# Open quantum system modeling of optically trapped nanoparticles

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In this thesis we develop an open quantum system model for a levitating particle trapped in an optical cavity by external optical tweezers.

First we define optical forces and see how they can be used to trap particles in optical tweezers. We study Stokes and Anti-Stokes processes and show that blue detuned optical cavities can be used to cool trapped particles to their quantum ground states.

We derive the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation for a general open quantum system, and present its quantum state diffusion (QSD) unravelling into an ensemble of pure states.

Then we derive the GKSL equation for our system, and use the QSD equations to find differential equations for parameters of an ansatz state. We find the time dependent norm for the pure states in the QSD ensemble and show how we can use it to calculate the expectation values for observables in our system.

Keywords: quantum optics, optomechanics, optical trapping, levitodynamics, electromagnetic forces, open quantum system

# Contents

# Introduction

Optomechanics studies the effects of light-matter interactions on the motion of a particle. The first mention of this field is from 1619, when Johannes Kepler used radiation pressure to explain why the tail of a comet always points away from the sun [1]. The invention of the laser in 1960 made controlling particles with light easier and led to the development of optical tweezers by Arthur Ashkin in 1970 [2], a feat that awarded him a joint Nobel Prize in Physics in 2018.

Levitodynamics is the study of dynamics of a nano- or micro-sized particle when it is trapped and suspended in an electromagnetic field. There are three basic methods to achieve this: the particle can be trapped by an optical, an electrical, or a magnetic field [3]. In this thesis we are interested mainly in optical trapping, which, as the name suggests, uses a coherent optical field, obtained from a laser, to trap particles.

Levitated particles are extremely sensitive to external forces, and because they have a relatively high mass, they are ideal to use in force and acceleration sensing, to detect minute forces like gravity, or rotational forces.

Today optomechanical applications are used in many fields. In biology and biochemistry optical tweezers are used to trap individual viruses and bacteria [4], and even single DNA-molecules [5].

In physics the possible applications are numeorous, ranging from quantum mechanical experiments to gravity research. The LIGO gravitational wave detector is an optomechanical device, that monitors the position of a mechanical oscillator via its coupling to an optical cavity [6].

Optomechanics is an excellent tool in creating and controlling mechanical quantum states, giving rise to the field of quantum optomechanics [7]. Optomechanics gives new ways to implement quantum systems in theory and experiments. A quantum optical system where two optical modes interact via a non-linear medium can be mapped to an optomechanical system where an optical and a mechanical mode interact via photon momentum tranfer [8]. Trapping a particle in vacuum leads to a highly isolated system diminishing the unwanted effects of its environment. Feedback and cavity back-action cooling can be used to cool particles to their quantum mechanical ground state [9]. Both of these conditions are important when doing experiments, where quantum effects may arise. Optical cavity coupled to a mechanical oscillator can even produce entanglement between the mechanical modes and optical modes [10].

There are many promising research areas where the use of levitated particles could lead to new breakthroughs in the future. For example finding new physics beyond the standard model of particle physics with highly sensitive detection of high-energy physics at short distances that allows for the exclusion of dark matter models, being able to prepare macroscopic superpositions of nanoparticles, or implementing far-from-equilibrium processes. However there are still many technical challenges to overcome before these advancements can be reached [3].

A recent work by Vijayan et al. [11] demonstrated that it is possible to control a cavity-mediated interaction of nanoparticles in a multiparticle optical system. They showed for the first time that creating programmable cavity-mediated interactions between nanoparticles in vacuum is possible. By controlling the system parameters like optical frequencies, cavity detuning, and the position and mehcanical frequencies of the particles, allowed them to choose which of the mechanical modes couple and to precisely tune the interaction strength, resulting in strong coupling that didn't decay with distance between particles. Their work is an important towards exploring many-body effects in interacting nano-particle arrays, and creating entanglement of motion.

In this work develop an open quantum system model to characterize an optically trapped nanoparticle in a cavity. We describe the physics behind optical trapping, introduce the theory of open quantum systems, and apply it in the study of our system.

First in section 1 we take a look at optical trapping: what are the forces used to trap particles, and how they can be calculated in general using the Maxwell stress tensor or using dipole approximation for a small particle. Then we see how these forces are used in optical tweezers to trap a particle, and lastly take a look at what optical cavities are and what they are used for in optomechanics.

In section 2 we take a look at the theory of open quantum systems. We derive the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation for a general open quantum system and we present also the Quantum State Diffusion (QSD) unravelling of the QKSL master equation into an ensemble of pure states.

In section 3 we introduce the system we are studying, derive the GKSL equation for the system from its general form and thus tie the real system into the theoretical models.

In section 4 we aim to solve the master equation of our system using the methods of open quantum systems. We use the QSD equation to find differential equation for parameters of our ansatz state. In particular we outline how analytical solutions for this important problem may be obtained.

The analytical model we derive for this system helps our understanding of the general principles and the physics behind how the system behaves.

# 1 Optical trapping of nanoparticles

## 1.1 Optical forces

A photon carries momentum, so it excerts a mechanical force when it interacts with a particle, whether it is absorbed, emitted, refracted or scattered. This force can be used to trap nano- and microparticles in an optical field and to control their mechanics. Solving the optomechanical forces acting on a particle is often a complicated problem that requires approximation methods. The size parameter $\xi = \frac{2\pi a n_m}{\lambda}$ of the particle, where $a$ is the radius of the particle, $n_m$ is the refractive index of the surrounding medium and $\lambda$ is the trapping wavelength in vacuum, determines the suitable approximate approach. For particles much smaller than the wavelength of light, $a \ll \lambda$ (Rayleigh regime), and we can use the dipole approximation, and for particles much larger than the wavelength of light, $a \gg \lambda$ (Geometric optics regime), we can use ray optics. However, when the size of the particle is comparable to the wavelength, $a \sim \lambda$ (Mie-Lorentz regime), no general approximation exists and we need to solve Maxwell's stress tensor of the system [12].

### Maxwell's stress tensor

Maxwell's stress tensor characterizes the interaction between electromagnetic fields and mechanical momentum. Its derivation can be found in full in [13].

To derive the stress tensor and the conservation law for momentum, we start from Maxwell's equations in vacuum

$$\nabla \times \mathbf{E}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\boldsymbol{r}, t), \tag{1}$$

$$\nabla \times \mathbf{B}(\boldsymbol{r}, t) = -\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\boldsymbol{r}, t) + \mu_0 \mathbf{j}(\boldsymbol{r}, t), \tag{2}$$

$$\nabla \cdot \mathbf{E}(\boldsymbol{r}, t) = \frac{1}{\epsilon_0}\rho(\boldsymbol{r}, t), \tag{3}$$

$$\nabla \cdot \mathbf{B}(\boldsymbol{r}, t) = 0, \tag{4}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic flux density, $\mathbf{j}$ is the current density, $\rho$ is the charge density, $c$ is the speed of light in vacuum, $\mu_0$ is the vacuum permeability, $\epsilon_0$ is the vacuum permittivity, $\boldsymbol{r}$ is the position vector and $t$ is time. We also need the electric displacement field $\mathbf{D} = \epsilon_0 \mathbf{E}$ and the magnetic field $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$ in vacuum.

Equation 1 is Faraday's law of induction which states that a changing magnetic field induces an electric field that is perpendicular to the magentic field. Equation 2 is Ampère's law which states that a temporally varying electric field and/or current density creates a magnetic field that is perpendicular to them. Equation 3 is Gauss's law for electricity that states that a charge density is the source for electric fields. Equation 4 is Gauss's law for magnetism that states that magnetic fields do not have sources or sinks due to the nonexistence of magnetic monopoles.

To get the Maxwell stress tensor we operate on equation 1 by $\times\epsilon_0\mathbf{E}$ and on equation 2 by $\times\mu_0\mathbf{H}$, whereafter we add them together:

$$\epsilon_0(\nabla\times\mathbf{E})\times\mathbf{E}+\mu_0(\nabla\times\mathbf{H})\times\mathbf{H}=-\epsilon_0\mu_0\frac{\partial}{\partial t}\mathbf{H}\times\mathbf{E}-\epsilon_0\mu_0\frac{\partial}{\partial t}\mathbf{E}\times\mathbf{H}+\mu_0\mathbf{j}\times\mathbf{H}, \tag{5}$$

where we have omitted arguments $\mathbf{r}$ and $t$ to simplify our notation. This relation subsequently leads to

$$\nabla\cdot[\epsilon_0\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mu_0\mathbf{H}\mathbf{H}^{\mathrm{T}}-\frac{1}{2}(\epsilon_0E^2+\mu_0H^2)\mathbf{I}]-\rho\mathbf{E}=\mathbf{j}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial}{\partial t}[\mathbf{H}\times\mathbf{E}], \tag{6}$$

where $E^2=\mathbf{E}^{\mathrm{T}}\mathbf{E}$ is the strength of the electric field, $H^2=\mathbf{H}^{\mathrm{T}}\mathbf{H}$ is the strength of the magnetic field, and $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ and $\mathbf{H}\mathbf{H}^{\mathrm{T}}$ are dyadic products, with the superscript T denoting the transpose. From the latter relation we define Maxwell's stress tensor $\mathbf{T}$ as

$$\mathbf{T}\equiv\epsilon_0\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mu_0\mathbf{H}\mathbf{H}^{\mathrm{T}}-\frac{1}{2}(\epsilon_0\mathrm{E}^2+\mu_0\mathrm{H}^2)\mathbf{I}, \tag{7}$$

where $\mathbf{I}$ is the identity tensor. The last term is the total energy density of the electromagnetic field. According to equations 6 and 7 the divergence of Maxwell's stress tensor fulfils

$$\nabla\cdot\mathbf{T}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c^2}[\mathbf{E}\times\mathbf{H}]+\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}. \tag{8}$$

Next we want to use the Maxwell stress tensor to calculate the mechanical force acting on the particle. To do this we first integrate equation 8 over an arbitrary volume $V$ that includes all sources $\rho$ and $\mathbf{j}$:

$$\int_V\nabla\cdot\mathbf{T}\mathrm{d}V=\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c^2}\int_V[\mathbf{E}\times\mathbf{H}]\mathrm{d}V+\int_V(\rho\mathbf{E}+\mathbf{j}\times\mathbf{B})\mathrm{d}V. \tag{9}$$

From this equation we recognise the Lorentz force law

$$\mathbf{F}(\boldsymbol{r},t) = q[\mathbf{E}(\boldsymbol{r},t) + \mathbf{v}(\boldsymbol{r},t) \times \mathbf{B}(\boldsymbol{r},t)] \tag{10}$$

$$= \int_V [\rho(\boldsymbol{r},t)\mathbf{E}(\boldsymbol{r},t) + \mathbf{j}(\boldsymbol{r},t) \times \mathbf{B}(\boldsymbol{r},t)]\mathrm{d}V, \tag{11}$$

where equation 10 expresses the force acting on a single particle of charge $q$ moving at velocity $\mathbf{v}$, while equation 11 represents the force for a distribution of charge density $\rho$ and current density $\mathbf{j}$. By now using Stokes theorem on the left hand side of equation 9 we get the conservation law for linear momentum

$$\int_{\partial V} \mathbf{T} \cdot \mathbf{n}\mathrm{d}a = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{c^2}\int_V [\mathbf{E} \times \mathbf{H}]\mathrm{d}V + \mathbf{F} \tag{12}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{G}_{\text{field}} + \mathbf{G}_{\text{mech}}], \tag{13}$$

where $\partial V$ is the surface of $V$, $\mathbf{n}$ is the unit vector perpendicular to it and $\mathrm{d}a$ is an infinitesimal surface element. In particular $\mathbf{G}_{\text{field}} = \int_V \frac{1}{c^2}[\mathbf{E} \times \mathbf{H}]\mathrm{d}V$ is the momentum of the electromagnetic field within volume $V$ and $\mathbf{G}_{\text{mech}}$ is the mechanical momentum, for which $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}_{\text{mech}} = \mathbf{F}$ gives the mechanical force.

We are interested only in the average force so we take a time average of the force by integrating it over time

$$\langle \mathbf{F} \rangle = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{F}(s)\mathrm{d}s, \tag{14}$$

where $T$ is the duration of one oscillation period. The light used in optical trapping is usually monochromatic (a laser), so $T$ is simply the duration of one wavelength of the light. From equations 13 and 14 we find that the average force is

$$\langle \mathbf{F} \rangle = \int_{\partial V} \langle \mathbf{T}(\boldsymbol{r},t) \rangle \cdot \mathbf{n}(\boldsymbol{r})\mathrm{d}a - \langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}_{\text{field}} \rangle. \tag{15}$$

The expectation value for $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}_{\text{field}}$ is zero over one period, so the average force is just

$$\langle \mathbf{F} \rangle = \int_{\partial V} \langle \mathbf{T}(\boldsymbol{r},t) \rangle \cdot \mathbf{n}(\boldsymbol{r})\mathrm{d}a. \tag{16}$$

Here we see that the force expression does not depend on the material parameters of the object, but only on the electromagnetic field on the surface of the arbitrary volume. The field however is now a superposition of the incident field and the field scattered from the object, and of course the scattered field depends on the properties of the object.

### Radiation pressure

Radiaton pressure is the pressure that electromagnetic radiation excerts on an object. It can be calculated using the mechanical force in equation 16 as [13]

$$P_R = \frac{\langle \mathbf{F} \rangle}{A} = \frac{1}{A} \int_A \langle \mathbf{T}(\boldsymbol{r}, t) \rangle \cdot \mathbf{n}(\boldsymbol{r}) \mathrm{d}a, \tag{17}$$

where the integration is done over an area $A$ perpendicular to $\mathbf{F}$.

As a special case, we consider an infinite plane irradiated at normal incindence by a linearly polarized monochromatic plane wave. We choose $\mathbf{n}(\boldsymbol{r}) = -\mathbf{n}_z$ and $\mathbf{E}||\mathbf{n}_x$, where the subscripts of the unit vectors refer to the x and z axes of a Cartesian coordinate system. The total electric field is the superposition of the incident and reflected parts

$$\mathbf{E}(\mathbf{r}, t) = E_0 \mathrm{Re}[(e^{ikz} + Re^{-ikz})e^{-i\omega t}]\mathbf{n}_x, \tag{18}$$

where $E_0$ is the amplitude, k is the wave number, and $\omega$ is the angular frequency of the electric field, and $R$ is a complex reflection coefficient with condition $|R| \leq 1$. The magnetic field can then be calculated from the electric field using Faraday's law in equation 1:

$$\begin{aligned}\mathbf{H}(\mathbf{r},t) =&\frac{1}{\mu_0}\mathbf{B}(\mathbf{r},t) = -\frac{1}{\mu_0}\int \nabla\times\mathbf{E}(\mathbf{r},t)\mathrm{d}t\\ =&-\frac{1}{\mu_0}\int E_0\frac{\partial}{\partial z}\mathrm{Re}[(e^{ikz}+Re^{-ikz})e^{-i\omega t}]\mathrm{d}t\mathbf{n}_y\\ =&-\frac{k}{\mu_0}E_0\int \mathrm{Re}[(ie^{ikz}-iRe^{-ikz})e^{-i\omega t}]\mathrm{d}t\mathbf{n}_y\\ =&\frac{k}{\mu_0}E_0\frac{1}{\omega}\mathrm{Re}[(e^{ikz}-Re^{-ikz})e^{-i\omega t}]\mathbf{n}_y\\ =&\sqrt{\frac{\epsilon_0}{\mu_0}}E_0\mathrm{Re}[(e^{ikz}-Re^{-ikz})e^{-i\omega t}]\mathbf{n}_y. \end{aligned}\tag{19}$$

Maxwell's stress tensor is easy to calculate from equations 18 and 19. For brevity we denote $\mathbf{E} = E\mathbf{n}_x$ and $\mathbf{H} = H\mathbf{n}_y$. The only nonzero components of the stress tensor are

$$T_{xx} = \frac{1}{2}\epsilon_0 E^2 - \frac{1}{2}\mu_0 H^2, \tag{20}$$

$$T_{yy} = -\frac{1}{2}\epsilon_0 E^2 - \frac{3}{2}\mu_0 H^2, \tag{21}$$

$$T_{zz} = -\frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2). \tag{22}$$

Only the $zz$-component impacts the pressure, because the other ones are parallel to the surface, and $\langle\mathbf{T}\rangle\cdot(-\mathbf{n}_z) = -\langle T_{zz}\rangle\mathbf{n}_z$. The averaged $zz$-component is

$$\langle T_{zz}\rangle = -\frac{I_0}{c}(1+|R|^2), \tag{23}$$

where $I_0 = \frac{\epsilon_0}{2}cE_0^2$ is the intensity of the plane wave. The full calculation of the $zz$-component can be found in appendix A. The radiation pressure is then according to equations 17 and 23

$$P_R\mathbf{n}_z = \frac{1}{A}\int_A(-\langle T_{zz}\rangle\mathbf{n}_z)\mathrm{d}a = \frac{\epsilon_0}{2}E_0^2(1+|R|^2)\mathbf{n}_z. \tag{24}$$

The radiation pressure depends on the energy of the electric field $E_0^2$. The higher the energies of the photons are, the higher is the force, and so also the pressure, they excert on an object. The pressure depends also on the reflectivity $|R|^2$: the minimum value $P_{R,min} = \frac{\epsilon_0}{2}E_0^2$ corresponds to a perfectly absorbing surface $|R|^2 = 0$, and the maximum value $P_{R,max} = \epsilon_0 E_0^2$ corresponds to a perfectly reflecting surface $|R|^2 = 1$.

### Dipole approximation

A particle much smaller than the wavelength of light can be approximated as a dipole. We consider an electric dipole with dipole moment $\boldsymbol{\mu} = q\mathbf{s}$, where $\mathbf{s}$ is the vector between the two charges of the dipole, and whose center of mass coordinate is $\mathbf{r}$. Owing to the interaction with the electromagnetic field, the dipole experiences the mechanical force

$$\mathbf{F}(\mathbf{r},t) = (\boldsymbol{\mu}\cdot\nabla)\mathbf{E}(\mathbf{r},t) + \dot{\boldsymbol{\mu}}\times\mathbf{B}(\mathbf{r},t) + \dot{\mathbf{r}}\times(\boldsymbol{\mu}\cdot\nabla)\mathbf{B}(\mathbf{r},t). \tag{25}$$

Here the first term is the force acting on a dipole due to an inhomogeneous electric field, the second term is the force acting on a changing dipole due to a magnetic field and the last term is the force acting on a moving dipole due to an inhomogeneous magnetic field. For non-relativistic speeds ($|\dot{\mathbf{r}}| \ll c$) the last term is very small compared to the others [13] and is therefore henceforth left out.

Using the product rule and Faraday's law in equation 1, the second term in the force expression in equation 25 can be written as

$$\begin{aligned}\dot{\boldsymbol{\mu}}\times\mathbf{B} &= -\boldsymbol{\mu}\times\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}\times\mathbf{B}) \\ &= \boldsymbol{\mu}\times(\nabla\times\mathbf{E}) + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}\times\mathbf{B}),\end{aligned} \tag{26}$$

and so the force becomes

$$\begin{aligned}\mathbf{F} &= (\boldsymbol{\mu}\cdot\nabla)\mathbf{E} + \boldsymbol{\mu}\times(\nabla\times\mathbf{E}) + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}\times\mathbf{B}) \\ &= \sum_i \mu_i\nabla E_i + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}\times\mathbf{B}).\end{aligned} \tag{27}$$

The time average of the force is then simply

$$\langle\mathbf{F}\rangle = \sum_i \langle\mu_i\nabla E_i\rangle, \tag{28}$$

because $\langle \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu} \times \mathbf{B}) \rangle$ is zero.

Next we take a closer look at the scenario with a monochromatic field of angular frequency $\omega$ as that is the case we are specifically interested in when later developing our model. For a monochromatic field the real electric and magnetic fields are [13]

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}[\underline{\mathbf{E}}(\mathbf{r}) e^{-i\omega t}], \tag{29}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathrm{Re}[\underline{\mathbf{B}}(\mathbf{r}) e^{-i\omega t}], \tag{30}$$

where $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are the complex amplitudes of the fields. Here

$$\underline{\mathbf{E}}(\mathbf{r}) = E_0(\mathbf{r}) e^{i\phi(\mathrm{r})} \mathbf{n}_E, \tag{31}$$

$$\underline{\mathbf{B}}(\mathbf{r}) = B_0(\mathbf{r}) e^{i\phi(\mathrm{r})} \mathbf{n}_B, \tag{32}$$

where $E_0$ and $B_0$ arer the real amplitudes of the electric and magnetic fields, respectively, and $\phi$ is the phase of the complex amplitude. Considering a linear interaction and isotropic particle with no static dipole moment, the induced dipole moment is proportional to the complex polarizability $\alpha(\omega) = \alpha_r(\omega) + i\alpha_i(\omega)$ and the electric field [13]:

$$\boldsymbol{\mu}(\mathbf{r}, t) = \mathrm{Re}[\alpha(\omega) \underline{\mathbf{E}}(\mathbf{r}) e^{-i\omega t}] = \mathrm{Re}[\underline{\boldsymbol{\mu}} e^{-i\omega t}]. \tag{33}$$

From equations 28-33 we then find that the average force over one cycle is

$$\langle \mathbf{F} \rangle = \frac{\alpha_r}{4} \nabla E_0^2 + \frac{\alpha_i}{2} E_0^2 \nabla \phi = \frac{\alpha_r}{2} \nabla \langle \mathbf{E}^2 \rangle + \alpha_i \omega \langle \mathbf{E} \times \mathbf{B} \rangle, \tag{34}$$

where the first term is the gradient (or dipole) force and the second term is the scattering force.

The gradient force arises from the inhomogeneities of the field. Particles with a positive real polarizability are pulled towards high-intensity regions of the field while particles with a negative real polarizability are pushed out of them. At the extremum of the optical field intensity the force is zero. Because of this the gradient force can be used to trap particles (of positive polarizability) in the high-intensity region of

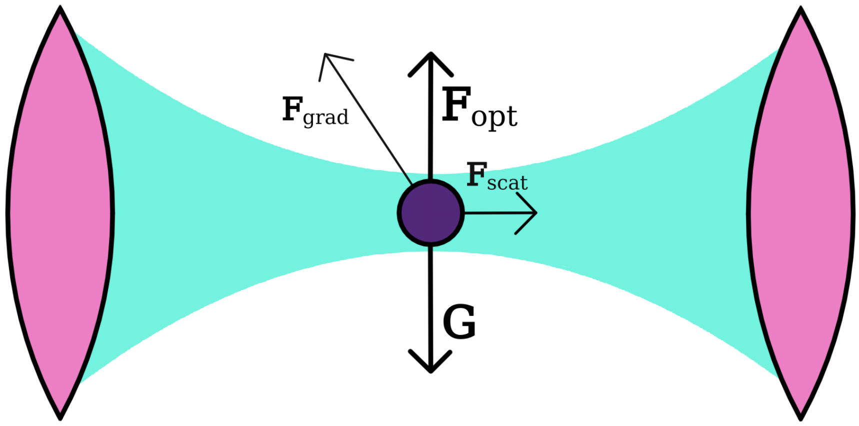


Figure 1. Illustration of an optical tweezer. The particle in the middle (purple) is trapped with a highly focused light field (blue). The light is focused with lenses (pink). The forces acting on the particle are gravity ($\mathbf{G}$) and the optomechanical force, which consists of the gradient force and the scattering force ($\mathbf{F}_{\text{opt}} = \mathbf{F}_{\text{grad}} + \mathbf{F}_{\text{scat}}$). In the focus of the optical field the net force is zero and the particle is levitating.

the field, and that is exactly how particles are trapped in optical tweezers. The scattering force is caused by the momentum transfer from the field to the particle and is proportional to the field momentum $\mathbf{G}_{\text{field}}$. It always pushes the particle in the direction of the beam due to the positive imaginary polarizability, whereupon it can be used to slow down (cool) the motion of the particle.

## 1.2 Optical tweezers

Optical tweezers are devices used to trap nano- and microscopic particles using focused lasers. The method of optical tweezing was first proposed by Arthur Ashkin in 1970 [2] and later demonstrated experimentally by him and his team in 1986 [14]. Since then optical tweezers have become an important tool in many fields, like in microbiology for trapping singular cells [15].

In order to trap a particle in an optical field, there has to be a point where the

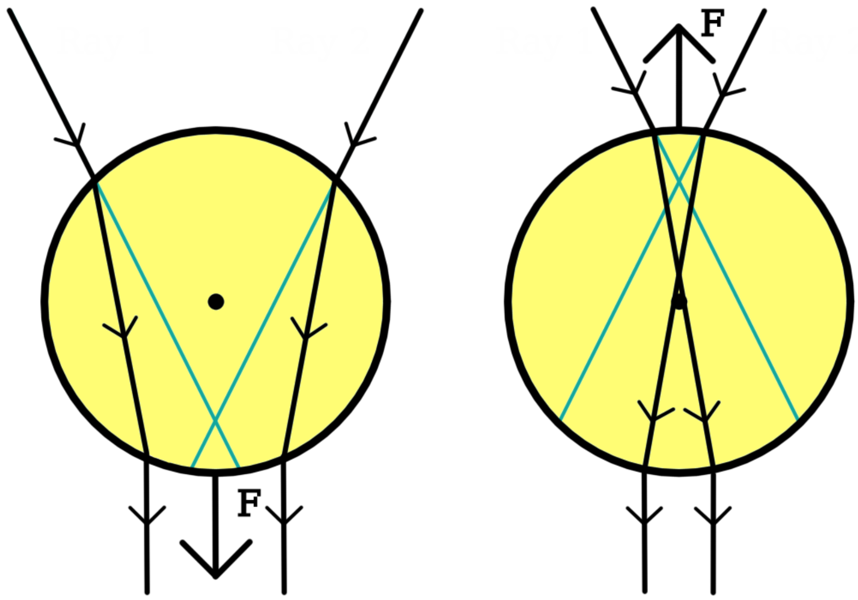


Figure 2. Illustration of the optical trapping force according to ray optics for particles bigger than the wavelength of the laser. Here are shown two rays and the force **F** acting on the particle. The particle is always pulled towards the focus of the rays (the crossing of the blue lines).

net force is zero. As we showed before the gradient force of an optical field has exactly this property and it is the force that traps the particle in an optical tweezer towards to the focus of the beam. Figure 1 shows an illustration of optical tweezers. The laser is tightly focused using lenses and at the focus optical forces perfectly counteract gravity which makes the particle levitate. In practice the particle is not at rest but oscillates around the focus point.

For particles smaller than the wavelength of the laser, we can use the dipole approximation and the trapping force is simply the gradient force in equation 34. For particles bigger than the wavelength of the laser, the trapping force can be calculated using ray optics. An illustration of the optical trapping force is shown in figure 2. The illustration shows two light rays and how their trajectories change when they interact with the particle. The change in the rays' momenta have to be matched by the change in the particle's momentum, and so the particle experiences a force that always pulls it towards the focus of the beam. Again, the force consists of two parts: the scattering force, in the direction of light propagation, and the gradient force, which is perpendicular to it [12]. For intermediate particles the full Maxwell stress tensor is needed to calculate the optomechanical forces.

The trapping force in an optical tweezer is [13]

$$\langle \mathbf{F}(\mathbf{r}) \rangle = \mathbf{Q}(\mathbf{r}) \frac{\epsilon_s^2 P_t}{c}, \tag{35}$$

where $P_t$ is the power of the trapping beam, $c$ is the vacuum speed of light (as before), and $\epsilon_s$ is the relative permittivity of the surrounding medium. We work in vacuum, where $\epsilon_s = 1$. $\mathbf{Q}(\mathbf{r})$ is the trapping efficiency, which in the Rayleigh regime and with no particle losses depends on the normalized gradient of the light intensity and the polarizability of the particle $\alpha$ [13]. The polarizability of a homogeneous and isotropic spherical particle is the Clausius-Mossotti polarizability [12]

$$\alpha(\omega) = 3\epsilon_0 V_0 \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 2}, \tag{36}$$

where $\epsilon_0$ is the vacuum permittivity, $V_0$ is the volume of the particle, and $\epsilon(\omega)$ is the relative permittivity of the particle. Moreover, for small deviations $x$ from the focus, the restoring force is linear as $\langle F \rangle = -kx$, where $k$ is the trap stiffness. This corresponds to a harmonic trapping potential $U = \frac{1}{2}kx^2$.

Figure 3 shows a real photo of a levitating nanoparticle in optical tweezers, taken by the Quantum Nanophotonics Group of the Vamivakas Lab in the Institute of Optics at the University of Rochester.

## 1.3 Cavities in optical trapping

The basic model of an optical cavity is presented in figure 4. It is composed of two parabolic mirrors facing each other. Between the mirrors is a standing wave, the cavity mode with frequency $\omega_c$. A nanoparticle placed in the cavity couples to

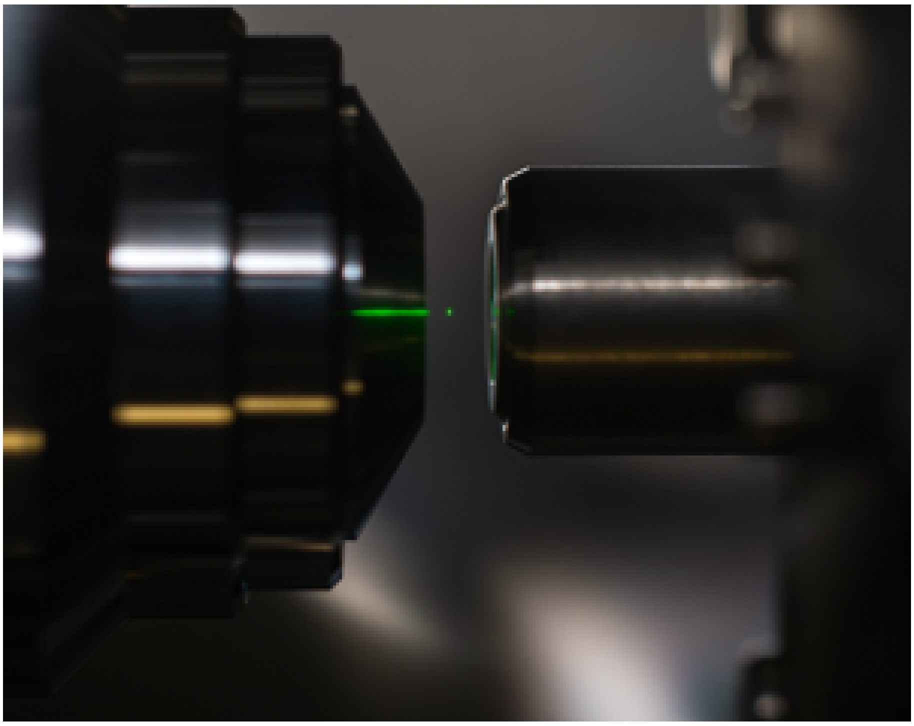

Figure 3. Photo of a levitated nanoparticle by J. Adam Fenster, University of Rochester.
https://labsites.rochester.edu/vamivakaslab/research/levitated-optomechanics/

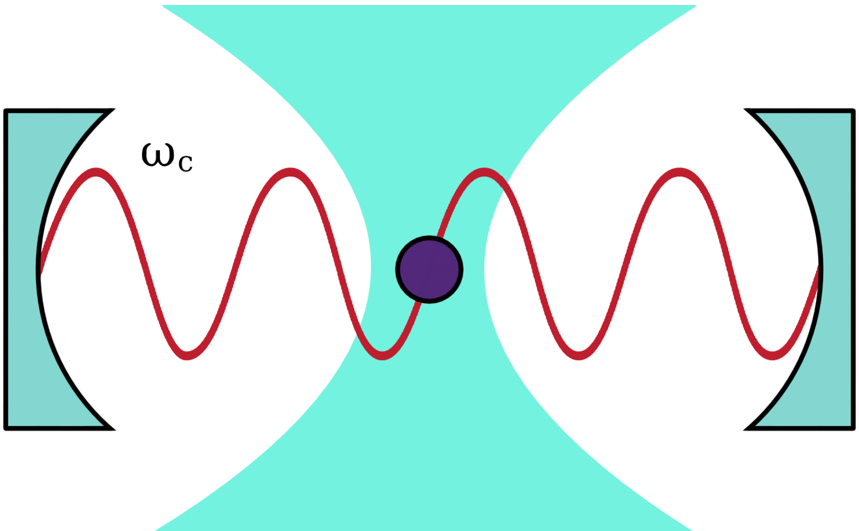


Figure 4. Illustration of a particle in an optical cavity. The particle is trapped in optical field (blue area) and placed between mirrors in a cavity. The particle couples to the cavity mode with frequency $\omega_c$.

the cavity mode and this interaction can be used to cool down the particle to its quantum ground state.

The cavity can consist of two modes, one for trapping of the particle and one for cooling, or the particle can be trapped in an external optical tweezer and placed inside a cavity with only one mode, so that the trapping field and the cavity field are orthogonal [6]. We are interested in the latter case.

In a cavity the light field is an active participant in the dynamics. Close to equilibrium the combined system can be modeled as two coupled quantum harmonic oscillators, instead of a mechanical oscillator in an external potential, as was the case in optical tweezers without cavity.

Optical cavities are used for cavity assisted optomechanical cooling, where the center of mass motion is cooled as photons are scattered off from the particle into the blue-detuned cavity [16].

Optomechanical quantum control, preparing and controlling quantum states using optomechanical tools, requires that the particle is near its quantum ground state, where thermal energy is much smaller than the mechanical energy of the oscillator: $k_B T \ll \hbar\Omega_M$. Here $k_B$ is the Boltzmann constant, $T$ is temperature, $\hbar$ is the reduced Planck constant, and $\Omega_M$ is the mechanical frequency. This condition can be reached using cavity assisted cooling [9].

In a blue-detuned cavity, where the resonance frequency of the cavity is higher than the center of mass motion's, photons scattering from the particle due to optomechanical interaction tend to scatter to higher energies in order to enter the cavity resonance. This increase in photon energy comes from mechanical motion of the particle, meaning that the particle slows and cools down. This process is called the Anti-Stokes process.

The opposite process, where the photon scatters red-shifted, i.e. with a lower energy that it had before, is called the Stokes process. This process increases the energy of the mechanical oscillator and therefore heats it up. These processes can happen simultaneously and in order to cool the particle the Anti-Stokes process must be dominant.

The rate of Anti-Stokes processes is $A^-$, so the transition rate from $n$ to $n-1$ phonons is

$$\Gamma_{n\to n-1} = nA^-. \tag{37}$$

Similarly the rate of the Stokes process is $A^+$ and the transition rate from $n$ to $n+1$ phonons is

$$\Gamma_{n\to n+1} = (n+1)A^+. \tag{38}$$

When the cavity is blue-detuned the rate of the Stokes process is smaller than the rate of the Anti-Stokes process.

Both the Anti-Stokes and the Stokes processes contribute to the full optomechanical damping rate

$$\Gamma_{\text{opt}} = A^- - A^+. \tag{39}$$

The average phonon number $\bar{n} = \sum_{n=0}^{\infty} nP_n$ is affected by the processes, so

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}t} = (\bar{n}+1)A^+ - \bar{n}A^- \tag{40}$$

In the steady state $\frac{\mathrm{d}\bar{n}}{\mathrm{d}t} = 0$ this gives

$$\bar{n} = \frac{A^+}{A^- - A^+}. \tag{41}$$

The rates of (Anti-)Stokes process can be calculated using Fermi's golden rule, using the optomechanical interaction Hamiltonian that we'll calculate a bit later in section 3 (equation 95) $\hat{H}_I = g_0\hat{a}^\dagger\hat{a}(\hat{b}+\hat{b}^\dagger)$, where $g_0$ is the coupling strength, and $\hat{a}^\dagger$ ($\hat{a}$) and $\hat{b}^\dagger$ ($\hat{b}$) are the creation (annihilation) operators for the cavity mode and the mechanical mode, respectively. In the weak coupling regime the rates can be calculated using quantum noise spectrum

$$S_{FF}(\omega) = G^2 \int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \langle \hat{a}^\dagger \hat{a}(t) \hat{a}^\dagger \hat{a}(0) \rangle = G^2 S_{NN}(\omega), \tag{42}$$

where $S_{NN}$ is the photon number noise spectrum, that gives us the energy eigenstates that the photons can scatter into, and is calculated in [17] to be

$$S_{\mathrm{NN}}(\omega) = \bar{n}_{\mathrm{cav}} \frac{\kappa}{\kappa^2/4 + (\Delta + \omega)^2}, \tag{43}$$

where $\bar{n}_{cav}$ is the average photon number of the cavity mode, $\kappa$ is photon dissipation rate, and $\Delta = \omega_L - \omega_{cav}$ is the detuning between laser frequency $\omega_L$ and cavity frequency $\omega_{cav}$.

Using the spectrums, we get that the (Anti-)Stokes process rates are

$$A^{\pm} = x_{\mathrm{ZPF}^2} S_{\mathrm{FF}}(\omega = \mp\Omega_m) = g_0^2 S_{NN}(\omega = \mp\Omega_m) = \quad g_0^2 \bar{n}_{\mathrm{cav}} \frac{\kappa}{\kappa^2/4 + (\Delta \mp \Omega_m)^2}, \tag{44}$$

where $x_{ZPF}$ is the zero point fluctuation amplitude of the position operator $\hat{b} + \hat{b}^\dagger$, and $g_0 = Gx_{ZPF}$ is the single-photon optomechanical coupling strength [9].

Inserting them into equation 41 we get that the steady state the phonon number is

$$\bar{n} = (\frac{A^-}{A^+} - 1)^{-1} = (\frac{\kappa^2/4 + (\Delta - \Omega_m)^2}{\kappa^2/4 + (\Delta + \Omega_m)^2} - 1)^{-1}. \tag{45}$$

The phonon number can be minimised by varying the detuning $\Delta$. When $\kappa \ll \Omega_m$, the minimum is

$$\bar{n}_{min} = (\frac{\kappa}{4\Omega})^2 \ll 1 \tag{46}$$

meaning that ground state cooling is possible.

# 2 Theory of open quantum systems

Some systems, where interactions with environment are insignificant, we can approximate as closed systems fully isolated from their environments. But in reality, no system can be fully isolated, and we need the theory of open quantum systems to describe them.

Open quantum system theory can be used to analyse many different kinds of systems in many different fields [18]. For example, in quantum optics it is needed to describe light sources and dissipation of photons [19]. It can also be used to describe phenomena in condensed matter physics [20], molecular physics [21], and in quantum information protocols [22].

Open quantum system interacts with an external environment. Therefore when we are interested in its dynamics we have to take into account not only the system itself but also the dynamics of the environment and the interactions between the system and the environment.

In open quantum systems the total system (with Hilbert space $\mathcal{H}_T$, density matrix $\rho_T$ and Hamiltonian $H_T$) is closed and consists of the system of interest ($\mathcal{H}_S$, $\rho_S$, $H_S$) and the environment ($\mathcal{H}_E$, $\rho_E$, $H_E$), as shown in figure 5. The total Hilbert space is a composite of the Hilbert spaces of the system and the environment $\mathcal{H}_T = \mathcal{H}_S \otimes \mathcal{H}_E$ and each of the reduced density matrices of the subsystems can be calculated as a partial trace of the total density matrix: $\rho_S(t) = \mathrm{Tr}_E[\rho_T(t)]$ and $\rho_E(t) = \mathrm{Tr}_S[\rho_T(t)]$. The total Hamiltonian consists of a bare system term, a bare environment term, and an interaction term:

$$H_T = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + \alpha H_I, \tag{47}$$

where $\alpha$ is the strength of the system-environment interaction and the interaction Hamiltonian can be decomposed as

$$H_I = \sum_i S_i \otimes E_i, \tag{48}$$

where $S_i \in B(\mathcal{H}_S)$ and $E_i \in B(\mathcal{H}_E)$ are bounded operators of $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively. In practice, the boundedness assumption is often relaxed.

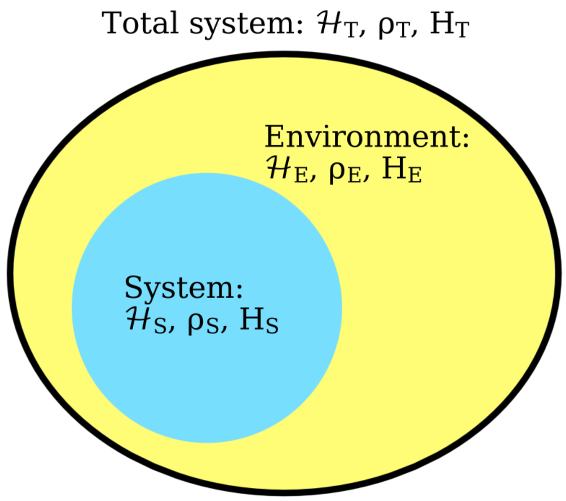


Figure 5. The total system is composed of the system we are interested in and the environment it interacts with.

The equation of motion for the total system is given by the von Neumann equation

$$\dot{\rho}_T(t) = -i[H_T, \rho_T(t)]. \tag{49}$$

The next goal is to reduce this equation to find the equation of motion for only the system, effectively removing the environment from the equation and making it easier to solve. This way we find the Gorini-Kossakowski-Sudarshan-Lindblad master equation (GKSL) for the system [23, 24].

## 2.1 Gorini-Kossakowski-Sudarshan-Lindblad master equation

Lindblad equation is the most general generator for Markovian dynamics in an open quantum system. We derive it here from microscopic dynamics in a system described in figure 5. The derivation follows [25].

We work in the interaction picture, where both density matrices and operators depend on time, as opposed to Schrödinger picture where only density matrices depend on time, and Heisenberg picture where only operators depend on time. In the interaction picture, an arbitrary operator $\mathrm{A} \in B(\mathcal{H}_T)$ is represented as a time-dependent operator $\hat{\mathrm{A}}(t) = e^{i(H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E)t} \mathrm{A} e^{-i(H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E)t}$ and evolves due to the system and environment Hamiltonians, while the time evolution of the density matrix comes from the interaction Hamiltonian, as given by equation

$$\frac{\mathrm{d}\hat{\rho}_T(t)}{\mathrm{d}t} = -i\alpha[\hat{H}_I(t), \hat{\rho}_T(t)]. \tag{50}$$

Integrating this over time, we get the equation

$$\hat{\rho}_T(t) = \hat{\rho}_T(0) - i\alpha \int_0^t \mathrm{d}s[\hat{H}_I(s), \hat{\rho}_T(s)]. \tag{51}$$

Then we insert this $\hat{\rho}_T(t)$ back into equation 50

$$\frac{\mathrm{d}\hat{\rho}_T(t)}{\mathrm{d}t} = -i\alpha[\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t \mathrm{d}s[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(s)]], \tag{52}$$

and then do the same thing again to $\hat{\rho}_T(s)$ in the integral, using $\hat{\rho(t)}$ as the initial condition and integrating backwards in time,

$$\begin{aligned}
\frac{\mathrm{d}\hat{\rho}_T(t)}{\mathrm{d}t} =& - i\alpha[\hat{H}_I(t), \hat{\rho}_T(0)] \\
& - \alpha^2 \int_0^t \mathrm{d}s[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(t) - i\alpha \int_t^s \mathrm{d}s'[\hat{H}_I(s'), \hat{\rho}_T(s')]]] \\
=& - i\alpha[\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t \mathrm{d}s[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(t)]] \\
& + i\alpha^3 \int_0^t \mathrm{d}s \int_t^s \mathrm{d}s'[\hat{H}_I(t), [\hat{H}(s), [\hat{H}_I(s'), \hat{\rho}_T(s')]]] \\
=& - i\alpha[\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t \mathrm{d}s[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(t)]],
\end{aligned} \tag{53}$$

where on the last line we have assumed that the interaction between system and environment is weak, i.e. $\alpha$ is small, and therefore the term of order $\alpha^3$ is very small and can be left out.

This equation 53 is for the total system, and to find the equation of motion for only the system we are interested in, we have to take a partial trace over the environment

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} =& \mathrm{Tr}_E[\frac{\mathrm{d}\hat{\rho}_T(t)}{\mathrm{d}t}] \\ =& -i\alpha \mathrm{Tr}_E[\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t \mathrm{d}s \mathrm{Tr}_E[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(t)]]. \end{aligned} \tag{54}$$

This still depends on the total density matrix. We assume that at time $t = 0$ there are no correlations between the system and the environment and so the total system is in a separable state $\hat{\rho}_T(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$, and that initially environment is in a thermal state

$$\hat{\rho}_E(0) = \frac{e^{-\frac{H_E}{k_B T}}}{\mathrm{Tr}[e^{-\frac{H_E}{k_B T}}]}, \tag{55}$$

where $T$ is the temperature and $k_B$ is the Boltzmann constant. Going forward we set $k_B = 1$. Using the form 48 for the interaction Hamiltonian $H_I$, we can calculate that the first term in 54 is

$$\begin{aligned}\mathrm{Tr}_E[\hat{H}_I(t), \hat{\rho}_T(0)] =& \mathrm{Tr}_E[\sum_i \hat{S}_i \otimes \hat{E}_i, \hat{\rho}_T(0)] \\ =& \mathrm{Tr}_E(\sum_i \hat{S}_i \hat{\rho}_S(0) \otimes \hat{E}_i \hat{\rho}_E(0) - \sum_i \hat{\rho}_S(0) \hat{S}_i \otimes \hat{\rho}_E(0) \hat{E}_i) \\ =& \sum_i (\hat{S}_i \hat{\rho}_S(0) \mathrm{Tr}(\hat{E}_i \hat{\rho}_E(0)) - \hat{\rho}_S(0) \hat{S}_i \mathrm{Tr}(\hat{\rho}_E(0) \hat{E}_i)) \qquad (56) \\ =& 0, \qquad (57) \end{aligned}$$

where we have assumed $\langle E_i \rangle = \mathrm{Tr}[E_i \hat{\rho}_E(0)] = \mathrm{Tr}[\hat{\rho}_E(0) E_i] = 0$, $\forall i$. If $\langle E_i \rangle$ is not zero, we can rewrite the interaction Hamiltonian as

$$H_I' = \sum_i S_i \otimes (E_i - \langle E_i \rangle \mathbb{I}_E), \tag{58}$$

where clearly now $\langle E_i - \langle E_i \rangle \mathbb{I}_{\mathbb{E}} \rangle = 0$. To cancel out the effect that this change has in the system dynamics we need to add a driving term $\alpha \sum_i \langle E_i \rangle S_i$ to the original system Hamiltonian. This way the total Hamiltonian remains unchanged:

$$\begin{aligned}H_T =&H'_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + \alpha H'_I \\ =&(H_S + \alpha \sum_i \langle E_i \rangle S_i) \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + \alpha \sum_i S_i \otimes (E_i - \langle E_i \rangle \mathbb{I}_{\mathbb{E}}) \\ =&H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + \alpha \sum_i S_i \otimes E_i - \alpha \langle E_i \rangle \sum_i S_i \otimes \mathbb{I}_E + \alpha \langle E_i \rangle \sum_i S_i \otimes \mathbb{I}_E \\ =&H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + \alpha H_I, \end{aligned} \tag{59}$$

and we can assume that $\langle E \rangle_i = 0$.

Because the interaction between system and environment is very weak, we can assume that the system and the environment are always noncorrelated and therefore the environment is always in thermal state. Corrections to this assumptions are higher order in $\alpha$. The total state at time $t$ is then $\hat{\rho}_T(t) = \hat{\rho}_S(t) \otimes \hat{\rho}_E(0)$ and equation 54 becomes

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = -\alpha^2 \int_0^t \mathrm{d}s \mathrm{Tr}_E[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_S(t) \otimes \hat{\rho}_E(0)]]. \tag{60}$$

By taking the upper limit to infinity and changing the integration variable $s \to t-s$, we obtain Redfield equation

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = -\alpha^2 \int_0^\infty \mathrm{d}s \mathrm{Tr}_E[\hat{H}_I(t), [\hat{H}_I(s-t), \hat{\rho}_S(t) \otimes \hat{\rho}_E(0)]]. \tag{61}$$

To ensure the complete positivity of the evolution given by the master equation, we must do a rotating wave approximation. Before that we define a superoperator $\tilde{H}A \equiv [H_S, A]$, $\forall A \in B(\mathcal{H})$. Operators fulfiling

$$\tilde{H}S_i(\omega) =[H_S, S_i(\omega)] = -\omega S_i(\omega), \tag{62}$$

$$\tilde{H}S_i^\dagger(\omega) =[H_S, S_i^\dagger(\omega)] = \omega S_i^\dagger(\omega) \tag{63}$$

are the eigenoperators of the superoperator and form a complete basis of $B(\mathcal{H})$. Then we can write the system operator $S_i$ from the decomposition of interaction

Hamiltonian 48 in this basis

$$S_i = \sum_\omega S_i(\omega). \tag{64}$$

We change back to the Schrödinger picture for the system part of the interaction Hamiltonian by expression

$$\hat{S}_k = e^{itH_S} S_k e^{-itH_S} = e^{itH_S} e^{-it(H_S+\omega)} S_k = e^{-i\omega t} S_k. \tag{65}$$

We apply these changes for the interaction Hamiltonian in equation 61, and after some calculations [25] we get

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = \sum_{\omega',\omega,k,l} &(e^{-i(\omega-\omega')t}\Gamma_{kl}(\omega)[S_l(\omega)\hat{\rho}_S(t), S_k^\dagger(\omega')] \\ &+ e^{i(\omega-\omega')t}\Gamma_{lk}^*(\omega')[S_l(\omega), \hat{\rho}_S(t) S_k^\dagger(\omega')]),\end{aligned} \tag{66}$$

where factors $\Gamma_{kl}(\omega) = \int_0^\infty \mathrm{d}s e^{i\omega s} \mathrm{Tr}[\hat{E}_k^\dagger(t)\hat{E}_l(t-s)\hat{\rho}_E(0)]$ include the effects of the environment.

Here $|\omega - \omega'|$ gives the frequency of the oscillation for each term. If $|\omega - \omega'| \gg \alpha^2$, the term oscillates much faster than the system evolves, since $\alpha^2$ is the strength of the interaction and so $\frac{1}{\alpha^2}$ gives the time scale of evolution. Such terms do not contribute to the overall time-evolution of the system and we can leave them out of consideration.

We have already assumed that the interaction between the system and environment is weak and $\alpha$ is small. Taking $\alpha \to 0$ we are left with only the resonant terms $\omega = \omega'$ in equation 66:

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = \sum_{\omega,k,l}(\Gamma_{kl}(\omega)[S_l(\omega)\hat{\rho}_S(t), S_k^\dagger(\omega)] + \Gamma_{lk}^*(\omega)[S_l(\omega), \hat{\rho}_S(t) S_k^\dagger(\omega)]). \tag{67}$$

This is the rotating wave approximation.

We divide the operators $\Gamma_{kl}(\omega)$ into Hermitian and non-Hermitian parts as

$$\Gamma_{kl}(\omega) = \frac{1}{2}\gamma_{kl}(\omega) + i\pi_{kl}(\omega), \tag{68}$$

where the Hermitian part is

$$
\begin{aligned}
\gamma_{kl}(\omega) \equiv& \Gamma_{kl}(\omega) + \Gamma_{kl}(\omega)^* \\
=& \int_0^{\infty} \mathrm{d}s e^{i\omega s} \mathrm{Tr}[\hat{E}_k^{\dagger}(t)\hat{E}_l(t-s)\hat{\rho}_E(0)] + \int_0^{\infty} \mathrm{d}s e^{-i\omega s} \mathrm{Tr}[\hat{E}_k^{\dagger}(t)\hat{E}_l(t-s)\hat{\rho}_E(0)] \\
=& \int_0^{\infty} \mathrm{d}s e^{i\omega s} \mathrm{Tr}[\hat{E}_k^{\dagger}(t)\hat{E}_l(t-s)\hat{\rho}_E(0)] - \int_0^{-\infty} \mathrm{d}s e^{i\omega s} \mathrm{Tr}[\hat{E}_k^{\dagger}(t)\hat{E}_l(t+s)\hat{\rho}_E(0)] \\
=& \int_{-\infty}^{\infty} \mathrm{d}s e^{i\omega s} \mathrm{Tr}[\hat{E}_k^{\dagger}(s)E_l\hat{\rho}_E(0)]
\end{aligned} \tag{69}
$$

and the non-Hermitian part is $\pi_{kl}(\omega) \equiv \frac{-i}{2}(\Gamma_{kl}(\omega) - \Gamma_{kl}(\omega)^*)$. Using this form for $\Gamma_{kl}(\omega)$ equation 67 becomes

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} =& \sum_{\omega,k,l}((\frac{1}{2}\gamma_{kl}(\omega) + i\pi_{kl}(\omega))[S_l(\omega)\hat{\rho}_S(t), S_k^{\dagger}(\omega)] \\
&+ (\frac{1}{2}\gamma_{kl}(\omega) - i\pi_{kl}(\omega))[S_l(\omega), \hat{\rho}_S(t)S_k^{\dagger}(\omega)]) \\
=& \sum_{\omega,k,l}(\gamma_{kl}(\omega)(S_l(\omega)\hat{\rho}_S(t)S_k^{\dagger}(\omega) \\
&- \frac{1}{2}\{S_k^{\dagger}(\omega)S_l(\omega), \hat{\rho}_S(t)\}) - i\pi_{kl}[S_k^{\dagger}(\omega)S_l(\omega), \hat{\rho}_S(t)]).
\end{aligned} \tag{70}
$$

Then we change fully back into the Schrödinger picture using $\hat{\rho}_S = e^{itH_S}\rho_S e^{-itH_S}$. On the left hand side we get

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\rho}_S}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}(e^{itH_S}\rho_S e^{-itH_S}) \\
=& iH_S e^{itH_S}\rho_S e^{-itH_S} + e^{itH_S}\frac{\mathrm{d}\rho_S}{\mathrm{d}t}e^{-itH_S} - e^{itH_S}\rho_S e^{-itH_S} iH_S \\
=& e^{itH_S}\frac{\mathrm{d}\rho_S}{\mathrm{d}t}e^{-itH_S} + ie^{itH_S}[H_S, \rho_S]e^{-itH_S},
\end{aligned} \tag{71}
$$

and, because operators $S_k$ are eigen operators for $\tilde{H}$, we can take out the exponential factors in each term on the right hand side in equation 70

$$
\begin{aligned}
S_k^{\dagger}(\omega)S_l(\omega)e^{itH_S}\rho_S e^{-itH_S} =& S_k^{\dagger}(\omega)e^{itH_S}e^{i\omega t}S_l(\omega)\rho_S e^{-itH_S} \\
=& e^{itH_S}e^{-i\omega t}S_k^{\dagger}(\omega)e^{i\omega t}S_l(\omega)\rho_S e^{-itH_S} \\
=& e^{itH_S}S_k^{\dagger}(\omega)S_l(\omega)\rho_S e^{-itH_S}.
\end{aligned} \tag{72}
$$

The calculation is almost identical for the other two terms. Then we can cancel out the exponential factors on both sides of the equation 70. This gives us the master equation on Schrödinger picture

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = - i[H_S + H_L, \rho_S(t)] + \sum_{\omega,k,l}(\gamma_{kl}(\omega)(S_l(\omega)\rho_S(t)S_k^\dagger(\omega) - \frac{1}{2}\{S_k^\dagger(\omega)S_l(\omega), \rho_S(t)\})), \tag{73}$$

where $H_L = \sum_{\omega,k,l} \pi_{kl} S_k^\dagger(\omega)S_l(\omega)$ is called the Lamb shift Hamiltonian. We denote the full Hamiltonian $H = H_S + H_L$.

To get the Lindblad equation we still need to write this in diagonal form. $\gamma(\omega) = (\gamma_{kl}(\omega))$ is a positive matrix so it can be diagonalized using a unitary transformation $O$ as

$$O\gamma(\omega)O^\dagger = \begin{pmatrix} d_1(\omega) & 0 & \dots & 0 \\ 0 & d_2(\omega) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_N(\omega) \end{pmatrix}. \tag{74}$$

Then we define operators $L_k = \sum_l O_{lk} S_l$, which form another orthonormal basis. Using these operators we rewrite the master equation in a diagonal form:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = - i[H, \rho_S(t)] + \sum_{\omega,k} d_k(\omega)(L_k(\omega)\rho_S(t)L_k^\dagger(\omega) - \frac{1}{2}\{L_k^\dagger(\omega)L_k(\omega), \rho_S(t)\}). \tag{75}$$

$L_k$ are the jump operators, which describe the stochastic part of the dynamics based on how the environment affects the system.

If there is only one relevant frequency $\omega$ Lindblad equation simplifies to

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = - i[H, \rho_S(t)] + \sum_k d_k(L_k\rho_S(t)L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_S(t)\}). \tag{76}$$

## 2.2 Unravellings (Quantum State Diffusion)

We start with an ensemble of state vectors, that satisfy a stochastic differential equation, and whose density matrix satisfies a deterministic master equation. This is called an unravelling of the master equation. Each unravelling corresponds to only on master equation, while a master equation can have many different possible unravellings. One of the unravellings for Lindblad master equation 76 is the Quantum State Diffusion (QSD) [26, 27].

In QSD the ensemble consists of normalized pure states $|\psi_t\rangle$ and the density matrix is the ensemble average of the projection operators for these states:

$$\rho(t) = \langle |\psi(t)\rangle\langle\psi(t)| \rangle. \tag{77}$$

The QSD equation can be derived by first writing the stochastic differential equation for state vector $|\psi_t\rangle$ in Itô form [28, 29]:

$$|\mathrm{d}\psi\rangle = |v\rangle \mathrm{d}t + \sum_k |u_k\rangle \mathrm{d}\xi_k. \tag{78}$$

where the first term is the drift of the state vector, and the sum is the stochastic fluctuations. Here $\mathrm{d}\xi_k$ are independent complex Wiener increments, with independent and equal fluctuations in their real and imaginary parts, and which satisfy

$$\langle \mathrm{d}\xi_k \rangle = 0,\ \langle \mathrm{d}\xi_k \mathrm{d}\xi_l \rangle = 0,\ \langle \mathrm{d}\xi_k^* \mathrm{d}\xi_l \rangle = \delta_{kl}\mathrm{d}t. \tag{79}$$

Also the fluctuations at different times are assumed to be independent, so the process is Markovian.

The fluctuations must be orthogonal to the state vector $\langle\psi_t|u_k\rangle = 0$ to preserve the normalization of the state vector. This means that

$$\langle |\mathrm{d}\psi\rangle \rangle = |v\rangle \mathrm{d}t \text{ and } \langle |\mathrm{d}\psi\rangle\langle \mathrm{d}\psi| \rangle = 2\sum_k |u_k\rangle\langle u_k| \mathrm{d}t. \tag{80}$$

The change in the density matrix $\rho$ is in Itô formalism

$$\begin{aligned} \mathrm{d}\rho =& \mathrm{d}\langle |\psi\rangle\langle\psi| \rangle = \langle |\mathrm{d}\psi\rangle\langle\psi| + |\psi\rangle\langle \mathrm{d}\psi| + |\mathrm{d}\psi\rangle\langle \mathrm{d}\psi| \rangle \\ \to \dot{\rho} =& |v\rangle\langle\psi| + |\psi\rangle\langle v| + 2\sum_k |u_k\rangle\langle u_k|. \end{aligned} \tag{81}$$

The stochastic terms are orthogonal to $|\psi\rangle\langle\psi|$, so using the Lindblad master equation 76 for $\dot{\rho}$ and initial density matrix $\rho_\psi = |\psi\rangle\langle\psi|$, that projects onto a pure state, we see that they are

$$\begin{aligned}
2\sum_k |u_k\rangle\langle u_k| =&(\mathbb{I} - |\psi\rangle\langle\psi|)\dot{\rho}(\mathbb{I} - |\psi\rangle\langle\psi|) \\
=&(\mathbb{I} - |\psi\rangle\langle\psi|)(-[H,\rho] + \sum_k (L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}))(\mathbb{I} - |\psi\rangle\langle\psi|) \\
=&\sum_k (\mathbb{I} - |\psi\rangle\langle\psi|)L_k|\psi\rangle\langle\psi|L_k^\dagger(\mathbb{I} - |\psi\rangle\langle\psi|),
\end{aligned} \tag{82}$$

and from this we see that

$$|u_k\rangle = ((L_k) - \langle L_k\rangle_\psi)|\psi\rangle. \tag{83}$$

The drift is given by

$$\begin{aligned}
\dot{\rho}|\psi\rangle =&|v\rangle + |\psi\rangle\langle v|\psi\rangle \\
\langle\psi|\dot{\rho}|\psi\rangle =&\langle v|\psi\rangle + \langle v|\psi\rangle = 2\mathrm{Re}\langle\psi|v\rangle,
\end{aligned} \tag{84}$$

and so

$$|v\rangle =\dot{\rho}|\psi\rangle - \langle\psi|v\rangle|\psi\rangle = \dot{\rho}|\psi\rangle - (\frac{1}{2}\langle\psi|\dot{\rho}|\psi\rangle + ic)|\psi\rangle, \tag{85}$$

where $ic = i\mathrm{Im}(\langle\psi|v\rangle)$ is a purely imaginary phase change function that does not affect the physics of the system and is by convention set to zero so that the equation simplifies back to Schrödinger equation in absence of an environment.

Similarly as before we insert $\dot{\rho}$ from Lindblad equation 76 and using initial density matrix $\rho_\psi$ we can solve $|v\rangle$

$$\begin{aligned}|v\rangle =& -i(H|\psi\rangle\langle\psi|\psi\rangle - |\psi\rangle\langle H\rangle_\psi)\\ &+ \sum_k (L_k|\psi\rangle\langle L_k^\dagger\rangle - \frac{1}{2}(L_k^\dagger L_k|\psi\rangle\langle\psi|\psi\rangle + |\psi\rangle\langle L_k^\dagger\rangle\langle L_k\rangle))\\ &- \frac{1}{2}(-i\langle H\rangle_\psi|\psi\rangle + i\langle H\rangle_\psi|\psi\rangle\\ &+ \sum_k (\langle L_k\rangle_\psi\langle L_k^\dagger\rangle_\psi|\psi\rangle - \frac{1}{2}(\langle L_k^\dagger\rangle\langle L_k\rangle\langle\psi|\psi\rangle|\psi\rangle + \langle L_k^\dagger\rangle\langle L_k\rangle\langle\psi|\psi\rangle|\psi\rangle)))\\ =& -iH|\psi\rangle + \sum_k [\langle L_k^\dagger\rangle L_k - \frac{1}{2}L_k^\dagger L_k - \frac{1}{2}\langle L_k^\dagger\rangle\langle L_k\rangle]|\psi\rangle. \end{aligned} \tag{86}$$

The full stochastic differential equation for the state vector is therefore

$$\begin{aligned}|\mathrm{d}\psi\rangle =& -iH|\psi\rangle\mathrm{d}t + \sum_k [\langle L_k^\dagger\rangle L_k - \frac{1}{2}L_k^\dagger L_k - \frac{1}{2}\langle L_k^\dagger\rangle\langle L_k\rangle]|\psi\rangle\mathrm{d}t\\ &+ \sum_k [L_k - \langle L_k\rangle]|\psi\rangle\mathrm{d}\xi_k. \end{aligned} \tag{87}$$

This is the non-linear QSD equation in Itô form.

However, for our system we want to use a linear QSD equation in Stratonovich form, so we can solve the equation we using normal rules of calculus. For this reason we next write the linear QSD equation by simply excluding the non-linear terms [29]

$$|\mathrm{d}\psi\rangle = -iH|\psi\rangle\mathrm{d}t - \sum_k \frac{1}{2}L_k^\dagger L_k|\psi\rangle\mathrm{d}t + \sum_k L_k|\psi\rangle\mathrm{d}\xi_k. \tag{88}$$

This linear form no longer preserves the norm of the state for individual states, but on average produces the GKSL evolution.

In this case, the linear QSD equation is the same in both Itô and Stratonovich forms, but in Stratonovich form it is usually written in differential form

$$\frac{\partial}{\partial t}|\psi\rangle = -iH|\psi\rangle - \sum_k \frac{1}{2}L_k^\dagger L_k|\psi\rangle + \sum_k \eta_k L_k|\psi\rangle, \tag{89}$$

where the new noise $\eta_k = \frac{\mathrm{d}\xi_k}{\mathrm{d}t}$ has properties

$$\langle\eta_k(t)\rangle = 0,\ \langle\eta_k(t)\eta_l(s)\rangle = 0,\ \langle\eta_k(t)\eta_k^*(s)\rangle = \kappa\delta(t-s). \tag{90}$$

This form of writing implies that we can use normal rules of calculus when solving the equation.

# 3 System under study

We are interested in a system, where a nanoparticle is trapped in optical tweezers and placed in an optical cavity. We use a Febry-Pérot cavity, where one mirror is movable, making it possible to adjust the length of the cavity. This system includes two modes, a mechanical mode for the center of mass motion of the particle and a cavity mode for the optical cavity, that interact. We do not take the polarization of the optical field into account, so we have simply assumed that the field is a scalar.

The Hamiltonian part of the dynamics must therefore include both modes and the interaction between them, and the non-Hamiltonian part of the dynamics, given by jump operators $L_m$ and $L_{cav}$, describe the decay of the modes.

To write the GKSL equation for this system we need to know the Hamiltonian $H$ and the jump operators $L_{cav}$, $L_m$. We use a similar approach as in [9].

The tweezer field creates a harmonic potential to trap the particle, meaning that the Hamiltonian for the mechanical oscillator trapped in the tweezer field, but not coupled to the cavity field is

$$H_{m,0} = \omega_t \hat{b}^\dagger \hat{b}, \tag{91}$$

where $\omega_t$ is the mechanical frequency (the frequency of the harmonic trap), and $\hat{b}$ ($\hat{b}^\dagger$) is the annihilation (creation) operator for the mechanical mode [7].

The cavity mode can also be described by a harmonic oscillator, so its Hamiltonian is

$$\hat{H}_{cav} = \omega_{cav}(\hat{x})\hat{a}^\dagger \hat{a}, \tag{92}$$

where $\omega_{cav}(\hat{x})$ is the cavity mode frequency, which depends on the length of the cavity, and $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operator for the cavity mode. Creation and annihilation operators satisfy the usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1 = [\hat{b}, \hat{b}^\dagger]$.

The presence of the particle disturbs the cavity field and mimics a change in cavity length [6]. The cavity frequency depends on the length of the cavity and in this way also depends on the displacement of the particle $\hat{x}$. We can write the displacement dependence as

$$\omega_{cav}(\hat{x}) \approx \omega_{cav} + \hat{x}\frac{\partial \omega_{cav}}{\partial \hat{x}} + ..., \tag{93}$$

where we are only interested in linear terms. We define optical frequency shift per displacement $G = \frac{\partial \omega_{cav}}{\partial \hat{x}}$ and use it to write the cavity Hamiltonian as

$$\omega_{cav}(\hat{x})\hat{a}^\dagger\hat{a} \approx \omega_{cav}\hat{a}^\dagger\hat{a} - \hat{x}G\hat{a}^\dagger\hat{a} = \omega_{cav}\hat{a}^\dagger\hat{a} + g_0(\hat{b} + \hat{b}^\dagger)\hat{a}^\dagger\hat{a}, \tag{94}$$

where $\hat{x} = x_{ZPF}(\hat{b} + \hat{b}^\dagger)$ is the position operator for a particle in a harmonic trap, $x_{ZPF}$ is the zero point fluctuation amplitude, and $g_0 = Gx_{ZPF}$ is the single-photon optomechanical coupling strength. This gives us the free cavity mode Hamiltonian $\hat{H}_{cav,0} = \omega_{cav}\hat{a}^\dagger\hat{a}$ and the interaction Hamiltonian

$$\hat{H}_I = g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger). \tag{95}$$

Our full Hamiltonian is then

$$\hat{H} = \hat{H}_0 + \hat{H}_I = \omega_{cav}\hat{a}^\dagger\hat{a} + \omega_m\hat{b}^\dagger\hat{b} + g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger), \tag{96}$$

where we have denoted the full free Hamiltonian of the system $\hat{H}_0 = \hat{H}_{m,0} + \hat{H}_{cav,0}$.

We change into a frame rotating at the frequency of the driving laser $\omega_L$ with a unitary transformation $\hat{U} = e^{i\omega_L\hat{a}^\dagger\hat{a}t}$, where the new Hamiltonian is

$$\begin{aligned}\hat{H} =& \hat{U}(\hat{H}_0 + \hat{H}_I)\hat{U}^\dagger + i\frac{\partial \hat{U}}{\partial t}\\ =& \hat{H}_0 + \hat{H}_I - \omega_L\hat{a}^\dagger\hat{a}e^{i\omega_L\hat{a}^\dagger\hat{a}t}\\ =& -\delta\hat{a}^\dagger\hat{a} + \omega_m\hat{b}^\dagger\hat{b} + g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger),\end{aligned} \tag{97}$$

where $\delta = \omega_L - \omega_{cav}$ is the detuning of laser frequency from cavity frequency.

We divide $\hat{a}$ into classical part and quantum part $\hat{a} = \bar{\alpha} + \hat{a}'$, where $\bar{\alpha} = \langle \hat{a} \rangle$ is the average coherent amplitude and $\hat{a}'$ is a fluctuating term that going forward we will denote with simply $\hat{a}$. The interaction Hamiltonian is then

$$\hat{H}_I = g_0(\bar{\alpha} + \hat{a})^\dagger(\bar{\alpha} + \hat{a})(\hat{b} + \hat{b}^\dagger) \tag{98}$$

$$= g_0(|\alpha|^2 + \bar{\alpha}\hat{a}^\dagger + \bar{\alpha}^*\hat{a} + \hat{a}^\dagger\hat{a})(\hat{b} + \hat{b}^\dagger)$$

$$\approx g_0(\bar{\alpha}\hat{a}^\dagger + \bar{\alpha}^*\hat{a})(\hat{b} + \hat{b}^\dagger), \tag{99}$$

where in the last equation the first term is left out, because it implies an average radiation pressure $\bar{F} = G|\bar{\alpha}|^2$ that can be omitted by shifting displacement's origin by $\delta\bar{x} = \frac{\bar{F}}{m_{eff}\omega_m^2}$. The last term was left out because $\hat{a} \ll \bar{\alpha}$.

We assume $\bar{\alpha} = \sqrt{\bar{n}_{cav}} \in \mathbb{R}$ and define the optomechanical coupling strength $g = g_0\sqrt{\bar{n}_{cav}}$. Then the interaction Hamiltonian is

$$\hat{H}_I = g(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger). \tag{100}$$

The full cavity optomechanics Hamiltonian $H$ is

$$\hat{H} = -\delta\hat{a}^\dagger\hat{a} + \omega\hat{b}^\dagger\hat{b} + g(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger). \tag{101}$$

It describes the coherent coupling between cavity and mechanical modes [7].

We have two modes so we have two jump operators: $L_{cav} = \hat{a}$ the cavity annihilation operator, and $L_m = \hat{q} = \hat{b} + \hat{b}^\dagger$ the mechanical position operator for a particle in harmonic trap ($\hat{q}^\dagger = \hat{q},\ (\hat{q}^2)^\dagger = \hat{q}^2$). These give us the dissipation term for cavity

$$\mathcal{L}_{cav}[\rho] = \kappa(\hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}), \tag{102}$$

which describes the decay of cavity mode at rate $\kappa$ due to the presence of the particle and photon losses caused by mirror imperfections, and the diffusion term for mechanical mode

$$\mathcal{L}_m = \Gamma(\hat{q}\rho\hat{q} - \frac{1}{2}\{\hat{q}^2, \rho\}), \tag{103}$$

which generates decoherence of the mechanical motion at rate $\Gamma$ due to recoil heating caused by light scattering off the particle [7].

The full Lindblad master equation that describes our system is then

$$\dot{\rho} = i[\rho, \hat{H}] + \kappa(\hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}) + \Gamma(\hat{q}\rho\hat{q} - \frac{1}{2}\{\hat{q}^2, \rho\}). \tag{104}$$

# 4 Particle dynamics

## 4.1 Solving the stochastic differential equation

We have a master equation [30]

$$\dot{\rho} = i[\rho, \hat{H}] + \kappa(\hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}) + \Gamma(\hat{q}\rho\hat{q} - \frac{1}{2}\{\hat{q}^2, \rho\}), \tag{105}$$

where $H$ is given by equation 101, and we want to solve the dynamics of the system. We use quantum state diffusion to change the problem from a master equation for a density matrix to a stochastic differential equation for state vectors. The QSD equation in this case is

$$\frac{\partial}{\partial t}|\psi_t\rangle = -i\hat{H}|\psi_t\rangle + \xi_t^*\hat{a}|\psi_t\rangle - \frac{\kappa}{2}\hat{a}^\dagger\hat{a}|\psi_t\rangle + \eta_t^*\hat{q}|\psi_t\rangle - \frac{\Gamma}{2}\hat{q}^2|\psi_t\rangle, \tag{106}$$

where the noises $\xi_t^*$ and $\eta_t^*$ have properties

$$\langle\xi_t^*\rangle = 0,\ \langle\xi_t\xi_s\rangle = 0,\ \langle\eta_t^*\rangle = 0,\ \langle\eta_t\eta_s\rangle = 0,$$
$$\langle\xi_t\xi_s^*\rangle = \kappa\delta(t-s),\ \langle\eta_t\eta_s^*\rangle = \Gamma\delta(t-s). \tag{107}$$

We aim to solve $\rho$ that satisfies the master equation as the expectation value

$$\rho = \langle|\psi_t\rangle\langle\psi_t|\rangle. \tag{108}$$

We use the Stratonovich method to solve the stochastic differential equation, so we can use the product rule

$$\frac{\partial}{\partial t}|\psi_t\rangle\langle\psi_t| = |\dot{\psi_t}\rangle\langle\psi_t| + |\psi_t\rangle\langle\dot{\psi_t}|. \tag{109}$$

The derivative of state $\rho$ is then

$$\begin{aligned}\frac{\partial}{\partial t}\rho =&\frac{\partial}{\partial t}\langle|\psi_t\rangle\langle\psi_t|\rangle = \langle|\dot{\psi}_t\rangle\langle\psi_t| + |\psi_t\rangle\langle\dot{\psi}_t|\rangle \\ =&\langle -i\hat{H}|\psi_t\rangle\langle\psi_t| + \xi_t^*\hat{a}|\psi_t\rangle\langle\psi_t| - \frac{\kappa}{2}\hat{a}^\dagger\hat{a}|\psi_t\rangle\langle\psi_t| + \eta_t^*\hat{q}|\psi_t\rangle\langle\psi_t| - \frac{\Gamma}{2}\hat{q}^2|\psi_t\rangle\langle\psi_t| \\ &+ i|\psi_t\rangle\langle\psi_t|\hat{H}^\dagger + \xi_t|\psi_t\rangle\langle\psi_t|\hat{a}^\dagger - \frac{\kappa}{2}|\psi_t\rangle\langle\psi_t|\hat{a}^\dagger\hat{a} + \eta_t|\psi_t\rangle\langle\psi_t|\hat{q} - \frac{\Gamma}{2}|\psi_t\rangle\langle\psi_t|\hat{q}^2\rangle \\ =&-i[\hat{H},\rho] - \frac{\kappa}{2}\{\hat{a}^\dagger\hat{a},\rho\} + \hat{a}\langle\xi_t^*|\psi_t\rangle\langle\psi_t|\rangle + \langle\xi_t|\psi_t\rangle\langle\psi_t|\rangle\hat{a}^\dagger \\ &- \frac{\Gamma}{2}\{\hat{q}^2,\rho\} + \hat{q}\langle\eta_t^*|\psi_t\rangle\langle\psi_t|\rangle + \langle\eta_t|\psi_t\rangle\langle\psi_t|\rangle\hat{q}^\dagger. \end{aligned} \tag{110}$$

Comparing the terms in this equation with the terms in the master equation 105, we get the equation

$$\kappa\hat{a}\langle|\psi_t\rangle\langle\psi_t|\rangle\hat{a}^\dagger = \hat{a}\langle\xi_t^*|\psi_t\rangle\langle\psi_t|\rangle + \langle\xi_t|\psi_t\rangle\langle\psi_t|\rangle\hat{a}^\dagger, \tag{111}$$

which can be solved by using the Furutsu-Novikov theorem. We get the solution

$$\langle\xi_t^*|\psi_t\rangle\langle\psi_t|\rangle = \frac{\kappa}{2}\langle|\psi_t\rangle\langle\psi_t|\rangle\hat{a}^\dagger. \tag{112}$$

Similarly for the mechanical mode we get equation

$$\Gamma\hat{q}\langle|\psi_t\rangle\langle\psi_t|\rangle\hat{q}^\dagger = \hat{q}\langle\eta_t^*|\psi_t\rangle\langle\psi_t|\rangle + \langle\eta_t|\psi_t\rangle\langle\psi_t|\rangle\hat{q}^\dagger \tag{113}$$

with solution

$$\langle\eta_t^*|\psi_t\rangle\langle\psi_t|\rangle = \frac{\Gamma}{2}\langle\eta_t|\psi_t\rangle\langle\psi_t|\rangle\hat{q}^\dagger. \tag{114}$$

Next we assume that the state is of the form

$$|\psi_t\rangle = e^{\hat{X}}|0,0\rangle, \tag{115}$$

where $\hat{X}$ is an operator consisting of the annihilation operators of cavity mode and center of mass motion

$$\hat{X} = \lambda\hat{a}^\dagger + \mu\hat{b}^\dagger - \frac{1}{2}r(\hat{a}^\dagger)^2 - \frac{1}{2}s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger. \tag{116}$$

Taking a derivative of this state we get

$$\frac{\partial}{\partial t}|\psi_t\rangle = \frac{\dot{N}}{N}|\psi_t\rangle + \dot{\lambda}\hat{a}^\dagger|\psi_t\rangle + \dot{\mu}\hat{b}^\dagger|\psi_t\rangle - \frac{1}{2}\dot{r}(\hat{a}^\dagger)^2|\psi_t\rangle - \frac{1}{2}\dot{s}(\hat{b}^\dagger)^2|\psi_t\rangle - \dot{z}\hat{a}^\dagger\hat{b}^\dagger|\psi_t\rangle. \quad (117)$$

We want to calculate the stochastic differential equation 106 explicitly in terms of the parameters in our ansatz ($\lambda$, $\mu$, $r$, $s$, $z$, and $N$) and the creation operators in order to compare it to equation 117 and in that way find individual differential equations for each of the parameters. If we can solve the time dependency of all the parameters in our state, we then know how our state changes in time. Here we present only the main parts of the calculation. More detailed calculations can be found in appendix B.

To do this we need to first calculate how the creation operators $\hat{a}$ and $\hat{b}$ operate on the state $|\psi\rangle$. We define a new operator $\hat{a}(\theta)$ as

$$\hat{a}(\theta) = e^{-\theta\hat{X}}\hat{a}e^{\theta\hat{X}}. \quad (118)$$

Its derivative is

$$\frac{\partial}{\partial\theta}\hat{a}(\theta) == -e^{-\theta\hat{X}}[\hat{X},\hat{a}]e^{\theta\hat{X}}, \quad (119)$$

where the commutator is

$$[\hat{X},\hat{a}] = -\lambda + r\hat{a}^\dagger + z\hat{b}^\dagger. \quad (120)$$

We do the same for the mechanical mode annihilation operator $\hat{b}$, whose commutator is

$$[\hat{X},\hat{b}] = -\mu + s\hat{b}^\dagger + z\hat{a}^\dagger. \quad (121)$$

The creation operators commute with operator $\hat{X}$ and its exponential:

$$[\hat{X},\hat{a}^\dagger] = 0 = [e^{\hat{X}},\hat{a}^\dagger], \quad (122)$$

$$[\hat{X},\hat{b}^\dagger] = 0 = [e^{\hat{X}},\hat{b}^\dagger], \quad (123)$$

so the derivative 119 becomes

$$\frac{\partial}{\partial\theta}\hat{a}(\theta) = \lambda - r\hat{a}^\dagger - z\hat{b}^\dagger \tag{124}$$

By integrating this over $\theta$ we see that

$$\begin{aligned} \hat{a}(1) - \hat{a}(0) &= \int_0^1 d\theta \frac{\partial}{\partial\theta}\hat{a}(\theta) = \lambda - r\hat{a}^\dagger - z\hat{b}^\dagger \\ \rightarrow e^{-\hat{X}}\hat{a}e^{\hat{X}} &= \hat{a} + \lambda - r\hat{a}^\dagger - z\hat{b}^\dagger, \end{aligned} \tag{125}$$

and by operating with this on $|\psi_t\rangle$ we get the result of how the annihilation operator operates on the state:

$$\begin{aligned} e^{-\hat{X}}\hat{a}e^{\hat{X}}|0\rangle &= \hat{a}|0\rangle + (\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)|0\rangle \\ \hat{a}e^{\hat{X}}|0\rangle &= (\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)e^{\hat{X}}|0\rangle \\ \hat{a}|\psi_t\rangle &= (\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)|\psi_t\rangle. \end{aligned} \tag{126}$$

For operator $\hat{b}$ we get the corresponding solution

$$\hat{b}|\psi_t\rangle = (\mu - s\hat{b}^\dagger - z\hat{a}^\dagger)|\psi_t\rangle. \tag{127}$$

Now that we know how the annihilation operators $\hat{a}$ and $\hat{b}$ operate on the state, we can calculate how different quadratic combinations of creation and annihilation operators operate on it:

$$\begin{aligned} \hat{a}^\dagger\hat{a}|\psi_t\rangle =& (\lambda\hat{a}^\dagger - r(\hat{a}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \\ \hat{b}^\dagger\hat{b}|\psi_t\rangle =& (\mu\hat{b}^\dagger - s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \\ \hat{a}\hat{b}|\psi_t\rangle =& [\lambda\mu - z - (r\mu + z\lambda)\hat{a}^\dagger - (s\lambda + z\mu)\hat{b}^\dagger + rz(\hat{a}^\dagger)^2 + sz(\hat{b}^\dagger)^2 \\ &+ (sr + z^2)\hat{a}^\dagger\hat{b}^\dagger]|\psi_t\rangle, \\ \hat{a}\hat{b}^\dagger|\psi_t\rangle =& (\lambda\hat{b}^\dagger - z(\hat{b}^\dagger)^2 - r\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \\ \hat{a}^\dagger\hat{b}|\psi_t\rangle =& (\mu\hat{a}^\dagger - z(\hat{a}^\dagger)^2 - s\hat{a}^\dagger\hat{a}^\dagger)|\psi_t\rangle \\ \hat{b}^\dagger\hat{b}|\psi_t\rangle =& (\mu\hat{b}^\dagger - s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle \\ \hat{b}^2|\psi_t\rangle =& (\mu^2 - s - 2z\mu\hat{a}^\dagger - 2s\mu\hat{b}^\dagger + z^2(\hat{a}^\dagger)^2 + s^2(\hat{b}^\dagger)^2 + 2sz\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle. \end{aligned} \tag{128}$$

Using these, we can write the QSD equation 106 in terms of the ansatz state parameters:

$$
\begin{aligned}
\frac{\partial}{\partial t}|\psi_t\rangle =& -i\hat{H}|\psi_t\rangle + \xi_t^* \hat{a}|\psi_t\rangle - \frac{\kappa}{2}\hat{a}^\dagger\hat{a}|\psi_t\rangle + \eta_t^*\hat{q}|\psi_t\rangle - \frac{\Gamma}{2}\hat{q}^2|\psi_t\rangle \\
=&[\xi_t^*\lambda + \eta_t^* - ig(\lambda\mu - z) - \frac{1}{2}\Gamma(\mu^2 + 1 - s)]|\psi_t\rangle \\
&+ [i\delta\lambda - ig(-z\lambda + \mu(1-r)) - \xi_t^* r - \eta_t^* z - \frac{1}{2}\kappa\lambda + \Gamma z\mu]\hat{a}^\dagger|\psi_t\rangle \\
&+ [-i\omega\mu - ig(-z\mu + \lambda(1-s)) - \xi_t^* z - \eta_t^*(1-s) - \Gamma\mu(1-s)]\hat{b}^\dagger|\psi_t\rangle \\
&+ [-i\delta r + igz(1-r) + \frac{1}{2}\kappa r - \frac{1}{2}\Gamma z^2](\hat{a}^\dagger)^2|\psi_t\rangle \\
&+ [i\omega s + igz(1-s) - \frac{1}{2}\Gamma(1-s)^2](\hat{b}^\dagger)^2|\psi_t\rangle \\
&+ [iz(\omega - \delta) - ig(z^2 + rs - r - s) + \frac{1}{2}\kappa z + \Gamma z(1-s)]\hat{a}^\dagger\hat{b}^\dagger|\psi_t\rangle. \qquad (129)
\end{aligned}
$$

By comparing the terms with the same combination of annihilation operators in this equation and in equation 117 we get differential equations for each of the parameters in the ansatz state

$$
\begin{aligned}
\frac{\dot{N}}{N} =& \xi_t^*\lambda + \eta_t^* - ig(\lambda\mu - z) - \frac{1}{2}\Gamma(\mu^2 + 1 - s), && (130) \\
\dot{\lambda} =& (i\delta + igz - \frac{1}{2}\kappa)\lambda - ig\mu(1-r) - \xi_t^* r - \eta_t^* z + \Gamma z\mu, && (131) \\
\dot{\mu} =& (-i\omega + igz - \Gamma(1-s))\mu - ig\lambda(1-s) - \xi_t^* z - \eta_t^*(1-s), && (132) \\
\dot{r} =& (2i\delta + 2igz - \kappa)r - 2igz + \Gamma z^2, && (133) \\
\dot{s} =& -2i\omega s - 2igz(1-s) + \Gamma(1-s)^2, && (134) \\
\dot{z} =& (i\delta - i\omega - \frac{1}{2}\kappa - \Gamma(1-s))z + ig(z^2 + rs - r - s). && (135)
\end{aligned}
$$

These equations give us the time dependence for the parameters of the ansatz. This shows that the solution of the QSD equation is of the form $|\psi_t\rangle = N_t e^{\hat{X}_t}|0,0\rangle$.

## 4.2 Norm of the state

We want to be able to calculate the expectation values for creation and annihilation operators and their combinations. We can do this by taking a derivative of the norm of the state $\langle\psi_t|\psi_t\rangle = |N|^2\langle 0|e^{\hat{X}^\dagger}e^{\hat{X}}|0\rangle$ with respect to one of the parameters in operator $\hat{X} = \lambda\hat{a}^\dagger + \mu\hat{b}^\dagger - \frac{1}{2}r(\hat{a}^\dagger)^2 - \frac{1}{2}s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger$. For example, we get the expectation value for operators $\hat{a}^\dagger$ by taking a derivative of the norm with respect to the parameter $\lambda$

$$\frac{\partial}{\partial\lambda}\langle\psi_t|\psi_t\rangle = \frac{\partial}{\partial\lambda}|N|^2\langle 0|e^{\hat{X}^\dagger}e^{\hat{X}}|0\rangle = |N|^2\langle 0|e^{\hat{X}^\dagger}\hat{a}^\dagger e^{\hat{X}}|0\rangle = \langle\psi_t|\hat{a}^\dagger|\psi_t\rangle = \langle\hat{a}^\dagger\rangle. \quad (136)$$

This can naturally be expanded for expectation values for combinations of the operators. For example

$$\langle\hat{a}\hat{b}^\dagger\rangle = \frac{\partial^2}{\partial\lambda^*\partial\mu}\langle\psi_t|\psi_t\rangle. \quad (137)$$

To calculate the norm we first define new state vectors

$$||u\rangle = e^{u\hat{a}^\dagger}|0\rangle,\ ||v\rangle = e^{v\hat{b}^\dagger}|0\rangle,\ ||u,v\rangle = e^{u\hat{a}^\dagger + v\hat{b}^\dagger}|0\rangle, \quad (138)$$

that have the property

$$\int d^2u\frac{1}{\pi}e^{-|u|^2}\int d^2v\frac{1}{\pi}e^{-|v|^2}||u,v\rangle\langle u,v|| = \mathbb{I}. \quad (139)$$

Then we want to know how the creation and annihilation operators operate on the vector $\langle u,v||$. We find that

$$\frac{\partial}{\partial u^*}\langle u,v|| = \frac{\partial}{\partial u^*}(\langle 0|e^{u^*\hat{a}+v^*\hat{b}}) = \langle 0|e^{u^*\hat{a}+v^*\hat{b}}\hat{a} = \langle u,v||\hat{a}, \quad (140)$$

$$\begin{aligned}\langle u,v||\hat{a}^\dagger =& \langle 0|e^{u^*\hat{a}}\hat{a}^\dagger e^{v^*\hat{b}} = \langle 0|\sum_{k=0}^{\infty}\frac{(u^*\hat{a})^k}{k!}\hat{a}^\dagger e^{v^*\hat{b}}\\ =& \langle 0|(\hat{a}^\dagger + u^*(\hat{a}^\dagger\hat{a}+1) + \frac{(u^*)^2}{2}(\hat{a}^\dagger\hat{a}^2 + a) + \ldots)e^{v^*\hat{b}}\\ =& \langle 0|u^*\sum_{k=0}^{\infty}\frac{(u^*\hat{a})^k}{k!}\hat{a}^\dagger e^{v^*\hat{b}} = u^*\langle 0|e^{u^*\hat{a}+v^*\hat{b}} = u^*\langle u,v||, \quad (141)\end{aligned}$$

with corresponding equations for $\hat{b}$ and $v$:

$$\langle u,v||\hat{b} = \frac{\partial}{\partial v^*}\langle u,v||, \quad (142)$$

$$\langle u,v||\hat{a}^\dagger = u^*\langle u,v||. \quad (143)$$

By reorganising the terms in equations 126 and 127 we find that $|\psi_t\rangle$ is an eigenvector for operators $\hat{a}+r\hat{a}^\dagger+z\hat{b}^\dagger$ and $\hat{b}+s\hat{b}^\dagger+z\hat{a}^\dagger$ with

$$(\hat{a}+r\hat{a}^\dagger+z\hat{b}^\dagger)|\psi_t\rangle=\lambda|\psi_t\rangle, \tag{144}$$

$$(\hat{b}+s\hat{b}^\dagger+z\hat{a}^\dagger)|\psi_t\rangle=\mu|\psi_t\rangle. \tag{145}$$

Using these properties we get equations

$$\lambda\langle u,v||\psi_t\rangle=\langle u,v||(\hat{a}+r\hat{a}^\dagger+z\hat{b}^\dagger)|\psi_t\rangle=(\frac{\partial}{\partial u^*}+ru^*+zv^*)\langle u,v||\psi_t\rangle, \tag{146}$$

$$\mu\langle u,v||\psi_t\rangle=(\frac{\partial}{\partial v^*}+sv^*+zu^*)\langle u,v||\psi_t\rangle. \tag{147}$$

Let's define $f(u^*,v^*)=\langle u,v||\psi_t\rangle$ and assume that it is of form $f(u^*,v^*)=e^{g(u^*,v^*)}$. Then from equations 146 and 147 we get a pair of partial differential equations

$$\frac{\partial}{\partial u^*}g+ru^*+zv^*-\lambda=0, \tag{148}$$

$$\frac{\partial}{\partial v^*}g+sv^*+zu^*-\mu=0, \tag{149}$$

from which we can solve $g(u^*,v^*)$ and therefore $f(u^*,v^*)$:

$$g(u^*,v^*)=\lambda u^*+\mu v^*-\frac{1}{2}r(u^*)^2-\frac{1}{2}s(v^*)^2-zu^*v^*, \tag{150}$$

$$f(u^*,v^*)=e^{\lambda u^*+\mu v^*-\frac{1}{2}r(u^*)^2-\frac{1}{2}s(v^*)^2-zu^*v^*}, \tag{151}$$

and finally use this form to calculate

$$\begin{aligned}|\langle u,v||\psi_t\rangle|^2=&|f(u^*,v^*)|^2\\ =&\exp[\lambda u^*+\lambda^*u+\mu v^*+\mu^*v-\frac{1}{2}r(u^*)^2-\frac{1}{2}r^*u^2\\ &-\frac{1}{2}s(v^*)^2-\frac{1}{2}s^*v^2-zu^*v^*-z^*uv].\end{aligned} \tag{152}$$

Then we can write the norm of the state as

$$\begin{aligned}\langle\psi_t|\psi_t\rangle =& \int d^2u \int d^2v \frac{1}{\pi^2} e^{-|u|^2-|v|^2} \langle\psi_t||u,v\rangle\langle u,v||\psi\rangle \\ =& \int d^2u \int d^2v \frac{1}{\pi^2} e^{-|u|^2-|v|^2} |f(u^*,v^*)|^2 \\ =& \frac{1}{\pi^2} \int d^2v e^{-|v|^2+\mu v^*+\mu^* v-\frac{1}{2}s(v^*)^2-\frac{1}{2}s^*v^2} \\ & \int d^2u e^{-|u|^2+(\lambda-zv^*)u^*+(\lambda^*-z^*v)u-\frac{1}{2}r(u^*)^2-\frac{1}{2}r^*u^2}. \end{aligned} \tag{153}$$

Both the integral over $u$ and integral over $v$ are of the same form

$$\frac{1}{\pi} \int d^2u e^{-|u|^2-\frac{1}{2}b(u^*)^2-\frac{1}{2}b^*u^2+\beta u^*+\beta^* u}. \tag{154}$$

We solve it in this general form, so the solution can be easily applied to both integrals.

First we want to switch to a matrix notation. We define a vector

$$\boldsymbol{u} = \begin{pmatrix} u \\ u^* \end{pmatrix}, \tag{155}$$

and a matrx

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}, \tag{156}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}$. Then the product

$$\frac{1}{2}\boldsymbol{u}^\dagger \boldsymbol{A} \boldsymbol{u} = a|u|^2 + \frac{1}{2}b(u^*)^2 + \frac{1}{2}b^*u^2, \tag{157}$$

is almost the same form as the exponents in equation 153.

With the help of the eigenvalues of matrix $\boldsymbol{A}$, we aquire the diagonal form

$$\boldsymbol{A} = \boldsymbol{C}\boldsymbol{D}\boldsymbol{C}^\dagger,$$

where

$$\boldsymbol{D} = \begin{pmatrix} a-|b| & 0 \\ 0 & a+|b| \end{pmatrix} \text{ and } \boldsymbol{C} = \frac{1}{\sqrt{2}|b|} \begin{pmatrix} -b & b \\ |b| & |b| \end{pmatrix}.$$

To achieve the wanted form, we need to add terms that are linear in $u$. To do this we define another vector $\boldsymbol{\beta} = \begin{pmatrix} \beta \\ \beta^* \end{pmatrix}$ and use it to write the linear terms as a vector product

$$\beta^* u + \beta u^* = \boldsymbol{\beta}^\dagger \boldsymbol{u}. \tag{158}$$

The exponent in the norm 153 can then be written in matrix form as

$$\boldsymbol{Q} = -\frac{1}{2}\boldsymbol{u}^\dagger \boldsymbol{C}\boldsymbol{D}\boldsymbol{C}^\dagger \boldsymbol{u} + \boldsymbol{\beta}^\dagger \boldsymbol{u}. \tag{159}$$

Next we transform the vectors with $C^\dagger$

$$\tilde{\boldsymbol{u}} = \begin{pmatrix} \tilde{u} \\ \tilde{u}^* \end{pmatrix} = \boldsymbol{C}^\dagger \boldsymbol{u} \tag{160}$$

$$\tilde{\boldsymbol{\beta}} = \begin{pmatrix} \tilde{\beta} \\ \tilde{\beta}^* \end{pmatrix} = \boldsymbol{C}^\dagger \boldsymbol{\beta} \tag{161}$$

so the exponent is then

$$\boldsymbol{Q} = -\frac{1}{2}\tilde{\boldsymbol{u}}^\dagger \boldsymbol{D}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\beta}}^\dagger \tilde{\boldsymbol{u}}. \tag{162}$$

We can write it also in the form

$$\boldsymbol{Q} = -\frac{1}{2}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}})\boldsymbol{D}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}) + \frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}, \tag{163}$$

because

$$\begin{aligned} -\frac{1}{2}(\tilde{\boldsymbol{u}}^\dagger - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}})^\dagger \boldsymbol{D}(\tilde{\boldsymbol{u}}^\dagger - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}) &= -\frac{1}{2}\tilde{\boldsymbol{u}}^\dagger \boldsymbol{D}\tilde{\boldsymbol{u}} + \frac{1}{2}(\tilde{\boldsymbol{u}}^\dagger \tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{u}}\tilde{\boldsymbol{\beta}}^\dagger) - \frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}\tilde{\boldsymbol{\beta}} \\ &= -\frac{1}{2}\tilde{\boldsymbol{u}}^\dagger \boldsymbol{D}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\beta}}^\dagger \tilde{\boldsymbol{u}} - \frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}\tilde{\boldsymbol{\beta}}, \end{aligned} \tag{164}$$

and $Q$ remains unchanged due to the added term $\frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}$.

Now we can write the integral in matrix form

$$\frac{1}{\pi}\int d^2u e^{-|u|^2 - \frac{1}{2}b(u^*)^2 - \frac{1}{2}b^* u^2 + \beta u^* + \beta^* u} = \frac{1}{\pi}\int d^2u e^{-\frac{1}{2}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}})\boldsymbol{D}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}) + \frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}}. \tag{165}$$

To solve it, we change the integration parameter $d^2u$ to $d^2\tilde{u}$. We can write $u$ in terms of $\tilde{u}$ as

$$\boldsymbol{u} = \boldsymbol{C}\tilde{\boldsymbol{u}} = \frac{1}{\sqrt{2}|b|}\begin{pmatrix} b(\tilde{u}^* - \tilde{u}) \\ |b|(\tilde{u}^* + \tilde{u}) \end{pmatrix}, \tag{166}$$

and use it to calculate the Jacobian for the tranformation

$$J = \begin{vmatrix} \frac{\partial}{\partial \tilde{u}} u & \frac{\partial}{\partial \tilde{u}^*} u \\ \frac{\partial}{\partial \tilde{u}} u^* & \frac{\partial}{\partial \tilde{u}^*} u^* \end{vmatrix} = -\frac{b}{\sqrt{2}|b|}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{b}{\sqrt{2}|b|} = -\frac{b}{|b|}. \tag{167}$$

So $d^2u = -\frac{b}{|b|}d^2\tilde{u}$ and the integral 165 becomes

$$\begin{aligned} \frac{1}{\pi}\int d^2u e^{-\frac{1}{2}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}})\boldsymbol{D}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}) + \frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}} &= \frac{b}{\pi|b|}e^{\frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}}\int d^2\tilde{u} e^{-\frac{1}{2}(\tilde{\boldsymbol{u}} - D^{-1}\tilde{\boldsymbol{\beta}})\boldsymbol{D}(\tilde{\boldsymbol{u}} - \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}})} \\ &= \frac{b}{\pi|b|}e^{\frac{1}{2}\tilde{\boldsymbol{\beta}}^\dagger \boldsymbol{D}^{-1}\tilde{\boldsymbol{\beta}}}\frac{2\pi}{\sqrt{\det D}} \\ &= \frac{2b}{|b|}\frac{1}{\sqrt{a^2 - |b|^2}}e^{\frac{1}{2}\boldsymbol{\beta}^\dagger \boldsymbol{C}\boldsymbol{D}^{-1}\boldsymbol{C}^{dag}\boldsymbol{\beta}} \\ &= \frac{2b}{|b|}\frac{1}{\sqrt{a^2 - |b|^2}}e^{\frac{1}{a^2 - |b|^2}(a|\beta|^2 - \frac{1}{2}b(\beta^*)^2 - \frac{1}{2}b^*\beta^2)}, \end{aligned} \tag{168}$$

where

$$\begin{aligned} \frac{1}{2}\boldsymbol{\beta}^\dagger \boldsymbol{C}\boldsymbol{D}^{-1}\boldsymbol{C}^\dagger\boldsymbol{\beta} &= \frac{1}{2}\boldsymbol{\beta}^\dagger \boldsymbol{A}^{-1}\boldsymbol{\beta} = \frac{1}{2}\boldsymbol{\beta}^\dagger\frac{1}{a^2 - |b|^2}\begin{pmatrix} a & -b \\ -b^* & a \end{pmatrix}\boldsymbol{\beta} \\ &= \frac{1}{a^2 - |b|^2}(a|\beta|^2 - \frac{1}{2}b(\beta^*)^2 - \frac{1}{2}b^*\beta^2). \end{aligned} \tag{169}$$

Now we can solve the norm in full. In the integral over $u$ in equation 153 our parameters are $a = 1$, $b = r$, and $\beta = \lambda - zv^*$, so the solution for the integral is

$$\begin{aligned} &\frac{1}{\pi}\int d^2u e^{-|u|^2 + (\lambda - zv^*)u^* + (\lambda^* - z^*v)u - \frac{1}{2}r(u^*)^2 - \frac{1}{2}r^*u^2} \\ &= \frac{2r}{|r|\sqrt{1 - |r|^2}}\exp[\frac{1}{1 - |r|^2}(|\lambda - zv^*|^2 - \frac{1}{2}r(\lambda^* - z^*v)^2 - \frac{1}{2}r^*(\lambda - zv^*))] \\ &= \frac{2r}{|r|\sqrt{1 - |r|^2}}\exp[\frac{1}{1 - |r|^2}(|\lambda|^2 - \frac{1}{2}r(\lambda^*)^2 - \frac{1}{2}r^*\lambda^2 - |z|^2|v|^2 \\ &\quad + (r^*\lambda - \lambda^*)zv^* + (r\lambda^* - \lambda)z^*v - \frac{1}{2}r(z^*)^2v^2 - \frac{1}{2}r^*z^2(v^*)^2)], \end{aligned} \tag{170}$$

and putting this back into the full integral of the norm gives us

$$\begin{aligned}
\langle\psi_t|\psi_t\rangle =& \frac{2r}{|r|\sqrt{1-|r|^2}} e^{\frac{1}{1-|r|^2}(|\lambda|^2-\frac{1}{2}r(\lambda^*)^2-\frac{1}{2}r^*\lambda^2)} \\
&\times \frac{1}{\pi}\int d^v \exp[-(1+\frac{|z|^2}{1-|r|^2})|v|^2 + (\mu + \frac{r^*\lambda z - \lambda^* z}{1-|r|^2})v^* \\
&+ (\mu^* + \frac{r\lambda^* z^* - \lambda z^*}{1-|r|^2})v - \frac{1}{2}(s + \frac{r^* z^2}{1-|r|^2})(v^*)^2 - \frac{1}{2}(s^* + \frac{r(z^*)^2}{1-|r|^2})v^2] \\
=& \frac{2r}{|r|\sqrt{1-|r|^2}} \exp[\frac{1}{1-|r|^2}(|\lambda|^2 - \frac{1}{2}r(\lambda^*)^2 - \frac{1}{2}r^*\lambda^2)] \\
&\times \frac{2b}{|b|\sqrt{a^2-|b|^2}} \exp[\frac{1}{a^2-|b|^2}(a|\beta|^2 - \frac{1}{2}b(\beta^*)^2 - \frac{1}{2}b^*\beta^2)],
\end{aligned} \tag{171}$$

where the parameters are now $a = 1 + \frac{|z|^2}{1-|r|^2}$, $b = s + \frac{r^* z^2}{1-|r|^2}$, and $\beta = \mu + \frac{r^*\lambda z - \lambda^* z}{1-|r|^2}$.

In the special case where there is no interaction between the particle and the cavity $z = 0$ and the norm is

$$\langle\psi_t|\psi_t\rangle = \frac{2r}{|r|\sqrt{1-|r|^2}} e^{\frac{1}{1-|r|^2}(|\lambda|^2-\frac{1}{2}r(\lambda^*)^2-\frac{1}{2}r^*\lambda^2)} \frac{2s}{|s|\sqrt{1-|s|^2}} e^{\frac{1}{1-|s|^2}(|\mu|^2-\frac{1}{2}s(\mu^*)^2-\frac{1}{2}s^*\mu^2)}. \tag{172}$$

# 5 Conclusions and outlook

Using optical fields to trap and control particles has proven to be a useful tool in many research fields. It has so far been used among other things to trap individual cells, bacteria and viruses, and in gravitational detectors. As the technology is developed further, it may find use in even more wide-ranged applications, like in highly sensitive commercial acceleration sensors.

Quantum optomechanics is a promising tool quantum mechanics research. It can be used to create highly isolated, quantum entangled states and using feedback and cavity cooling particles can be cooled to their quantum ground states. In the future, it could be used to generate quantum many-particle systems and macroscopic quantum superpositions.

In this work we described optical forces and how they are used in optical trapping.

We saw how the optical force can be calculated from Maxwell stress tensor in general case, or by using the dipole approximation in the Rayleigh regime.

We went over the basics of open quantum system theory and derived the Lindblad master equation and the corresponding quantum state diffusion equation for open quantum systems.

Then we applied these concepts to develop and open quantum system model for optically trapped nanoparticles. We found the Lindblad master equation for our system and solved it using stochastic methods for an ansatz state. We also calculated the time dependent norm for the state.

We found a way to analytically solve the master equation. It gives us insight to the principles of why the system behaves as it does.

In this thesis we studied a basic model of a nanoparticle trapped in an optical cavity. There are many ways this model could be expanded upon in the future. For example we could add an external control field to control the dynamics of the system.

We excluded the polarization effects by using non-polarized light in our model. By considering a model with polarized light and comparing it with the basic model we can study how the polarization affects the system dynamics.

Our model had a particle trapped in an optical cavity, but we could also consider a case with no cavity, simply a particle levitating in an optical trap. Comparing these systems would show us how the presence of the cavity affects the dynamics of the particle.

Optical traps can be used to trap multiple particles at once, so we could study a system of two or more particles trapped in the field. This could give insight to the interactions between particles in the absence of external forces.

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# A zz-component of the average Maxwell stress tensor

We want to calculate the $zz$-component of the average Maxwell tensor for a plane wave. The electric field is now

$$\begin{aligned}\mathbf{E}(t) =&E_0\mathrm{Re}[(e^{ikz} + Re^{-ikz})e^{-i\omega t}]\mathbf{n}_x \\ =&E_0[\cos(kz-\omega t) + R_r\cos(kz+\omega t) + R_i\sin(kz+\omega t)]\mathbf{n}_x,\end{aligned} \tag{173}$$

where $E_0$ is the amplitude, k is the wave number, and $\omega$ is the angular frequency of the electric field, and $R = R_r + iR_i$ is a complex reflection coefficient, and the magnetic field is

$$\begin{aligned}\mathbf{H}(t) =&\sqrt{\frac{\epsilon_0}{\mu_0}}E_0\mathrm{Re}[(e^{ikz} - Re^{-ikz})e^{-i\omega t}]\mathbf{n}_y \\ =&\sqrt{\frac{\epsilon_0}{\mu_0}}E_0[\cos(kz-\omega t) - R_r\cos(kz+\omega t) - R_i\sin(kz+\omega t)]\mathbf{n}_y.\end{aligned} \tag{174}$$

The $zz$-component of the average Maxwell stress tensor is then

$$\begin{aligned}\langle T_{zz}\rangle =& -\frac{1}{2}\langle\epsilon_0 E^2 + \mu_0 H^2\rangle \\ =& -\frac{1}{2}\langle\epsilon_0 E_0^2[\cos(kz-\omega t) + R_r\cos(kz+\omega t) + R_i\sin(kz+\omega t)]^2 \\ &+ \mu_0\frac{\epsilon_0}{\mu_0}E_0^2[\cos(kz-\omega t) - R_r\cos(kz+\omega t) - R_i\sin(kz+\omega t)]^2\rangle \\ =& -\frac{1}{2}\langle\epsilon_0 E_0^2[\cos^2(kz-\omega t) + R_r^2\cos^2(kz+\omega t) + R_i^2\sin^2(kz+\omega t) \\ &+ 2R_r\cos(kz-\omega t)\cos(kz+\omega t) + 2R_i\cos(kz-\omega t)\sin(kz+\omega t) \\ &+ 2R_rR_i\cos(kz+\omega t)\sin(kz+\omega t) \\ &+ \cos^2(kz-\omega t) + R_r^2\cos^2(kz+\omega t) + R_i^2\sin^2(kz+\omega t)] \\ &- 2R_r\cos(kz-\omega t)\cos(kz+\omega t) - 2R_i\cos(kz-\omega t)\sin(kz+\omega t) \\ &+ 2R_rR_i\cos(kz+\omega t)\sin(kz+\omega t)\rangle \\ =& -\frac{1}{2}\langle\epsilon_0 E_0^2 2[\cos^2(kz-\omega t) + R_r^2\cos^2(kz+\omega t) + R_i^2\sin^2(kz+\omega t) \\ &+ 2R_rR_i\cos(kz+\omega t)\sin(kz+\omega t)]\rangle \\ =& -\epsilon_0 E_0^2\frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}}[\cos^2(kz-\omega t) + R_r^2\cos^2(kz+\omega t) + R_i^2\sin^2(kz+\omega t) \\ &+ 2R_rR_i\cos(kz+\omega t)\sin(kz+\omega t)]\mathrm{d}t \\ =& -\epsilon_0 E_0^2\frac{\omega}{\pi}\frac{\pi}{2\omega}(1 + R_r^2 + R_i^2) \\ =& -\frac{\epsilon_0}{2}E_0^2(1+|R|^2) \\ =& -\frac{I_0}{c}(1+|R|^2),\end{aligned} \tag{175}$$

where $I_0 = \frac{\epsilon_0}{2}cE_0^2$ is the intensity of the plane wave.

# B Derivation of the equations of motion for the ansatz state

The QSD equation for our system is

$$\frac{\partial}{\partial t}|\psi_t\rangle = -i\hat{H}|\psi_t\rangle + \xi_t^*\hat{a}|\psi_t\rangle - \frac{\kappa}{2}\hat{a}^\dagger\hat{a}|\psi_t\rangle + \eta_t^*\hat{q}|\psi_t\rangle - \frac{\Gamma}{2}\hat{q}^2|\psi_t\rangle, \tag{176}$$

where the noises $\xi_t^*$ and $\eta_t^*$ have properties

$$\begin{aligned}\langle\xi_t^*\rangle = 0,\ \langle\xi_t\xi_s\rangle = 0,\ \langle\eta_t^*\rangle = 0,\ \langle\eta_t\eta_s\rangle = 0,\\ \langle\xi_t\xi_s^*\rangle = \kappa\delta(t-s),\ \langle\eta_t\eta_s^*\rangle = \Gamma\delta(t-s).\end{aligned} \tag{177}$$

We try to solve it using an ansatz state

$$|\psi_t\rangle = e^{\hat{X}}|0,0\rangle, \tag{178}$$

where $\hat{X}$ is

$$\hat{X} = \lambda\hat{a}^\dagger + \mu\hat{b}^\dagger - \frac{1}{2}r(\hat{a}^\dagger)^2 - \frac{1}{2}s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger. \tag{179}$$

The time derivative of this state is

$$\frac{\partial}{\partial t}|\psi_t\rangle = \frac{\dot{N}}{N}|\psi_t\rangle + \dot{\lambda}\hat{a}^\dagger|\psi_t\rangle + \dot{\mu}\hat{b}^\dagger|\psi_t\rangle - \frac{1}{2}\dot{r}(\hat{a}^\dagger)^2|\psi_t\rangle - \frac{1}{2}\dot{s}(\hat{b}^\dagger)^2|\psi_t\rangle - \dot{z}\hat{a}^\dagger\hat{b}^\dagger|\psi_t\rangle. \tag{180}$$

We define a new operator:

$$\hat{a}(\theta) = e^{-\theta\hat{X}}\hat{a}e^{\theta\hat{X}} \tag{181}$$

and take the time derivative of it:

$$\frac{\partial}{\partial\theta}\hat{a}(\theta) = -e^{-\theta\hat{X}}\hat{X}\hat{a}e^{\theta\hat{X}} + e^{-\theta\hat{X}}\hat{a}\hat{X}e^{\theta\hat{X}} = -e^{-\theta\hat{X}}[\hat{X},\hat{a}]e^{\theta\hat{X}}. \tag{182}$$

The commutator $[\hat{X},\hat{a}]$ is easily calculated:

$$\begin{aligned}[\hat{X},\hat{a}] &= \lambda[\hat{a}^\dagger,\hat{a}] - \frac{1}{2}r[(\hat{a}^\dagger)^2,\hat{a}] - z\hat{b}^\dagger[\hat{a}^\dagger,\hat{a}]\\ &= -\lambda - \frac{1}{2}r(\hat{a}^\dagger\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{a}^\dagger)\\ &= -\lambda - \frac{1}{2}r(\hat{a}^\dagger[\hat{a}^\dagger,\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a}^\dagger) + z\hat{b}^\dagger\\ &= -\lambda + r\hat{a}^\dagger + z\hat{b}^\dagger,\end{aligned} \tag{183}$$

and we can clearly see that the other commutators are

$$[\hat{X},\hat{b}] = -\mu + s\hat{b}^\dagger + z\hat{a}^\dagger \tag{184}$$

$$[\hat{X},\hat{a}^\dagger] = 0 = [e^{\hat{X}},\hat{a}^\dagger] \tag{185}$$

$$[\hat{X},\hat{b}^\dagger] = 0 = [e^{\hat{X}},\hat{b}^\dagger]. \tag{186}$$

So the derivative is then

$$\frac{\partial}{\partial\theta}\hat{a}(\theta) = \lambda e^{-\theta\hat{X}}e^{\theta\hat{X}} - re^{-\theta\hat{X}}\hat{a}^\dagger e^{\theta\hat{X}} - ze^{-\theta\hat{X}}\hat{b}^\dagger e^{\theta\hat{X}} = \lambda - r\hat{a}^\dagger - z\hat{b}^\dagger \tag{187}$$

By integrating the derivative over $\theta$ we see that

$$\begin{aligned} \hat{a}(1) - \hat{a}(0) &= \int_0^1 d\theta \frac{\partial}{\partial\theta}\hat{a}(\theta) = \int_0^1 d\theta(\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger) = \lambda - r\hat{a}^\dagger - z\hat{b}^\dagger \\ &\to e^{-\hat{X}}\hat{a}e^{\hat{X}} = \hat{a} + \lambda - r\hat{a}^\dagger - z\hat{b}^\dagger. \end{aligned} \tag{188}$$

Then we operate on $|\psi_t\rangle$ on both sides and get the result

$$\begin{aligned} e^{-\hat{X}}\hat{a}e^{\hat{X}}|0\rangle &= \hat{a}|0\rangle + (\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)|0\rangle \\ \hat{a}e^{\hat{X}}|0\rangle &= (\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)e^{\hat{X}}|0\rangle \\ \hat{a}|\psi_t\rangle &= (\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)|\psi_t\rangle. \end{aligned} \tag{189}$$

For operator $\hat{b}$ we get the corresponding solution

$$\hat{b}|\psi_t\rangle = (\mu - s\hat{b}^\dagger - z\hat{a}^\dagger)|\psi_t\rangle, \tag{190}$$

and using these solutions we can calculate how different quadratic combinations of creation and annihilation operators operate on the state $|\psi_t\rangle$:

$$\begin{aligned} \hat{a}^\dagger\hat{a}|\psi_t\rangle =& (\lambda\hat{a}^\dagger - r(\hat{a}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \\ \hat{b}^\dagger\hat{b}|\psi_t\rangle =& (\mu\hat{b}^\dagger - s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \\ \hat{a}\hat{b}|\psi_t\rangle =& \hat{a}(\mu - s\hat{b}^\dagger - z\hat{a}^\dagger)|\psi_t\rangle \\ =& [(\mu - s\hat{b}^\dagger)(\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger) - z(1 + \hat{a}^\dagger(\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger))]|\psi_t\rangle \\ =& [\lambda\mu - z - (r\mu + z\lambda)\hat{a}^\dagger - (s\lambda + z\mu)\hat{b}^\dagger + rz(\hat{a}^\dagger)^2 + sz(\hat{b}^\dagger)^2 \\ & + (sr + z^2)\hat{a}^\dagger\hat{b}^\dagger]|\psi_t\rangle, \\ \hat{a}\hat{b}^\dagger|\psi_t\rangle =& (\lambda\hat{b}^\dagger - z(\hat{b}^\dagger)^2 - r\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \\ \hat{a}^\dagger\hat{b}|\psi_t\rangle =& (\mu\hat{a}^\dagger - z(\hat{a}^\dagger)^2 - s\hat{a}^\dagger\hat{a}^\dagger)|\psi_t\rangle \\ \hat{b}^\dagger\hat{b}|\psi_t\rangle =& (\mu\hat{b}^\dagger - s(\hat{b}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle \\ \hat{b}^2|\psi_t\rangle =& \hat{b}(\mu - s\hat{b}^\dagger - z\hat{a}^\dagger)|\psi_t\rangle = [(\mu - s\dagger b - z\hat{a})(\mu - s\hat{b}^\dagger - z\hat{a}^\dagger) - s]|\psi_t\rangle \\ =& (\mu^2 - s - 2z\mu\hat{a}^\dagger - 2s\mu\hat{b}^\dagger + z^2(\hat{a}^\dagger)^2 + s^2(\hat{b}^\dagger)^2 + 2sz\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle. \end{aligned} \tag{191}$$

The first term in the stochastic differential equation 176 is

$$\begin{aligned} -i\hat{H}|\psi_t\rangle =& -i(-\delta\hat{a}^\dagger\hat{a} + \omega_b\hat{}^\dagger\hat{b} + g(\hat{a}\hat{b} + \hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b} + \hat{a}^\dagger\hat{b}^\dagger))|\psi_t\rangle, \\ =& -i[g(\lambda\mu - z) - (\delta\lambda + g(r\mu + z\lambda - \mu))\hat{a}^\dagger + (\omega\mu - g(s\lambda + z\mu - \lambda))\hat{b}^\dagger \\ & + (r\delta + gz(r-1))(\hat{a}^\dagger)^2 + (-s\omega + gz(s-1))(\hat{b}^\dagger)^2 \\ & + (z(\delta - \omega) + g(z^2 + rs - r - s))\hat{a}^\dagger\hat{b}^\dagger]|\psi_t\rangle, \end{aligned} \tag{192}$$

the second, third and fourth terms are

$$\xi_t^* \hat{a}|\psi_t\rangle = \xi_t^*(\lambda - r\hat{a}^\dagger - z\hat{b}^\dagger)|\psi_t\rangle, \tag{193}$$

$$-\frac{\kappa}{2}\hat{a}^\dagger\hat{a}|\psi_t\rangle = -\frac{\kappa}{2}(\lambda\hat{a}^\dagger - r(\hat{a}^\dagger)^2 - z\hat{a}^\dagger\hat{b}^\dagger)|\psi_t\rangle, \tag{194}$$

$$\eta_t^*\hat{q}|\psi_t\rangle = \eta_t^*(\mu + (1-s)\hat{b}^\dagger - z\hat{a}^\dagger)|\psi_t\rangle, \tag{195}$$

and the last term is

$$\begin{aligned}
-\frac{1}{2}\Gamma\hat{q}^2|\psi_t\rangle &= -\frac{1}{2}\Gamma(\hat{b}^2 + \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} + (\hat{b}^\dagger)^2)|\psi_t\rangle = -\frac{1}{2}\Gamma(\hat{b}^2 + 1 + 2\hat{b}^\dagger\hat{b} + (\hat{b}^\dagger)^2)|\psi_t\rangle \\
&= -\frac{1}{2}\Gamma[\mu^2 + 1 - s - 2z\mu\hat{a}^\dagger - 2\mu(1-s)\hat{b}^\dagger + z^2(\hat{a}^\dagger)^2 \\
&\quad + (1 - 2s + s^2)(\hat{b}^\dagger)^2 + 2z(s-1)\hat{a}^\dagger\hat{b}^\dagger]|\psi_t\rangle. 
\end{aligned} \tag{196}$$

Combining these we get the full differential equation for the state

$$\begin{aligned}
\frac{\partial}{\partial t}|\psi_t\rangle &= -i\hat{H}|\psi_t\rangle + \xi_t^*\hat{a}|\psi_t\rangle - \frac{\kappa}{2}\hat{a}^\dagger\hat{a}|\psi_t\rangle + \eta_t^*\hat{q}|\psi_t\rangle - \frac{\Gamma}{2}\hat{q}^2|\psi_t\rangle \\
&= [\xi_t^*\lambda + \eta_t^* - ig(\lambda\mu - z) - \frac{1}{2}\Gamma(\mu^2 + 1 - s)]|\psi_t\rangle \\
&\quad + [i\delta\lambda - ig(-z\lambda + \mu(1-r)) - \xi_t^* r - \eta_t^* z - \frac{1}{2}\kappa\lambda + \Gamma z\mu]\hat{a}^\dagger|\psi_t\rangle \\
&\quad + [-i\omega\mu - ig(-z\mu + \lambda(1-s)) - \xi_t^* z - \eta_t^*(1-s) - \Gamma\mu(1-s)]\hat{b}^\dagger|\psi_t\rangle \\
&\quad + [-i\delta r + igz(1-r) + \frac{1}{2}\kappa r - \frac{1}{2}\Gamma z^2](\hat{a}^\dagger)^2|\psi_t\rangle \\
&\quad + [i\omega s + igz(1-s) - \frac{1}{2}\Gamma(1-s)^2](\hat{b}^\dagger)^2|\psi_t\rangle \\
&\quad + [iz(\omega - \delta) - ig(z^2 + rs - r - s) + \frac{1}{2}\kappa z + \Gamma z(1-s)]\hat{a}^\dagger\hat{b}^\dagger|\psi_t\rangle.
\end{aligned} \tag{197}$$