)	(0.8733)
5	0.2960	1.7037	5.9197	7.4678
	(0.9977)	(0.8884)	(0.3141)	(0.1881)
10	0.5436	4.7365	10.3551	11.2882
	(0.9999)	(0.9081)	(0.4112)	(0.3355)

All parameters in the ARMA-GARCH model were tested at a 5% significance level against the null hypothesis that their true value is zero, which would indicate they do not notably influence the model. The autoregressive ϕ, moving average θ, and GARCH constant ω coefficients are consistently non-significant, thus failing to reject the null hypothesis. Conversely, the ARCH β parameter is the only one showing consistent significance, which could indicate long run persistence of volatility over time. The mean return μ and shape parameter υ show variability in their significance; they are consistently significant for OMXH GI, but for the F1 Treasury Index, they only exhibit significance in one of the two examined windows. (Harrell 2015, 67) emphasizes that in statistical analysis, one should not overstate the importance of parameter significance levels, since statistical tests are designed to test hypotheses rather than to select variables. To further examining the model's adequacy, the Ljung-Box and Engle tests are utilized, aligning with practices used in Lu et al. (2011).

Ljung-Box test created by Ljung and Box (1978) is used to test for autocorrelation remaining in the return residuals after fitting a model to a time series. The null hypothesis is that there is no autocorrelation in residuals after fitting the model. The results suggest that, at a 5% significance level, there is no indication of autocorrelation at the first and fifth lags for either index. However, autocorrelation appears to be present at the tenth lag for the first fit of the OMXH GI and in the last fit for both indices.

For Engle's ARCH test created by Robert Engle (1982), the null hypothesis is that there is no autoregressive conditional heteroskedasticity known as the ARCH effects in the residuals. Engle's ARCH test indicates high p-values across nearly all lags and fits for both indices, suggesting that there is no significant ARCH effects present, and hence the variance seems to be adequately modeled by the GARCH component.

These insights suggest the models are largely, but not perfectly, capturing the dynamics of the of the data and the models are adequately fitted to marginal distributions of both time series.

4.1 Copulas modelling

After fitting the ARMA(1,1)-GARCH(1,1) model for marginal distributions, we derived standardized residuals from both margins, as outlined in Chapter 3.1. These standardized residuals, initially used for evaluating the ARMA-GARCH model, are now employed to identify the most suitable copula among the four previously introduced: the Gaussian copula, Student's t copula, Gumbel copula, and Clayton copula.

Table 4: Copula estimation and test results

Table 4 presents the parameter estimations for each copula, the log-likelihood functions, and the results of the Akaike Information Criterion (AIC) selection criteria. As parameters vary between elliptical copulas and Archimedean copulas, see Chapter 3.2 for parameter specification. The Student's t copula's degree of freedom is in parentheses. The best-fitting copula is highlighted in bold.

OMXHGI-FI Treasury Index	parameter	$L(\theta)$	AIC
Gaussian	-0.3301	62.1876	-122.3751
Student's t	**-0.3372 (7.0285)**	**72.5202**	**-143.0404**
Gumbel	1	-0.0001	2.0001
Clayton	-0.1229	19.2551	-36.5103

Table 4 presents the outcomes of the copula fitting analysis for the joint distribution of OMXH GI and FI Treasury Index residuals. The results indicate that the Student's t copula offers the best fit, as evidenced by its lowest AIC value. This finding is consistent with existing financial literature and study's, which frequently acknowledges t copulas efficiency in capturing the dependencies of assets returns, see, for example, Ning (2010) and (Charfeddine and Benlagha, 2016).

The Gaussian copula also fits the model quite well, closely following the t copula in terms of AIC value. The adequacy of the Gaussian and t copulas likely stems from the ARMA-GARCH model's ability to capture the primary characteristics of the data, leaving residuals that adhere closely to a normal distribution but with the added complexity of fatter tails—traits that the t copula can accommodate. This is illustrated in figure 1, which shows that the joint distribution of the standardized residuals (marked with red dots) resembles the randomly generated data from a standardized normal distribution (marked with black dots).

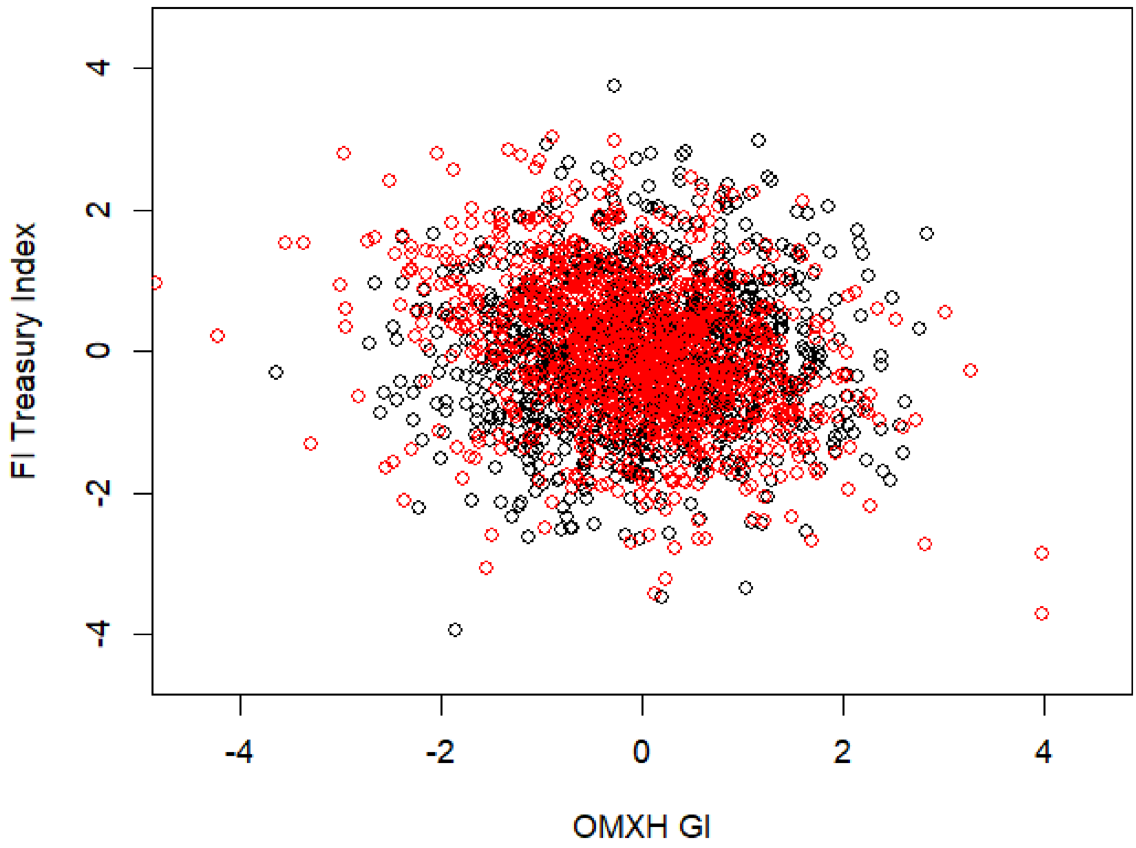

Figure 1: Residuals obtained from ARMA-GARCH model versus normal distribution

On the contrary, the Gumbel and Clayton copulas demonstrate a poor fit when applied to the residuals. The Gumbel copula, with a parameter estimate of 1, suggests no upper tail dependence, whereas the negative parameter estimate for the Clayton copula not only contradicts theoretical expectations, since the parameters' lower bound is 0, but also indicates a complete lack of fit, as it fails to model any lower tail dependence. These findings underscore the need for selecting a copula that aligns with the underlying data structure. As these findings all considered student's t is chosen for the VaR calculations as it has demonstrated best fit of the examined copulas.

4.2 VaR evaluation

After determining the most suitable copula for the models, we use it to calculate the one-day 95% and 99% VaR for the out-of-sample window of 248 estimations, by employing the Monte Carlo method as detailed in Chapter 3.5. In parallel, we estimated the traditional VaR, assuming that the returns follow a normal distribution. These forecasts were then backtested using the two methods outlined in Chapter 3.6. The results from these backtests are presented in Table 5.

Table 5: VaR backtest results

Table 5 presents forecast results for both VaRs on $\alpha = 0.95$ and $\alpha = 0.99$, denoted as $\text{VaR}_{95\%}$ and $\text{VaR}_{99\%}$. The Unconditional coverage ratio is denoted as LR_{uc}, and the Conditional coverage ratio as LR_{cc}. Both backtests were evaluated at the 5% significance level.

	Copula VaR		Traditional VaR	
	$\text{VaR}_{95\%}$	$\text{VaR}_{99\%}$	$\text{VaR}_{95\%}$	$\text{VaR}_{99\%}$
Expected exceedance	12	2	12	2
Actual VaR exceedance	7	1	18	8
LR_{uc}	0.0876	0.2830	0.1252	0.0051
LR_{cc}	0.1895	0.5597	0.2565	0.0153

Table 5 presents results that align with expectations and others that come as a surprise. Both backtests were evaluated at the 5% significance level, as with all previous tests. The null hypothesis for for LR_{uc} posits that the model accurately predicts the correct number of exceedances, and the hypothesis for LR_{cc} adds that the timing between exceedances is independent. For the Copula VaR model, both tests produced p-values greater than 0.05, so the null hypotheses are not rejected, suggesting the model captures risk dynamics well. However, the actual exceedances being lower than expected indicates that the Copula VaR model is conservative, possibly overestimating risk. Such caution might be preferred in risk management, as the number one concern is the possible losses, as noted before. Banks could also be penalized if they have excess amount of capital requirement violations (Danielsson 2011, 146). An overly conservative risk measure can also be problematic. For instance, banks may be restricted in their lending capacity due to higher capital held as a buffer, which could naturally lower profits. The conservative tendency of the Copula VaR model could stem from the influence of the 2008 financial crisis on the estimation period. This period can be causing the model to project a higher degree of caution in its forecasts for 2009, a period marked by yet very large volatility but also recovery and generally positive market returns.

Conversely, the Traditional VaR, relying on the normal distribution, holds up at the 95% VaR level without rejecting the null hypothesis, signifying a reasonable prediction of less severe risk levels. The number of actual exceedances is higher than the expected but not significantly, which is surprising finding. However, at the $\text{VaR}_{99\%}$ the null hypothesis is rejected indicating significant underestimation of risk level. The logical conclusion for the different outcomes between the VaR levels, is that the normal distribution assumptions holds up reasonably well for less extreme events but fall short at the tails of the distribution, where extreme events lie.

5 Conclusion

This study examined how the VaR based on the copula and ARMA(1,1)-GARCH(1,1) model performed against the traditional VaR. The study was based on an equally weighted portfolio constructed from the Bloomberg Finland Treasury Bond Index and OMXH GI. As there have been no studies on the Copula approach to VaR with these assets, it cannot be said that the results are directly in line with previous studies. However, the study was able to find results that were in line with the theory behind this comparison. The copula VaR gave more conservative results compared to traditional VaR, thus able to take the dynamics of asset returns better into account. For this reason it did not underestimate the risk and performed better under extreme volatility. As the Copula VaR did not reject the null hypotheses at either confidence level, while traditional VaR did reject, it leads us not to reject the null hypothesis of this study; *Value at Risk based on copulas and ARMA-GARCH model gives better estimation than Value at Risk based on normal distribution assumption.* However, as the study included data only from two indexes and estimated VaR under certain market conditions, we cannot state that the model is generally better than traditional in more normal market conditions and with different assets.

The study also aimed to find the best copula to fit the data and it was indeed much in line with previous studies. Studies have mainly found that the t-copula is the best fitting copula for asset returns, which was the case in this study as well. However, the copula fitting process in this study was slightly different as it specifically regarded residuals of the ARMA-GARCH model, so it would be exciting to conduct a study to see if the 'raw data' from these indexes and time period, suggest the same results.

The data used in this study was not very extensive, and it would have been better if we could have accessed data from before April 2004. This way, the data for the model estimation would have been larger, and the estimation window could have also been more extensive, spanning from the start of 2008, thus providing estimates for the entire period of the financial crisis. Moving forward, it would be great to analyze a longer time frame, especially during recent volatile periods, like the COVID-19 crisis. Another good way to construct further study is to focus more on stocks, looking into how certain stocks depend on each other, building a portfolio based on these pairs, and then calculating VaR. It would also be interesting to see how the VaR performs based on multivariate or rotated copulas under market stress or just generally in the Finnish stock or bond markets. Since this study did not focus on estimating the autoregressive and moving average coefficients for the ARMA(1,1) component, nor the autoregressive and moving average coefficients for the volatility equation in the GARCH(1,1), another potential study could delve deeper into finding the best coefficients for the model.

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