TURUN YLIOPISTO
UNIVERSITY OF TURKU

TOPICS IN ANALYTIC NUMBER THEORY AND ADDITIVE COMBINATORICS

Yu-Chen Sun

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Department of Mathematics and Statistics
Mathematics
Doctoral programme in Exact Sciences (EXACTUS)

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ABSTRACT

This thesis comprises four articles in multiplicative and additive number theory, two subfields of analytic number theory, concerning e.g. the distribution of primes, multiplicative structures and additive structures.

In the first article, we consider a combination of two breakthroughs on prime gaps (small prime gaps and large prime gaps), and improve on a previous result given by Pintz. We also apply a similar strategy to improve on previous works on lower bounds for the least prime in an arithmetic progression. The proofs rely on a variant of the Maynard-Tao theorem and arguments used in proving long prime gaps.

In the second article, we study a lower bound for the L_1 norm of the exponential sum of the Möbius function over short intervals. This result extends the long interval version given by Balog and Ruzsa. The proofs are based on the Balog-Ruzsa structure and an improvement for a key lemma. In the improvement we use two different techniques — complex analysis and van der Corput's method.

In the third article, we study Vinogradov's three primes theorem with Piatetski-Shapiro primes. Our result significantly improves the existing results via applying the transference principle and Harman's sieve method. Besides, we improve on a Roth-type result for Piatetski-Shapiro primes given by Merik.

In the fourth article, we study d_k bounded multiplicative functions in almost all short intervals. Our results generalize the breakthrough given by Matomäki and Radziwiłł and improve on Mangerel's result. The proofs depend on the Matomäki-Radziwiłł method and introducing restrictions on prime factors.

KEYWORDS: prime gaps, the least prime in an arithmetic progression, van der Corput's method, Möbius function, Vinogradov's theorem, Piatetski-Shapiro primes, the circle method, the transference principle, Harman's sieve method, divisor function, short intervals, the Matomäki-Radziwiłł method, Dirichlet polynomials

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TIIVISTELMÄ

Tämä väitöskirja koostuu neljästä artikkelista, jotka koskevat multiplikatiivista ja additiivista lukuteoriaa, kahta analyyttisen lukuteorian osa-aluetta, jotka liittyvät esimerkiksi alkulukujen jakautumiseen sekä multiplikatiivisiin ja additiivisiin rakenteisiin.

Ensimmäisessä artikkelissa tarkastelemme alkulukujen lyhyisiin ja pitkiin etäisyyksiin liittyvien läpimurtojen yhdistämistä parantaen Pintzin aiempaa tulosta. Sovellamme samankaltaista strategiaa myös parantaaksemme aiempia tuloksia alarajalle pienimmästä alkuluvusta aritmeettisessa jonossa. Todistukset perustuvat Maynard-Tao -lauseen muunnelmaan ja alkulukujen pitkien välien tulosten todistuksiin.

Toisessa artikkelissa tutkimme alarajaa Möbiuksen funktion eksponenttisumman L_1-normille lyhyillä väleillä. Tämä tulos yleistää Balogin ja Ruzsan todistaman pitkien välien version. Todistukset perustuvat Balog-Ruzsan todistuksen rakenteeseen ja avainlemman parannukseen. Parannuksessa käytetään kahta eri tekniikkaa — kompleksianalyysiä ja van der Corputin menetelmää.

Kolmannessa artikkelissa tutkimme Vinogradovin kolmen alkuluvun lausetta Piatetski-Shapiron alkuluvuille. Tuloksemme parantaa merkittävästi olemassa olevia tuloksia soveltamalla transferenssiperiaatetta ja Harmanin seulamenetelmää. Lisäksi parannamme Merikin todistamaa Roth-tyyppistä tulosta Piatetski-Shapiron alkuluvuille.

Neljännessä artikkelissa tutkimme d_k -rajoitettuja multiplikatiivisia funktioita melkein kaikilla lyhyillä väleillä. Tuloksemme yleistävät Matomäen ja Radziwiłłin läpimurtoa ja parantavat Mangerelin tulosta. Todistukset hyödyntävät Matomäki-Radziwiłł -menetelmää ja alkutekijöiden lukumäärän rajoittamista.

ASIASANAT: alkulukujen etäisyydet, pienin alkuluku aritmeettisessa jonossa, van der Corputin menetelmä, Möbiuksen funktio, Vinogradovin lause, Piatetski-Shapiron alkuluvut, ympyrämenetelmä, transferenssiperiaate, Harmanin seulamenetelmä, tekijäfunktio, lyhyet välit, Matomäki-Radziwiłł -menetelmä, Dirichlet'n polynomit

Preface

Number theory, a field both simple and complex, has captivated many mathematicians. I feel immensely happy to be among those dedicated to number theory research. It is called simple because many of its problems are stated very straightforwardly, such as Goldbach's Conjecture: "Is every even number greater than or equal to 4 the sum of two prime numbers?" and the Twin Prime Conjecture: "Are there infinitely many pairs of prime numbers that differ by 2?" However, it is also called complex because solving these problems can be extremely difficult, requiring profound mathematical insights and tools, and sometimes they may remain unsolved. For instance, both of these conjectures are yet to be proven.

Five years ago, I was anxious about finding a doctoral position because my undergraduate and master's degrees were not in mathematics, making it challenging to find a Ph.D. position in mathematics/number theory. Fortunately, during my master's studies, I conducted some research in number theory, which served as the only proof of my mathematical abilities.

An academically outstanding advisor is likely to produce good students. Contacting Professor Kaisa Matomäki to be my advisor was a suggestion from my collaborator, Professor Hao Pan. I remember Hao Pan telling me, "Matomäki is very strong, and I believe she will become even stronger." Although I was not very familiar with multiplicative number theory (one Matomaki's research fields) at the time, I trusted Professor Pan's words, as he knew me well.

My supervisor, Professor Kaisa Matomäki, is a master of analytic number theory and one of the most mathematically talented individuals I have ever encountered. When we first started communicating via email, I always addressed her as Professor Matomäki. Later, she told me we could be more informal, so I began addressing her as Kaisa. After the COVID, we would meet weekly to discuss mathematics, sometimes for just a few minutes, sometimes for an hour. I was amazed by Kaisa's mathematical abilities, as even short meetings often provided me with profound insights. Kaisa quickly grasped what I was saying and pointed out the critical points. She is also an excellent teacher, capable of explaining profound mathematical theorems in the simplest language. One of the most memorable moments was when Kaisa explained the key parts of the transference principle to me in just over ten minutes.

During my Ph.D. studies, the pandemic was the most challenging period. For the first year and a half, my studies were overshadowed by the pandemic. In September

2020, I left China for Finland for the first time, feeling both worried about adapting to life and eager to discuss mathematics with many excellent mathematicians. Things turned out to be more challenging than expected. At the beginning of my Ph.D., I hadn't done mathematical research for a long time, had little knowledge of analytic number theory techniques, and nearly everyone was working remotely from home. Coupled with the Finnish winter, my emotions became very tense, and my academic progress was slow. After eight months of rest in China, I returned to Finland to continue my unfinished doctoral journey. During my time back in China, Professor Lilu Zhao invited me to Shandong University for a long-term visit, where we discussed mathematics, played billiards, and had an unforgettable time. During the pandemic, Kaisa and I kept in touch almost weekly through emails, discussing topics like the circle method, sieve methods, exponential sum estimates, and mean value theorems. Under her guidance, I completed a piece of work.

Upon returning to Finland, I worked my hardest until I finished my fourth paper. I found that I could quickly learn advanced mathematical techniques and understand deep mathematical ideas. These techniques and ideas have now become part of my toolkit. Reflecting on the time when I completed my first paper during my Ph.D., my English writing was very poor, and sometimes my English expression and sentence structure were chaotic. Kaisa patiently helped me revise it at least six times until it reached a submission-worthy version. For the subsequent papers, I paid more attention to detail each time. Although I couldn't achieve perfection, I tried to do better within my abilities.

In the spring of 2023, China's pandemic policies began to ease, making it easier to return home. After attending conferences in Bristol and Oxford in the summer of 2023, I stayed in China for two months. Returning to my homeland after two years was exhilarating. Many colleagues enthusiastically invited me for academic visits. After returning to Europe, I attended a conference in Bonn, Germany, and then busily prepared for postdoctoral applications. Writing materials, submitting resumes, and waiting for interview opportunities took a long time, and the interviews were nerve-wracking. Fortunately, in January 2024, I received a fantastic offer from Professor Ben Krause at the University of Bristol, and I hope my future academic journey will also be smooth.

In the following acknowledgment, I want to thank all my teachers, friends, and family who helped me during my Ph.D. studies.

First, I want to express my special thanks to my supervisor, Professor Kaisa Matomäki, for her patient and meticulous guidance in mathematics and paper writing during my Ph.D. studies, and for providing many valuable suggestions. I am grateful for her funding support, which allowed me to attend international conferences and academic visits, listen many wonderful talks, and meet many academic peers. Her concern for my well-being made me feel warm even in a foreign land.

I deeply appreciate the reviewers of my doctoral dissertation, Alexander Mangerel and Aled Walker, for carefully reading my thesis and providing excellent evaluations and useful suggestions. I also want to thank Oleksiy Klurman for agreeing to be my opponent.

I am sincerely thankful to Edufi, UTUGS Turku University Graduate School, and the Academy of Finland project no. 333707 for funding my Ph.D. studies.

Special thanks go to my colleagues in the number theory group at the University of Turku: Sebastian Zuniga Alterman, Martin Čech, Jesse Jääsaari, Sarvagya Jain, Olli Järviniemi, Mikko Jaskari, Stelios Sachpazis, Joni Teräväinen, and Mengdi Wang. I cherish the times we spent dining together, discussing mathematics, and having fun.

Additionally, I am grateful to the colleagues who invited me for talks or visits, including Wang Chen at Nanjing Forestry University, Lixia Dai at Nanjing Normal University, Bingrong Huang and Yongxiao Lin at Shandong University, Max Wenqiang Xu at Stanford University, Zikang Dong and Guangliang Zhou at Tongji University, Zhenyu Guo and Ping Xi at Xi'an Jiaotong University, Yuchen Ding at Yangzhou University as well as Jie Ma, Tuan Tran, and Lilu Zhao at the University of Science and Technology of China. Thank you for giving me the opportunity to present my work. Furthermore, I am thankful to Andrew Granville for discussions on mathematics and guidance.

I am deeply grateful to my parents for their upbringing and constant encouragement and support. They have always provided me with a loving and harmonious family. My mother has great foresight, and my father has a tenacious will; they are my role models. I also want to thank my family for their continued care and support.

Lastly, I want to thank all my friends who have accompanied me along the way. Special thanks to my beloved Minqin Zhang, who is like a treasure in my life. You have always been by my side, warming and supporting me through difficult times. You are like a little sun, brightening my world.

May 27, 2024
Yu-Chen Sun

前言

数论，这门既简单又复杂的学科，让无数数学家为之着迷。能够成为致力于数论研究的一员，我感到无比高兴。说它简单，是因为许多问题的陈述非常简洁，例如哥德巴赫猜想："是否任何一个大于等于4的偶数都可以写成两个质数的和？"以及孪生素数猜想："是否存在无穷多对差为2的质数？"但说它复杂，是因为解决这些问题极其困难，常常需要深刻的数学思想和工具，有时甚至可能无法解决。例如，这两个著名猜想至今仍未被证明。

五年前，我为博士职位感到焦虑，因为我的本科和硕士都不是数学专业，找一个数学/数论的博士岗位可能并不容易。幸运的是，在硕士期间，我进行了一些数论方面的研究，这成为我数学能力的唯一证明。

一个学术能力出众的导师，往往能培养出优秀的学生。联系Kaisa Matomäki教授做我的导师，是当时的合作者潘颢教授建议的。我记得潘颢教授对我说："Matomäki非常强，我相信她会变得更强。"尽管当时我对乘性数论（Matomäki的主要研究方向之一）知之甚少，但我相信潘颢教授的判断，他非常了解我。

我的导师Kaisa Matomäki教授是解析数论的专家，是我见过的最有数学天赋的人之一。在我们刚开始通过邮件沟通时，我一直称呼她为Matomäki教授。后来她告诉我可以随意一点，于是我开始称呼她为Kaisa。疫情后，我们每周见面讨论数学，有时十几分钟，有时一个小时。Kaisa的数学能力令我惊叹，哪怕是短短十几分钟的讨论，也能让我受益匪浅。她总是能迅速理解我所表达的内容，并指出关键问题。Kaisa也是一位优秀的老师，能够用最简单的语言解释深奥的数学定理。令我印象深刻的一次是，Kaisa仅用了十几分钟就向我讲解了转换原理的关键部分。

读博期间，疫情是最艰难的时期。在我博士的前一年半，几乎全程都笼罩在疫情阴影下。2020年9月，我第一次离开中国，前往芬兰留学，心情复杂，既担心无法适应新生活，又期待与许多优秀的数学家交流。事情比预想中困难得多，入学初期，由于很久没有从事数学研究，对解析数论的技术了解甚少，加之几乎所有人都在家远程工作，又赶上芬兰的冬天，让我的情绪变得非常紧张，学术进展甚微。回国休养了8个月后，我重新返回芬兰继续博士之旅。回国期间，赵立璐教授邀请我去山东大学进行几个月的长期访问，我们一起讨论数学、打桌球，度过了一段难忘的时光。疫情期间，我和Kaisa几乎每周通过邮件沟通，讨论圆法、筛法、指数和估计、均值定理等内容，并在她的指导下完成了一项工作。

回到芬兰后，直到我完成第四篇论文的那段时间，是我最努力的时候。我

发现自己能够迅速掌握一些高级的数学技术，理解深刻的数学思想。这些技术和思想现在已经成为我工具包的一部分。回想起博士期间完成第一篇论文时，我的英文写作水平很差，表达混乱，结构不清。Kaisa耐心地帮我修改了至少六遍，才使我的论文达到投稿标准。之后的几篇论文，每次写作时，我都比上次更仔细一点，虽然无法做到完美，但力求在能力范围内有所提升。

2023年春天，中国的疫情政策开始放松，回国变得容易。2023年夏天，我在布里斯托和牛津大学参加完会议后，回国待了两个月。时隔两年再次回到祖国，心情激动。许多同事热情邀请我进行学术访问。回到欧洲后，我先去德国波恩参加了一个会议，然后紧锣密鼓地准备博后申请。申请过程中，写材料、投简历耗费了大量时间，等待面试机会时也非常紧张。幸运的是，2024年1月，我接到了布里斯托大学Ben Krause教授的一份非常棒的offer，希望未来的学术之旅一切顺利。

在此，我想感谢在博士期间帮助过我的所有老师、朋友和家人。

首先，我要特别感谢我的导师Kaisa Matomäki教授。在博士期间，她在数学研究和论文写作上给予我耐心和细致的指导，并提供了许多宝贵的建议。感谢她的基金支持我参加国际会议和学术访问，让我有机会听到许多精彩的报告，结识许多学术同行。Kaisa一直关心我的生活，让我在异国他乡也感受到温暖。

此外，非常感谢博士论文的审稿人Alexander Mangerel和Aled Walker，仔细阅读了我的论文，给予了很好的评价和有用的修改意见。也要感谢Oleksiy Klurman同意担任我的答辩对手。

我深深感谢Edufi、UTUGS图尔库大学研究生院和芬兰研究院对我博士项目的资助。

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同时，我也要感谢那些邀请我进行报告或访问的同事们，包括南京林业大学的王晨，南京师范大学的戴立霞，山东大学的黄炳荣和林永晓，斯坦福大学的徐文强，同济大学的董自康和周广良，西安交通大学的郭振宇和郗平，扬州大学的丁煜宸以及中国科技大学的马杰、Tuan Tran和赵立璐。感谢你们给予我展示自己工作的机会。此外，我也要感谢Andrew Granville，多次与我讨论数学并给予指导。

我要感谢南京大学的孙智伟教授和潘颢教授在我攻读硕士期间教我数论，和我讨论数学并与我合作。他们的指导为我打下了坚实的基础，并让我有信心追求数学事业。

感谢我的父母，他们一直鼓励和支持我。感谢父母为我建立了一个充满爱的和谐家庭。我母亲有着卓越的远见，父亲有着坚韧的意志，他们是我学习的榜样。也要感谢我的家人一直以来的关心爱护和支持。

最后，感谢所有爱我的朋友们，你们的陪伴让我倍感温暖。特别感谢我的爱人，张敏勤，你像珍宝一样来到我的世界，在我遇到困境时总是陪伴我、

温暖我、支持我。你像小太阳一样，照亮了我的世界。

二〇二四年五月二十八日
孙宇宸

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List of Original Publications

This thesis contains the following four papers:

I Y.-C. Sun and H. Pan. On the gaps between consecutive primes. Forum Math., 2022; No. 4, 919–932.

II Y.-C. Sun. On the Balog-Ruzsa theorem in short intervals. Q. J. Math., 2023; No. 3, 1165–1185.

III Y.-C. Sun, S. S. Du and H. Pan. Vinogradov's theorem with Piatetski-Shapiro primes. Preprint, arXiv:1912.12572.

IV Y.-C. Sun. On divisor bounded multiplicative functions in short intervals. Preprint, arXiv:2401.08432

1 Notations

1.1 Sets

- $\mathbb{N}$ — the set of positive integers $\{1, 2, 3, \dots\}$.
- $\mathbb{Z}$ — the set of all integers.
- $\mathbb{R}$ — the set of all real numbers.
- $\mathbb{R}_{\geq 0}$ — the set of all non-negative real numbers.
- $\mathbb{C}$ — the set of all complex numbers.
- $[N]$ — the finite set $\{1, 2, \dots, N\}$
- $\mathcal{P}$ — the set of primes.
- $\mathcal{P}(z)$ — the product of primes less or equal than z, $\mathcal{P}(z) := \prod_{p<z} p$.
- $\mathbb{N}^c$ — the set $\{\lfloor n^c \rfloor : n \in \mathbb{N}\}$ for $c > 1$.
- $\mathcal{P}_c$ — the set of Piatetski-Shapiro primes, namely $\mathcal{P}_c := \mathcal{P} \cap \mathbb{N}^c$, for $c > 1$.
- $\pi_c(x)$ — the number of $p \in \mathcal{P}_c$ with $p \leq x^c$ and $c > 1$.

1.2 Functions

- $\mathbf{1}_S(n)$ or $\mathbf{1}_{n \in S}$ — the indicator function of the set S, which equals 1 if $n \in S$ and equals 0 otherwise.
- $\pi(x)$ — the number of primes no more than x.
- $p, p_1, p_2, \dots, p_n$ — prime numbers.
- $e(\alpha)$ — the additive character $e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R}$.
- $\phi(n)$ — Euler's totient function, counting the number of integers $1 \leq k \leq n$ with $(k, n) = 1$.
- $\Omega(n)$ — the number of prime factors of n counting the multiplicities.

- $\omega(n)$ — the number of distinct prime factors of n.
- $\Lambda(n)$ — the Von Mangoldt function, which is equal to $\log p$ if $n = p^k$ for some prime p and $k \geq 1$, and equal to 0 otherwise.
- $\lambda(n)$ — the Liouville function, given by $\lambda(n) := (-1)^{\Omega(n)}$.
- $\mu(n)$ — the Möbius function, given by $\mu(n) := \lambda(n)\mathbf{1}_{\text{squarefree numbers}}(n)$.
- $d(n)$ — the number of the representations for writing n as a product of two positive integers, $d(n) := \sum_{m_1 m_2 = n} 1$.
- $d_k(n)$ — the number of the representations for writing n as a product of k positive integers, $d_k(n) := \sum_{m_1 m_2 \dots m_k = n} 1$.
- $\mathbb{P}(\mathbf{X})$ — the probability of the event $\mathbf{X}$.
- $\rho^+(n), \rho^-(n)$ — upper bound and lower bound sieve weights satisfying $\rho^-(n) \leq \mathbf{1}_{\mathcal{P}}(n) \leq \rho^+(n)$.
- $\rho(n, z)$ — the indicator function of z-rough numbers, $\rho(n, z) := \mathbf{1}_{p|n \Rightarrow p > z}$.
- $\|\alpha\|$ — the distance from α to the integer(s) closest to α, $\|\alpha\| := \min_{n \in \mathbb{Z}} |\alpha - n|$.
- $\log_k x$ — the k-th iteration of the logarithm function, namely $\log_1(x) = \log x$ and $\log_k(x) := \log(\log_{k-1}(x))$, $k \geq 2$.

1.3 Analysis

Let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$.

- $f(x) = O(g(x))$ or $f(x) \ll g(x)$ — there exist some constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in \mathbb{R}$.
- $f(x) \gg g(x)$ — there exist some constant $C > 0$ such that $f(x) \geq Cg(x)$ for all $x \in \mathbb{R}$.
- $f(x) \asymp g(x)$ — we have $f(x) \ll g(x)$ and $g(x) \ll f(x)$.
- $f(x) = o(g(x))$ — we have $\lim_{x \to \infty} f(x)/g(x) \to 0$.
- $f(x) \sim g(x)$ — we have $\lim_{x \to \infty} f(x)/g(x) \to 1$.

1.4 Fourier language and norms

Let $f, h : [N] \to \mathbb{C}$ and $g : [0, 1) \to \mathbb{C}$.

- $\widehat{f}(\alpha)$ — the Fourier transform of f, namely, $\widehat{f}(\alpha) := \sum_{n \leq N} f(n)e(n\alpha)$.
- $f * h$ — the convolution of f and h, $(f * h)(n) := \sum_{n_1+n_2=n} f(n_1)h(n_2)$.
- $\|f\|_q$ — l_q norm of f, $\|f\|_q := (\sum_{1\leq n\leq N} |f(n)|^q)^{\frac{1}{q}}$.
- $\|g\|_q$ — L_q norm of g, $\|g\|_q := (\int_0^1 |g(\alpha)|^q d\alpha)^{\frac{1}{q}}$.

1.5 Miscellaneous

- $\Delta_B(A)$ — the relative upper density of A in B, namely $\Delta_B(A) := \limsup_{N\to\infty} \frac{|A\cap[N]|}{|B\cap[N]|}$.
- $\delta_B(A)$ — the relative lower density of A in B, namely $\delta_B(A) := \liminf_{N\to\infty} \frac{|A\cap[N]|}{|B\cap[N]|}$.
- $a \sim A$ — the range $A < a \leq 2A$.

2 Introduction

Additive number theory focuses on discovering additive structures within subsets of integers. A classical tool in analytic number theory to study these additive structures is the Hardy-Littlewood method, also known as the circle method. This approach utilizes Fourier analysis and exponential sum estimates. In recent years, several new methods have emerged to investigate additive structures. For instance, Green's transference principle incorporates ideas from harmonic analysis.

Multiplicative number theory centers on the distribution of primes, multiplicative functions, and multiplicative structures. Unlike additive number theory, the standard tool in multiplicative number theory is complex analysis. A fundamental concept in multiplicative number theory is the theory of the Riemann zeta function.

While these two research areas may seem distinct, there are instances where they become intertwined, and some of the most notorious open problems in number theory have elements from both areas. Our overarching objective in the thesis is to illustrate several instances of these intersecting phenomena.

One of the most famous open problems in number theory is the twin prime conjecture, which conjectures that there are infinitely many prime pairs with difference 2. In recent years, significant progress has been made towards this conjecture, notably by Zhang [85]. He proved that there are infinitely many prime pairs (p, q) such that $|p - q| \leq 7 \times 10^7$. Subsequently, the bound 10^7 has been improved multiple times by [54; 65]. The best-known unconditional record for the bound is 246. If one assumes a certain generalization of the Elliott–Halberstam conjecture, then 246 can be improved to 6.

Turning to the existence of large prime gaps and letting p_n be the n-th prime number, Cramér made a heuristic argument, conjecturing that $p_n - p_{n-1} \ll \log^2 n$ for all $n \geq 1$. This conjecture suggests that the gaps between consecutive primes do not become very large. In 2014, Ford, Green, Konyagin, Maynard and Tao [17] proved that there are infinitely many consecutive prime pairs (p_n, p_{n+1}) such that

$$p_{n+1} - p_n \gg \frac{\log n \log_2 n \log_4 n}{\log_3 n}. \tag{2.1}$$

In article [I], jointly with Hao Pan we study the distribution of the primes, where we combine the small and large prime gaps together by proving that for any fixed m, there are infinitely many $m+1$-tuples $(p_{n-m+1}, p_{n-m+2}, \ldots, p_n, p_{n+1})$ such that $p_n - p_{n-m+1} \ll 1$ and $p_{n+1} - p_n$ satisfies (2.1).

Another example that involves both multiplicative and additive number theory is the study of the exponential sum of the Möbius function. This encompasses both a multiplicative function, Möbius function and additive characters $e(n\alpha)$. In 1937, Davenport [15], proved that for any $\alpha \in [0, 1)$ and any fixed $A > 0$

$$\left| \sum_{n \le X} \mu(n) e(n\alpha) \right| \ll \frac{X}{\log^A X},$$

for all sufficiently large $X > 0$. Recently, Matomäki and Teräväinen [51] studied the short interval version of Davenport's theorem and showed that, for any $H \ge X^{3/5+\epsilon}$ with any $\epsilon > 0$,

$$\left| \sum_{X < n \le X+H} \mu(n) e(n\alpha) \right| = o(H),$$

improving a work of Zhan [84]. In 2001, Balog and Ruzsa [6] proved a lower bound for the L_1 norm for the exponential sum of the Möbius function by establishing that

$$\int_0^1 \left| \sum_{n \le X} \mu(n) e(n\alpha) \right| d\alpha \gg X^{1/6}.$$

In article [II], we extend the Balog-Ruzsa theorem to short intervals by proving that for sufficiently large X and $H \ge X^{9/17+\epsilon}$ with any $\epsilon > 0$, we have

$$\int_0^1 \left| \sum_{X < n \le X+H} \mu(n) e(n\alpha) \right| d\alpha \gg H^{1/6}.$$

One of the famous open problem in additive number theory is the binary Goldbach problem, which conjectures that every even integer $n \ge 4$ can be written as a sum of two primes. On the other hand, the weak version, the ternary Goldbach problem, asks whether every odd integer $n \ge 7$ can be written as a sum of three primes. This was proved by Vinogradov [79] for all sufficiently large odd integers. This theorem is known as Vinogradov's three primes theorem. In 2013, Helfgott [29] announced a proof of the ternary Goldbach problem.

In article [III], jointly with Du and Pan, we study Vinogradov's three primes theorem where primes are restricted in a sparse subset of primes—Piatetski-Shapiro primes (primes of the form $\lfloor n^c \rfloor$ for some fixed $c > 1$). Specifically, we proved Vinogradov's three primes theorem for Piatetski-Shapiro primes whenever $1 < c < 41/35 \approx 1.171$, improving upon the work of Jia [37].

Another interesting topic in additive combinatorics is Roth's theorem, which asserts that any $A \subset \mathbb{N}$ with positive upper density ($\Delta_{\mathbb{N}}(A) > 0$) contains non-trivial three term arithmetic progressions. In 2003, Green [25] proved a Roth-type theorem for primes by showing that any $A \subset \mathcal{P}$ with relative positive upper density ($\Delta_{\mathcal{P}}(A) > 0$) contains non-trivial three term arithmetic progressions. In article [III], we also proved that any subset of Piatetski-Shapiro primes, for $1 < c \leq 243/205 \approx 1.185$, with relative positive upper density contains non-trivial three term arithmetic progressions improving upon the work of Merik [56].

One topic studied in multiplicative number theory is the distribution of primes and the sums of arithmetic functions in short intervals. For instance, Huxley [31] proved that for all sufficiently large X and $H \geq X^{7/12+\epsilon}$

$$\sum_{X<n\leq X+H} \Lambda(n) = (1 + o(1))H.$$

Following this, Motohashi [57] and Ramachandra [68] independently proved that for all sufficiently large X and $H \geq X^{7/12+\epsilon}$

$$\sum_{X<n\leq X+H} \mu(n) = o(H).$$

Recently, Matomäki and Radziwiłł in their breakthrough paper [49] proved that for any 1-bounded multiplicative function, any sufficiently large X and $h \to \infty$ with $X \to \infty$,

$$\frac{1}{h} \sum_{x<n\leq x+h} f(n) - \frac{1}{x} \sum_{x<n\leq 2x} f(n) = o\left(\frac{1}{X} \sum_{X<n\leq 2X} |f(n)| \right)$$

holds for almost all $x \in [X, 2X]$. In article [IV], we study the Matomäki-Radziwiłł theorem for d_k-bounded multiplicative functions in almost all short intervals. For any fixed integer $k \geq 2$ and any sufficient large X and for all $h \geq (\log X)^{k \log k - k+1+\epsilon}$ with any $\epsilon > 0$, we are able to show that for almost all $x \in [X, 2X]$,

$$\frac{1}{h} \sum_{x<n\leq x+h} d_k(n) - \frac{1}{x} \sum_{x<n\leq 2x} d_k(n) = o(\log^{k-1} X) \tag{2.2}$$

holds. This improved on the work of Mangerel [47]. On the other hand, we proved that the exponent $k \log k - k + 1$ is optimal by showing that if $h \leq (\log X)^{k \log k - k+1-\epsilon}$ with any $\epsilon > 0$, then (2.2) fails for almost all $x \in [X, 2X]$.

3 Gaps between consecutive primes

3.1 Primes and prime gaps

3.1.1 Primes and primes in short intervals

A prime number is a positive integer greater than 1 that is not a product of two smaller natural numbers. For $x \geq 2$, recall that $\pi(x)$ denotes the number of primes up to x. The prime number theorem states that

$$\pi(x) = (1 + o(1))\frac{x}{\log x},$$

or equivalently

$$\sum_{n \leq x} \Lambda(n) = (1 + o(1))x.$$

To understand the distribution of primes better, one studies primes in short intervals. Specifically, it is of interest to study for how small $\theta > 0$ one can show the asymptotic formula

$$\sum_{x < n \leq x + x^\theta} \Lambda(n) = (1 + o(1))x^\theta. \tag{3.1}$$

By a zero-density argument, Huxley [31] showed that (3.1) holds for $\theta > 7/12$, which has been improved by Heath-Brown [28] by showing (3.1) holds for $\theta > 7/12 - \epsilon$ for some small $\epsilon > 0$ tending to 0 when x tends to infinity.

To determine whether there is a prime in $(x, x + x^\theta]$, we only need to find a lower bound for the left-hand side of (3.1). The best known result is due to Baker, Harman and Pintz [2], who show the lower bound

$$\sum_{x < n \leq x + x^\theta} \Lambda(n) \gg x^\theta,$$

for $\theta > 0.525$.

3.1.2 Small gaps between primes

Pairs $(p, p + 2)$, where both p and $p + 2$ are primes, are called twin primes. The recent breakthrough towards twin prime conjecture is due to Zhang [85] where he proved the existence of infinitely many prime pairs (p, q) with $|q - p| \leq 7 \times 10^7$.

Subsequently, this upper bound was refined by Maynard [54] and Tao (unpublished) to 600 and then was improved by Polymath project [65] to 246. Maynard and Tao applied almost the same method — multi-dimensional sieve method now also known as the Maynard-Tao sieve method. In fact, the Maynard-Tao method can detect primes from more general "admissible sets".

We say that a set of integers

$$\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$$

is an admissible set if for any prime p, there exist $a \pmod p$ such that $h_i \not\equiv a \pmod p$ for all $1 \le i \le k$. Let us state [54, Theorem 1.1] which concerns small prime gaps.

***Theorem* 3.1.** *Let $m \ge 2$. There exists a constant k_m depending on m such that the following holds. Suppose that $\mathcal{H} = \{h_1, h_2, \ldots, h_{k_m}\}$ is an admissible set. For every sufficiently large X, there exists an integer $n \in [X, 2X]$ such that $n + \mathcal{H}$ contains at least m primes.*

3.1.3 Large gaps between primes

For large prime gaps, there are two aspects to consider. One involves an upper bound for the prime gaps, also known as primes in short intervals as mentioned earlier. The other is a lower bound for the largest gap between two consecutive primes. Let p_n be the n-th prime number. From the prime number theorem $\pi(x) = (1 + o(1))\frac{x}{\log x}$, it is easy to see there are infinitely many prime pairs (p_n, p_{n+1}) such that $p_{n+1} - p_n \gg \log n$. Westzynthius [81] first proved the non-trivial result that there are infinitely many consecutive prime pairs (p_n, p_{n+1}) such that

$$p_{n+1} - p_n \gg f(n) \log n \tag{3.2}$$

where $f(n) \to \infty$ with $n \to \infty$. The lower bound in (3.2) was improved several times. Brauer-Zeitz [9] improved it to

$$\frac{\log n \log_3 n}{\log_4 n},$$

then it was improved by Erdős [16] to

$$\frac{\log n \log_2 n}{(\log_3 n)^2}.$$

Later, Rankin [69] showed that there are infinitely many prime pairs (p_n, p_{n+1}) such that

$$p_{n+1} - p_n \ge C \frac{\log n \log_2 n \log_4 n}{(\log_3 n)^2}, \tag{3.3}$$

for some constant $C > 0$. Erdős offered a $\$10,000$ prize for a proof or disproof of the claim that the constant C in the above inequality may be taken arbitrarily large.

Recently, Maynard [55] and Ford, Green, Konyagin and Tao[18] independently used different approaches and proved that $C = \infty$. By combining the two methods, Ford, Green, Konyagin Maynard and Tao [18] improved the results further by showing there are infinitely many integers n such that

$$p_{n+1} - p_n \gg \frac{\log n \log_2 n \log_4 n}{\log_3 n}. \tag{3.4}$$

Motivated by the recent breakthroughs on small and large prime gaps, in article [I] we proved the following theorem

***Theorem* 3.2.** *For any $m \geq 1$, there exist infinitely many n such that*

$$p_n - p_{n-m} \leq C_m \tag{3.5}$$

and

$$p_{n+1} - p_n \geq \frac{c_m \log n \log_2 n \log_4 n}{\log_3 n}, \tag{3.6}$$

where $C_m, c_m > 0$ are two constants depending only on m.

This theorem improves on Pintz's result [64].

3.2 The least prime in an arithmetic progression

Another topic related to large prime gaps is the least prime in an arithmetic progression. In fact, both large prime gaps and lower bounds for the least prime in an arithmetic progression are connected to studying lower bounds for Jacobsthal's function j, see [34]. If m is a positive integer then $j(m)$ is defined to be the maximal gap between integers coprime to m and studying the large prime gaps is closely related to studying lower bounds for $j(m)$ with $m = \prod_{p \leq x} p$ for some $x > 1$. Pomerance [66, Theorem] pointed out the lower bound for $j(m)$ also implies a lower bound for the least prime in an arithmetic progression.

For any positive integers k and l with $(k, l) = 1$, let $p(k, l)$ denote the least prime of the form $kn + l$ with $n \geq 1$. There are three popular directions to study $p(k, l)$. The most popular one is to investigate upper bounds for $P(k) := \max_{(l,k)=1} p(k, l)$. Linnik [45] proved that $P(k) \leq k^L$ with a large constant L. This result has been called the Linnik theorem and the value L has been improved many times see [60; 11; 38; 12; 23; 13; 80; 82]. The best known result is $L \leq 5$ due to Xylouris [83]. All proofs above heavily rely on information concerning zeros of Dirichlet L-functions.

Very recently, there are three different L-function-free proofs of the Linnik theorem. Granville, Harper and Soundararajan [24] applied pretentious approach to prove the Linnik theorem but did not give an explicit value of L. Friedlander and Iwaniec gave a sieve-theoretic proof of Linnik's theorem in [19, Chapter 24] and made it explicit in [20; 21], showing $L \le 7,574,400$. Matomäki, Merikoski and Teräväinen [52] developed a sieve that can detect primes in sets that are multiplicative structured in a certain sense to show that $L \le 350$. Assuming GRH (the Generalized Riemann Hypothesis), Chowla [14] observed that $L \le 2+\epsilon$ for any $\epsilon > 0$ and he further conjectured that $L \le 1+\epsilon$ for any $\epsilon > 0$.

The other two directions are studying lower bounds of $P(k)$ and $p(k,l)$ with fixed l. In fact, known lower bounds for these two are closely connected to lower bounds for large prime gaps.

Concerning lower bounds for $P(k)$, Pomerance [66] proved that, for any sufficiently large positive integer k which has no more than $\exp(\log_2 k/\log_3 k)$ distinct prime factors, we have

$$P(k) \gg \phi(k)\frac{\log k \log_2 k \log_4 k}{(\log_3 k)^2}$$

and this result was improved by Li, Pratt and Shakan [44] who proved that, for any sufficiently large positive integer k with no more than $\exp\left(\frac{1}{2}\frac{\log_2 k \log_4 k}{\log_3 k}\right)$ distinct prime factors, we have

$$P(k) \gg \phi(k)\frac{\log k \log_2 k \log_4 k}{\log_3 k}.$$

For the lower bound of $p(k,l)$ with fixed l, Prachar [67] and Schinzel [73] proved the existence of infinitely many k such that

$$p(k,l) \gg \frac{k \log k \log_2 k \log_4 k}{(\log_3 k)^2}.$$

In article [I], we improve the above lower bound to

$$p(k,l) \gg \frac{k \log k \log_2 k \log_4 k}{\log_3 k}.$$

3.3 Long prime gaps and a covering idea

In order to have long gaps between consecutive primes p_n and p_{n+1}, we need to ensure that all integers between p_n and p_{n+1} are composite numbers. Hence, the key to obtaining a better lower bound for long prime gaps is finding more consecutive composite numbers between two primes. To achieve this goal we utilize a "covering system" based on the following lemma

***Lemma* 3.1.** *Let $y > x > 0$ be sufficiently large integers and let $M = \prod_{p \le x} p$. If there exist residue classes $\{a_p \pmod p\}_{p\le x}$ such that all integers $n \in [x,y]$ are* *covered by* $\{a_p \pmod p\}_{p\le x}$ *meaning that for any* n *there is a prime* $p \le x$ *such that* $n \equiv a_p \pmod p$*, then there exists* $1 \le b \le M$ *such that* $n \in [M+x+b, M+y+b]$ *are all composite.*

Proof. By the Chinese reminder theorem, there exist $1 \le b \le M$ such that $b \equiv -a_p \pmod p$ for all $p \le x$. We claim that for all $n \in [x,y]$, $M+b+n$ are all composite. This is because, for each $n \in [x,y]$, there exists a prime $p \le x$ such that $n \equiv a_p \pmod p$ and thus $M+n+b \equiv 0 \pmod p$. Since $M+n+b > x$, these numbers are all composite. □

From the above lemma, we observe that choosing p_n to be the largest prime such that $p_n \le M+x+b$ results in $p_{n+1}-p_n \ge y-x$. By the prime number theorem, we have $\log M = (1+o(1))x$ which implies that $x = (1+o(1))\log p_n = (1+o(1))\log n$. The remaining task is to determine the largest possible y such that every $n \in [x,y]$ is covered by $\{a_p \pmod p\}_{p\le x}$. For instance, to prove (3.4), one needs to show that for

$$y \asymp \frac{x\log x \log_3 x}{\log_2 x}, \tag{3.7}$$

the condition of Lemma 3.1 holds. In the rest of this chapter we always let $y := y(x) \asymp \frac{x\log x\log_3 x}{\log_2 x}$.

3.4 Digging holes

To successfully combine small prime gaps with large prime gaps, we aim to dig some "holes" in the previously mentioned string of consecutive composite numbers. Then we insert primes into some of the holes. The distances between these holes need to be small to obtain small gaps between primes. By applying the Maynard-Tao method, primes can be detected in admissible sets. In the following, we will employ an admissible set to dig these holes. Recall the definition of admissible sets and let

$$\mathcal{H}_n = \{q_1, q_2, \ldots, q_n\} \text{ for any primes } q_i > n$$

which is admissible, since $q_i \not\equiv 0 \pmod p$ for all primes $p \le n$ and for all primes $p > n$ there is $a_p \pmod p$ such that $a_p \not\equiv q_i \pmod p$ for all $i = 1, 2, \ldots, n$.

Because we will dig holes in the interval $[x,y]$ and have bounded gaps between the holes, having an admissible set within the range $[x, x+O(1)]$ is essential. Thanks to the Maynard-Tao theorem, we can find an n-tuple of primes $(q_1, q_2, \ldots, q_n)$ with bounded gaps in any sufficient large interval $[X, 2X]$. Then we can choose $x = q_1$. Hence, we will employ $\mathcal{H}_n$ to create these gaps. The following lemma describes how to dig holes.

Lemma 3.2. *Let* $x > 0$ *be sufficiently large integer and* $y = y(x)$ *is as before. Let* $\mathcal{H} = \{h_1, h_2, \ldots, h_k\} \subset [x, 2x]$ *be an admissible set such that* $|h_i - h_j|$

are bounded for all $1 \leq i \leq j \leq k$*. Let* $M = \prod_{p\leq x} p$*. If there is a covering system* $\{a_p \pmod p\}_{p\leq x}$ *such that all integers* $n \in [x,y] \setminus \mathcal{H}$ *are covered by* $\{a_p \pmod p\}_{p\leq x}$*, then there exist* $1 \leq b \leq M$ *such that if*

$$n \equiv b \pmod M,$$

then $n + [x,y] \setminus n + \mathcal{H}$ *are all composite numbers.*

Proof. See the proof of Lemma 3.1. □

3.5 Overview of the proof

3.5.1 Small prime gaps (holes)

By looking at Lemma 3.2, we would like to find a prime tuple $(p_1, p_2, \ldots, p_m)$ from the set $n + \mathcal{H}$ where $n \equiv b \pmod M$. Hence the first question is how does n grows with M tending to infinity.

The original arguments for the Maynard-Tao theorem required that the modulus M is not too large, specifically, it needed to satisfy

$$M \leq \prod_{p \ll \log\log\log n} p. \tag{3.8}$$

Recall that $x = (1 + o(1)) \log M$ and (3.7). If the largest size of M is from (3.8), then the large prime gap we obtain is

$$y - x \gg \frac{x \log x \log_3 x}{\log_2 x} \gg \frac{\log_3 n \log_4 n \log_6 n}{\log_5 n}$$

which is much smaller than we want. Fortunately, a variant of Maynard-Tao theorem established by Banks, Freiberg and Maynard [7, Theorem 4.3] allows us to choose $M = \prod_{\substack{p\leq c\log n \\ b\nmid q_0}}$ where $c > 0$ is a small constant and q_0 is an exceptional modulus.

3.5.2 Large prime gaps

In order to choose the suitable $\{a_p \pmod p\}_{p\leq x}$ satisfying the condition of Lemma 3.1, Ford, Green, Konyagin Maynard and Tao [17] employ a probabilistic method to prove that with probability $1 - o(1)$, one can find such residue classes. In our case an additional requirement is imposed: $a_p \notin \mathcal{H} \pmod p$ for all $p \leq x$ with $p \nmid q_0$. With slight modification of the discussions in [17], we successfully proved such residue classes $\{a_p \pmod p\}_{p\leq x}$ exist with probability $1 - o(1)$.

In the following, we introduce the ideas for proving long prime gaps, which is also the most important part in article [I].

A sieve idea

Let
$$z := \exp\left(\frac{\log x \log_4 x}{\log_2 x}\right)$$
and recall that
$$y \asymp \frac{x \log x \log_3 x}{\log_2 x}.$$
For a fixed large constant $C > 0$, consider the following sieving sets
$$\begin{aligned}\mathcal{S} &:= \{s \in \mathcal{P} : \log^{10} x < s \le z\}\\ \mathcal{T} &:= \{t \in \mathcal{P} : \frac{x}{2C} < t \le \frac{x}{C}\}\\ \mathcal{Q} &:= \{q \in \mathcal{P} : \frac{x}{C} < q \le y\}.\end{aligned}$$

For residue classes $\mathbf{a}_{\mathcal{S}} = \{a_s \pmod s\}_{s\in\mathcal{S}}$ and $\mathbf{a}_{\mathcal{T}} = \{a_t \pmod t\}_{t\in\mathcal{T}}$, let
$$S(\mathbf{a}_{\mathcal{S}}) := \{n \in \mathbb{Z} : n \not\equiv a_s \pmod s \text{ for all } s \in \mathcal{S}\}$$
and
$$T(\mathbf{a}_{\mathcal{T}}) := \{n \in \mathbb{Z} : n \not\equiv a_t \pmod t \text{ for all } t \in \mathcal{T}\}.$$
The following proposition is the key in [17].

***Proposition* 3.1.** *There are residue classes $\mathbf{a}_{\mathcal{S}} := \{a_s \pmod s\}_{s\in\mathcal{S}}$ and $\mathbf{a}_{\mathcal{T}} := \{a_t \pmod t\}_{t\in\mathcal{T}}$ such that*
$$|\mathcal{Q} \cap S(\mathbf{a}_{\mathcal{S}}) \cap T(\mathbf{a}_{\mathcal{T}})| \le \frac{x}{2\log x}. \tag{3.9}$$

Let us explain why Proposition 3.1 implies long prime gaps. We use a sieve approach. Let $\mathbf{a}_{\mathcal{S}}$ and $\mathbf{a}_{\mathcal{T}}$ be as in Proposition 3.1 and define $\{b_p \pmod p\}_{p\le x/C}$ such that
$$b_p = \begin{cases} a_p, & \text{if } p \in \mathcal{S}\cup\mathcal{T},\\ 0, & \text{otherwise}\end{cases}$$
If $n \in [x/C, y]$ cannot be covered by $\{0 \pmod p\}_{\substack{p\le x/C\\ p\notin\mathcal{S}\cup\mathcal{T}}}$, then n is either z-smooth or has the form pd with $p \in \mathcal{T}\cup\mathcal{Q}$ and $1 \le d \le \frac{y}{x/2C}$ implying that $d = 1$.

Note that the number of the z-smooth numbers is $\le \frac{1}{100}\frac{x}{\log x}$ and $\#\mathcal{T} \le \frac{x}{C\log x}$. By Proposition 3.1, we have
$$|\mathcal{Q} \cap S(\mathbf{a}_{\mathcal{S}}) \cap T(\mathbf{a}_{\mathcal{T}})| \le \frac{x}{2\log x}.$$

Therefore, the number of integers in $[x/C, y]$ that cannot be covered by $\{b_p \pmod p\}_{p \le x/C}$ is no more than

$$\#\{n \in [x/C, y] : n \text{ is } z\text{-smooth}\} + \#\mathcal{T} + |\mathcal{Q} \cap S(\mathbf{a}_\mathcal{S}) \cap T(\mathbf{a}_\mathcal{T})|$$
$$\le \left(\frac{1}{100} + \frac{1}{C} + \frac{1}{2}\right)\frac{x}{\log x} < \left(1 - \frac{2}{C}\right)\frac{x}{\log x}$$

for sufficiently large $C > 0$. We can choose residue classes $\{a_p \pmod p\}_{x/C < p \le x}$ to cover those integers one-by-one. Then the long prime gaps follow from Lemma 3.1.

Probabilistic ideas

Now we introduce the probabilistic ideas in the proof of Proposition 3.1 that allow us to find $\mathbf{a}_\mathcal{S}$ and $\mathbf{a}_\mathcal{T}$ that cover most primes in $\mathcal{Q}$. We choose the random residue classes $\mathbf{a}_\mathcal{S} = \{a_s \pmod s\}_{s \in \mathcal{S}}$ by selecting each $a_s \pmod s$ uniformly at random from $\mathbb{Z}/s\mathbb{Z}$. Then we have the following lemma.

***Lemma* 3.3.** *With probability* $1 - o(1)$, *we have*

$$|\mathcal{Q} \cap S(\mathbf{a}_\mathcal{S})| \asymp \frac{x}{\log x} \log_2 x. \tag{3.10}$$

Proof. See [17, Corollary 5]. □

The next step is to choose $\mathbf{a}_\mathcal{T} = \{a_t \pmod t\}_{t \in \mathcal{T}}$ using probabilistic model which is the hardest part. In fact, Ford, Green, Konyagin, Maynard and Tao [17] used the hypergraph covering theorem [17, Theorem 3] in their proof. Let us just explain the idea behind its application to long prime gaps. One finds a probabilistic model and hypergraph such that the following conditions (non-rigorous) hold.

- (small edges): In [17], for all $t \in \mathcal{T}$, a random edge $\mathbf{e}_t$, with small size, corresponds to a subset of the random residue class $\mathbf{a}_t$. Later, they use a subset of $\cup_{t \in \mathcal{T}} \mathbf{e}_t$ to cover $\mathcal{Q} \cap S(\mathbf{a}_\mathcal{S})$;

- (sparsity): $\mathbb{P}(n \in \mathbf{e}_t)$ is "small" for all $n \in \mathcal{Q} \cap S(\mathbf{a}_\mathcal{S})$ and $t \in \mathcal{T}$. It means that the probabilistic model is not concentrated on a small number of edges.

- (uniform covering): $\sum_{t \in \mathcal{T}} \mathbb{P}(n \in \mathbf{e}_t) \gg 1$ for almost all $n \in \mathcal{Q} \cap S(\mathbf{a}_\mathcal{S})$, which means that almost all n can be covered by many $\mathbf{e}_p$.

- (small codegrees) For all distinct $n_1, n_2 \in \mathcal{Q} \cap S(\mathbf{a}_\mathcal{S})$, $\sum_{t \in \mathcal{T}} \mathbb{P}(n_1, n_2 \in \mathbf{e}_t)$ is small, which means that $|\{t \in \mathcal{T} : n_1 \in \mathbf{e}_t \wedge n_2 \in \mathbf{e}_t\}|$ is small.

Then, by the hypergraph theorem ([17, Corollary 4 and Theorem 4]), with probability $1 - o(1)$, (3.9) holds. The probabilistic model is chosen by the multidimensional sieve, see [17, Sections 6 and 7]. We have now provided an overview of the idea behind the proof, and we suggest the interested reader refer to article [I] for the structure and [17] for more details.

4 On the Balog-Ruzsa Theorem in short intervals

4.1 Exponential sums

Estimating exponential sums is one of the most important tasks in the circle method which can be used to study additive structures in subsets of integers. For example, for understanding additive structures in primes, one needs to estimate

$$\sum_{n\leq N}\Lambda(n)e(n\alpha), \tag{4.1}$$

for $\alpha\in[0,1)$. The following theorem ([58, Theorem 8.5]) claims that if α is close to a rational number with medium size denominator, then (4.1) is $o(N)$.

***Theorem* 4.1.** *Let $a,q,N\in\mathbb{N}$ and $1\leq a<q\leq N$. If $(a,q)=1$ and $\left|\alpha-\frac{a}{q}\right|\leq 1/q^2$, then,*

$$\sum_{n\leq N}\Lambda(n)e(n\alpha)\ll\left(\frac{N}{q^{1/2}}+N^{4/5}+N^{1/2}q^{1/2}\right)\log^4 N$$

It is natural to ask whether we can get some cancellation for other exponential sums, e.g. exponential sums of multiplicative functions.

Davenport [15] proved that for sufficiently large N, we have

$$\sum_{n\leq N}\mu(n)e(n\alpha)\ll_A\frac{N}{\log^A N},$$

for any $\alpha\in[0,1)$ and any $A>0$.

Baker and Harman [1], under the Generalized Riemann Hypothesis (GRH), proved that for sufficiently large N, we have

$$\sum_{n\leq N}\mu(n)e(n\alpha)\ll_\epsilon N^{3/4+\epsilon},$$

for any $\alpha\in[0,1)$ and any $\epsilon>0$.

In addition, another important task in the circle method is to estimate L_p norms of exponential sums, aiming to achieve power saving results for certain values of

$p > 0$. For example, if we employ the circle method to attack the tenary Goldbach problem, then it is useful to estimate the L_2 norm of the exponential sum for the von Mangoldt function. Thanks to Parserval's identity, we have

$$\int_0^1 \left| \sum_{n \le N} \Lambda(n) e(n\alpha) \right|^2 d\alpha \asymp N \log N.$$

However, when $p < 2$, such as $p = 1$, we can utilize the L_2 norm to control the L_1 norm. Using the Cauchy-Schwarz inequality, we obtain

$$\int_0^1 \left| \sum_{n \le N} \Lambda(n) e(n\alpha) \right| d\alpha \le \left(\int_0^1 \left| \sum_{n \le N} \Lambda(n) e(n\alpha) \right|^2 d\alpha \right)^{1/2} \ll N^{1/2} \log^{1/2} N.$$

However, we cannot ascertain whether the upper bound for L_1 norm is optimal. Motivated by this, it becomes imperative to delve into the exploration of the lower bound for the L_1 norm. For instance, Vaughan [78] proved that

$$\int_0^1 \left| \sum_{n \le N} \Lambda(n) e(n\alpha) \right| d\alpha \gg N^{1/2}.$$

For some other important arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ in number theory, one can also consider bounds for the L_1 norm of the exponential sum, i.e. study

$$\int_0^1 \left| \sum_{n \le N} f(n) e(n\alpha) \right| d\alpha. \tag{4.2}$$

If $f(n)$ is the von Mangoldt function, then the best known upper bound for (4.2) is $\ll N^{1/2} \log^{1/2} N$ while the best known lower bound is $\gg N^{1/2}$. When $f(n)$ is the divisor function $d(n)$, Pandey [61] gave an asymptotic formula of the size $N^{1/2}$ for (4.2) improving the work by Goldston and Pandey [22]. When $f(n)$ is the Liouville function $\lambda(n)$, Pandey and Radziwiłł [62] recently obtained (4.2) is $\gg N^{1/4}$, improving Balog and Perelli [5]. For $f(n) = \mu_r(n)$, the indicator function of r-free numbers, which equals 1 if for every prime $p \mid n$ we have $p^r \nmid n$, and equals 0 otherwise, Balog and Ruzsa obtained the correct magnitude $N^{\frac{1}{r+1}}$ for (4.2) improving earlier work by Brüdern, Granville, Perelli, Vaughan, and Wooley [10]. We now state their result for $r = 2$.

***Theorem* 4.2** (Balog-Ruzsa)*. Let* $N \ge 2$*. Then*

$$N^{\frac{1}{3}} \ll \int_{\mathbb{T}} \left| \sum_{n=1}^{N} \mu^2(n) e(n\alpha) \right| d\alpha \ll N^{\frac{1}{3}}.$$

As a corollary, they deduce that if $f(n) = \mu(n)$, the Möbius function, then (4.2) is $\gg N^{1/6}$.

Inspired by investigations on primes in short intervals, in article [II], we proved the short interval version of the Balog-Ruzsa theorem

***Theorem* 4.3.** *(i) Let $\epsilon > 0$ and $N \geq H \geq N^{\frac{9}{17}+\varepsilon}$. Then*

$$\int_{\mathbb{T}} \left| \sum_{|n-N|<H} \mu^2(n) e(n\alpha) \right| d\alpha \gg H^{\frac{1}{3}}.$$

(ii) Let $\epsilon > 0$ and $N \geq H \geq N^{\frac{18}{29}+\epsilon}$. Then

$$\int_{\mathbb{T}} \left| \sum_{|n-N|\leq H} \mu^2(n) e(n\alpha) \right| d\alpha \ll H^{\frac{1}{3}}.$$

As a corollary, we also show that when $H \geq N^{9/17+\epsilon}$,

$$\int_{\mathbb{T}} \left| \sum_{|n-N|<H} \mu(n) e(n\alpha) \right| d\alpha \gg H^{\frac{1}{6}}.$$

4.2 Outline of the proof of Theorem 4.3

The proof of Theorem 4.3 starts from converting the single exponential sum into a double exponential sum using the identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d).$$

Next, we split the double sum into two parts depending on the size of d, namely

$$\begin{aligned}\sum_{n=1}^{N} \mu^2(n) e(n\alpha) &= \sum_{n=1}^{N} \sum_{\substack{d^2|n \\ 1\leq d\leq N^{1/2}}} \mu(d) e(n\alpha) \\ &= \sum_{n=1}^{N} \sum_{\substack{d^2|n \\ 1\leq d\leq D}} \mu(d) e(n\alpha) + \sum_{n=1}^{N} \sum_{\substack{d^2|n \\ D<d\leq N^{1/2}}} \mu(d) e(n\alpha) \\ &=: T_1 + T_2. \end{aligned} \tag{4.3}$$

The main task is to achieve an effective saving for the L_2 norm of T_2. In fact, for trivial reasons, T_2 is relatively small for large D. We have

$$\sum_{n=1}^{N}\sum_{\substack{d^2|n\\ D<d\le N^{1/2}}}\mu(d)e(n\alpha)\ll\sum_{D<d\le N^{1/2}}\sum_{\substack{1\le n\le N\\ d^2|n}}1$$
$$\ll\sum_{D<d\le N^{1/2}}\sum_{\substack{1\le n\le N\\ d^2|n}}1\ll ND^{-1}+N^{1/2}.$$

Remark. In practice, we will not directly apply the above trivial arguments. Instead, we will apply L_2 norm to bound the L_1 norm of T_2 and apply Parseval's identity. We will choose $D=H^{1/3}$ in our short interval case.

4.2.1 The key lemma

The above argument suggests that the L_2 norm of T_2 should be small. Therefore, we can use the L_2 norm of T_2 to bound its L_1 norm.

***Lemma* 4.1.** *For any $1<K\le N$ and $y<d\le z$, we have*

$$\sum_{N-K<n\le N}\Big(\sum_{\substack{d^2|n\\ D<d\le N^{1/2}}}1\Big)^2\ll KD^{-1}+N^{1/2}\log^3 N. \tag{4.4}$$

Proof. See [6, Lemma 1]. □

For the long interval case, (4.4) is sufficient. However, if we consider the short interval case with a length of interval shorter than $N^{3/4}$, then (4.4) is not enough. Let us use the upper bound case to explain the reason.

Suppose that we study

$$\sum_{N<n\le N+H}\mu^2(n)e(n\alpha)=\sum_{N<n\le N+H}\sum_{\substack{d^2|n\\ 1\le d\le N^{1/2}}}\mu(d)e(n\alpha)$$
$$=\sum_{N<n\le N+H}\sum_{\substack{d^2|n\\ 1\le d\le D}}\mu(d)e(n\alpha)+\sum_{N<n\le N+H}\sum_{\substack{d^2|n\\ D<d\le N^{1/2}}}\mu(d)e(n\alpha)$$
$$=:T_1'+T_2', \tag{4.5}$$

and we can handle the L_1 norm of T_1' well by following Balog-Ruzsa's arguments. By the Cauchy-Schwarz inequality, Parserval's identity and Lemma 4.1 with $K=H$ and $D=H^{1/3}$, we have

$$\int_0^1|T_2'|d\alpha\ll\Big(\int_0^1|T_2'|^2d\alpha\Big)^{1/2}\ll\Big(H^{2/3}+N^{1/2}\log^3 N\Big)^{1/2}\ll H^{1/3}+N^{1/4+\epsilon}.$$

Recall the claim in Theorem 4.3. The first term above is corresponding to our expectation, but if we require the second term $N^{1/4+\epsilon} \ll H^{1/3}$, then $H \geq N^{3/4+\epsilon}$. Hence, in order to make the interval shorter than $N^{3/4+\epsilon}$, we have to improve on Lemma 4.2. The following improvement of Lemma 4.1 is [II, Lemma 3.1].

***Lemma* 4.2.** *Let* $1 \leq K < N$, $\epsilon > 0$ *and* $1 \leq y < \min\{z, K^{1/2-\epsilon}\}$. *We have*

$$\sum_{N-K<n\leq N} \Big(\sum_{\substack{d^2|n \\ y<d\leq z}} 1 \Big)^2 \ll Ky^{-1} + N^{\frac{12}{29}+\epsilon} y^{-\frac{10}{29}}. \tag{4.6}$$

This is the crucial ingredient in article [II]. Let us outline the proof of Lemma 4.2. Initially, observe that the left-hand side of (4.6) can be bounded by

$$\sum_{N-K<n\leq N} \sum_{\substack{[d_1^2,d_2^2]|n \\ y<d_i\leq z}} 1 \leq \sum_{N-K<n\leq N} \sum_{\substack{n=h^2 d_1^2 d_2^2 a \\ y<hd_i\leq z \\ (d_1,d_2)=1}} 1. \tag{4.7}$$

By a dyadic argument, we can assume $d_1 \sim D_1$ and $d_2 \sim D_2$ for some $D_1, D_2 > 0$. Without loss of generality, we further assume $D_1 \leq D_2$. We now apply two different approaches to get the upper bound. Let D' be an parameter that will be optimized later. When $D_2 \leq D'$, we apply the hyperbola method to bound (4.7) by

$$\sum_{\substack{d_1\sim D_1 \\ d_2\sim D_2}} \left(\sum_{\frac{y}{d_1} \leq h \leq (\frac{N}{d_1^2 d_2^2})^{\frac{1}{3}}} \sum_{\frac{N-K}{h^2 d_1^2 d_2^2} \leq a \leq \frac{N}{h^2 d_1^2 d_2^2}} 1 + \sum_{a \leq (\frac{N}{d_1^2 d_2^2})^{\frac{1}{3}}} \sum_{(\frac{N-K}{a d_1^2 d_2^2})^{\frac{1}{2}} \leq h \leq (\frac{N}{a d_1^2 d_2^2})^{\frac{1}{2}}} 1 \right). \tag{4.8}$$

Then we apply a standard Fourier expansion of $\{x\}$ and the van der Corput bound to yield that (4.8) is at most $N^{\frac{2(p+q)+1}{6(p+1)}+\epsilon} D'^{2-\frac{4p+4q+2}{3(p+1)}}$, where (p,q) is the exponent pair $(2/7, 1/14)$.

Remark. In the proof of Lemma 4.2, the exponent pair has to satisfy that $1-4p+2q \geq 0$, so it is convenient to choose $(p,q) = (2/7, 1/14)$. If one assumes the exponent pair conjecture, namely $(p,q) = (\eta,\eta)$ for any $\eta > 0$, then we can improve the second term in (4.6) to $N^{\frac{9}{22}+\epsilon} y^{-\frac{4}{11}}$ which can yield a shorter H in Theorem 4.3. Specifically, under the exponent pair conjecture, the length H in Theorem 4.3 (i) and (ii) can be improved to $N^{\frac{27}{52}+\epsilon} < H \leq N$ and $N^{\frac{81}{104}+\epsilon} < H \leq N$ respectively, and correspondingly, in the Möbius case, we can also improve H to $N^{\frac{27}{52}+\epsilon} < H \leq N$.

When $D_2 > D'$, we apply Perron's formula to rewrite (4.7) as the integral

$$\frac{1}{2\pi i} \int_{1+\varepsilon-iT_0}^{1+\varepsilon+iT_0} \frac{N^s - (N-K)^s}{s} \zeta(s) P(2s) ds + N^{\epsilon}, \tag{4.9}$$

where $T_0 \asymp N$ and

$$P(s) = \left(\sum_{d_1 \sim D_1} \frac{1}{d_1^s}\right)\left(\sum_{d_2 \sim D_2} \frac{1}{d_2^s}\right)\left(\sum_{h \sim H} \frac{1}{h^s}\right)$$

is a Dirichlet polynomial. By contour integration, we shift the integral in (4.9) to the $1/2$ line. It is not difficult to imagine that the residue at $s = 1$ contributes $O\left(\frac{K}{D_1 D_2 H}\right)$, the contributions of two horizontal lines can be absorbed into $O(N^\epsilon)$ and the contribution of $1/2$ line is $\ll N^{1/2+\epsilon} y^{-1/2} D'^{-\frac{1}{2}}$.

Finally, we choose $D' = N^{\frac{5}{29}+\epsilon} y^{-\frac{9}{29}}$ to balance the two upper bounds obtained above.

4.2.2 Balog-Ruzsa's ideas

The Fejér kernel

The Fejér kernel, see (4.10), is the Cesàro mean of the Dirichlet kernel $D_N(\alpha) := \sum_{|n| \le N} e(n\alpha)$. In some cases the Fejér kernel can be regarded as a smooth replacement for the Dirichlet kernel. In Balog-Ruzsa [6], they utilized the Fejér kernel, which can help save at least one $\log$ factor in the upper bound of L_1 norm. This will be explained in the following. Recall the definition of the Fejér kernel

$$F_N(\alpha) := \sum_{|n| \le N} \left(1 - \frac{|n|}{N}\right) e(n\alpha) = \frac{\sin^2(\pi N \alpha)}{N \sin^2(\pi \alpha)} \ll \min\left\{N, \frac{1}{N\|\alpha\|^2}\right\}. \quad (4.10)$$

By the uniform upper bound of $D_N(\alpha)$, the L_1 norm of the Dirichlet kernel is

$$\int_0^1 |D_N(\alpha)|\, d\alpha \ll \int_0^1 \min\left\{N, \frac{1}{\|\alpha\|}\right\} \ll \log N.$$

However, the L_1 norm of the Fejér kernel is

$$\int_0^1 |F_N(\alpha)| d\alpha \ll \int_0^1 \min\left\{N, \frac{1}{N\|\alpha\|^2}\right\} \ll 1,$$

This suggests that one may be able to apply the Fejér kernel to save $\log N$ in upper bounds of L_1 norms.

The upper bound case for the Balog-Ruzsa theorem

By the previous discussion, we only need to focus on L_1 norm of T_1. However, if we interchange the summations and bound the L_1 norm of T_1 by

$$\sum_{1 \le d \le D} \int_0^1 \left| \sum_{\substack{1 \le n \le N \\ d^2 | n}} e(n\alpha) \right| d\alpha,$$

then we can imagine that an extra $\log N$ comes from the integral, since the Dirichlet kernel appears. The idea to remove the $\log N$ is to utilize the Fejér kernel (other good/smooth kernel should also be fine) and actually Balog and Ruzsa smooth the indicator $1_{[1,N]}(n)$ by a trapezoidal function which can be regarded as a Fourier coefficient of $\frac{1}{N-K}(NF_N(\alpha) - KF_K(\alpha))$ where $K = (1-\eta)N$ for small $\eta > 0$.

The lower bound case for the Balog-Ruzsa theorem

To prove the lower bound for the L_1 norm, we utilize some ideas from the circle method. Thanks to Parseval's identity and Lemma 4.1 (or Lemma 4.2 for the short interval version), we can see that the L_1 norm of T_2 is much smaller than the L_1 norm of T_1. Let us focus on estimating the L_1 norm of T_1. By interchanging the summation, we have

$$T_1 = \sum_{1\le d\le D} \mu(d) \sum_{\substack{1\le n\le N\\ d^2|n}} e(n\alpha) =: \sum_{1\le d\le D} \mu(d) G_d(\alpha).$$

Let $1 \le a \le d^2$ with d sqaurefree. We choose suitably the major arcs

$$\mathfrak{M}_{d^2,a} \subset \left\{\alpha \in [0,1) : \left|\alpha - \frac{a}{d^2}\right| \le \frac{1}{N}\right\} \quad \text{and} \quad \mathfrak{M}_{d^2} := \bigcup_{\substack{1\le a\le d^2\\ (a,d^2)\text{ squarefree}}} \mathfrak{M}_{d^2,a}$$

and show that the following three conditions are satisfied:

- When $\alpha \in \mathfrak{M}_{d^2}$, $G_d(\alpha) \gg \frac{N}{d^2}$ is significantly larger than other $G_{d_0}(\alpha)$ with $d_0 \neq d$.
- The measure of $\mathfrak{M}_{d^2}$ is not too small, meaning that $|\mathfrak{M}_{d^2}| \gg \frac{d^2}{N}\prod_{p|d}\left(1-\frac{1}{p^2}\right) \gg \frac{d^2}{N}$.
- For distinct d_1, d_2, $\mathfrak{M}_{d_1^2} \cap \mathfrak{M}_{d_2^2} = \emptyset$.

Then one can show that

$$\int_0^1 |T_1| d\alpha \gg \sum_{1\le d\le D} |\mathfrak{M}_{d^2}| \min_{\alpha\in\mathfrak{M}_{d^2}} |G_d(\alpha)| \gg D = N^{1/3}$$

For more details the interested reader may refer to the article [II].

5 Vinogradov's theorem with Piatetski-Shapiro primes

5.1 Vinogradov's theorem and its variants

One of the most famous open problems in number theory is Goldbach's problem which ask whether any even number $n \geq 4$ can be written as a sum of two primes.

In 1937, Vinogradov (see e.g. [59, Chapter 8]) used the circle method to prove a weak version of Goldbach's problem by showing that any sufficiently large odd integer n can be written as a sum of three primes.

In recent years, many people have proved variants of Vinogradov's three primes theorem. From the combinatorics perspective, one can investigate Vinogradov's three primes theorem for subsets of primes with positive relative density. In 2010, Li and Pan [43] established that if A_1, A_2, A_2 are subsets of primes with positive relative lower densities ($\delta_{\mathcal{P}}(A_i) > 0$ for $i \in \{1, 2, 3\}$), and $\delta_{\mathcal{P}}(A_1) + \delta_{\mathcal{P}}(A_2) + \delta_{\mathcal{P}}(A_3) > 2$, then any sufficiently large odd integer n can be written as $n = p_1 + p_2 + p_3$ with $p_i \in A_i$. Subsequently, Shao [75] considered Vinogradov's three primes theorem for a single subset of primes with positive relative lower density and showed that if $A \subset \mathcal{P}$ with $\delta_{\mathcal{P}}(A) > 5/8$, then any sufficiently large odd integer can be written as a sum of three elements in A.

From the number theory perspective, number theorists have explored Vinogradov's three primes theorem for some special forms of primes. In 2017, Matomäki and Shao [50] demonstrated that any sufficiently large integer $n \equiv 3 \pmod 6$ can be written as a sum of three Chen primes (the set of primes p such that $p + 2$ has at most two prime factors). Additionally, for any fixed $m \geq 2$, they proved that there exists $H(m) \geq 0$ such that any sufficiently large odd integer can be written as a sum of three primes p_1, p_2, p_3 such that $[p_i, p_i + H(m)]$ contains m primes for $i = 1, 2, 3$. Many researchers have studied Vinogradov's three primes theorem with primes in short intervals. The shortest intervals have been reached by Matomäki, Maynard and Shao [48] who showed that for any $\epsilon > 0$, every sufficiently large odd integer n can be written as $n = p_1 + p_2 + p_3$ with $|p_i - n/3| \leq n^{0.55+\epsilon}$. Teräväinen [76] proved that Vinogradov's three primes theorem also holds for Linnik's primes (primes of the form $x^2 + y^2 + 1$). All these works rely on the transference principle which was first introduced by Green [25].

5.2 Piatetski-Shapiro primes and our results

Recall that

$$\mathbb{N}^c = \{\lfloor n^c \rfloor : n \in \mathbb{N}\},$$

for $c > 1$. There is another fascinating special form of primes known as Piatetski-Shapiro primes ($\mathcal{P}_c = \mathcal{P} \cap \mathbb{N}^c$ for some $c > 1$). One of the reasons to study Piatetski-Shapiro primes is that, when $c > 1$, these types of primes are remarkably rare. Recall that $\pi_c(x)$ denotes the number of Piateski-Shapiro primes up to x^c. In 1953, Piatetski-Shapiro [63] first proved that the asymptotic formula

$$\pi_c(x) \sim \frac{x}{c \log x}$$

holds for $1 < c < 1.1$. Over the years, several authors have improved the range of c for this asymptotic formula, and the record is due to Rivat and Sargos [70] who showed that the asymptotic formula holds for $1 < c < 2817/2426 \approx 1.161$, improving previous works [40; 27; 39; 46]. When considering the lower bound of $\pi_c(x)$, the record is that, for $1 < c < 243/205 \approx 1.185$, we have $\pi_c(x) \gg \frac{x}{\log x}$ as shown by Rivat and Wu [71] improving the previous results [36; 3; 35; 42].

In 1992, Balog and Friedlander [4] first proved Vinogradov's three primes theorem for Piatetski-Shapiro primes (for $1 < c < 1.05$). This result has been improved by Kumchev [41] who established the asymptotic formula for the number of representations of $n = p_1 + p_2 + p_3$ when $1 < c < 1.06$ and Jia [37] who applied a sieve method to establish a lower bound for the number of representations when $1 < c < 16/15 \approx 1.067$.

Let $B \subset \mathcal{P} \cap \mathbb{N}^c$ with positive upper density for $c > 1$. The Roth-type problem for Piatetski-Shapiro primes is to find the largest $c > 1$ such that B contains many non-trivial three term arithmetic progressions. (We say this is the Roth-type problem because Roth [72] first proved that any subset of integers with positive upper density contains a three term arithmetic progressions, see Theorem 5.3). Concerning the Roth-type theorem for Piatetski-Shapiro primes, Merik [56] proved that B contains non-trivial three-term arithmetic progressions when $1 < c < 72/71 \approx 1.014$.

In article [III], we proved the following theorems toward Vinogradov's three primes theorem and Roth-type theorem with Piatetski-Shapiro primes.

***Theorem* 5.1.** *For any $c_1, c_2, c_3 \in (1, \frac{41}{35})$, every sufficiently large odd N can be represented as*

$$N = p_1 + p_2 + p_3,$$

where $p_i \in \mathcal{P}^{c_i}$ for each $1 \leq i \leq 3$.

***Theorem* 5.2.** *For any $c \in (1, \frac{243}{205})$, any $B \subset \mathcal{P}_c$ with positive relative upper density contains nontrivial 3-term arithmetic progressions.*

Note that $\frac{41}{35} \approx 1.171$ and $\frac{243}{205} \approx 1.185$. In order to apply the transference principle to study Vinogradov's three primes theorem for Piatetski-Shapiro primes, we need a variant of the transference principle that was developed by Matomäki Maynard and Shao [48]. Let us start from the original version of Green's transference principle and try to illustrate the ideas behind Green's transference principle.

5.3 Green's transference principle and its variant

The original version of Green's transference principle was used to study 3-term arithmetic progressions in subsets of primes with positive relative upper density. In 2005, Green [25] proved that any subset of primes with positive relative upper density contains a non-trivial three term arithmetic progression. Before introducing the idea behind Green's transference principle, let us review the Roth theorem which claims that any subset of integers with positive upper density contains a non-trivial 3-term arithmetic progression. Varnavides [77] used a clever combinatorial argument to show a lower bound for the number of three term arithmetic progressions by proving the following theorem.

***Theorem* 5.3.** *Suppose that $N > 3$ is a sufficiently large integer and $\mathcal{A} \subset [N]$ with $|\mathcal{A}| \gg N$. Then the set $\mathcal{A}$ contains $\gg N^2$ non-trivial 3-term arithmetic progressions.*

Proof. See [77]. □

Let us turn back to Green's transference principle. Let A be a set of primes with positive relative upper density. The fundamental idea in the proof of Roth-type theorem for primes is transferring this problem to study a subset of integers with positive relative upper density. Now we formalize this idea although we do not give a very rigorous argument.

Let $f : [N] \to \mathbb{C}$. We aim to study three-term arithmetic progressions by the formula

$$\sum_{n_1+n_2=2n_3} f(n_1)f(n_2)f(n_3). \tag{5.1}$$

For instance, if $f(n) = f_A(n) := \log n \cdot \mathbf{1}_A(n)$, then (5.1) serves to detect 3-term arithmetic progressions in the set A. What we want to achieve next is to transfer f_A into another function $f_0 : [N] \to \mathbb{R}_{\geq 0}$ such that the uniform lower bound

$$f_0(n) \geq c_0 \mathbf{1}_{\mathcal{A}}(n) \tag{5.2}$$

holds for some $c_0 > 0$, where $\mathcal{A}$ is a subset of integers with positive relative upper density. Now (5.1) can be transferred to

$$\approx \sum_{n_1+n_2=2n_3} f_0(n_1)f_0(n_2)f_0(n_3) \gg \sum_{n_1+n_2=2n_3} \mathbf{1}_{\mathcal{A}}(n_1)\mathbf{1}_{\mathcal{A}}(n_2)\mathbf{1}_{\mathcal{A}}(n_3). \tag{5.3}$$

This is $\gg N^2$ by Theorem 5.3. Now we discuss how to interpret $\approx$ in (5.3). By rewriting (5.1) and the left-hand side of (5.3) using Fourier transform, what we want is

$$\int_0^1 \widehat{f_A}(\alpha)\widehat{f_A}(\alpha)\widehat{f_A}(-2\alpha)d\alpha \approx \int_0^1 \widehat{f}_0(\alpha)\widehat{f}_0(\alpha)\widehat{f}_0(-2\alpha)d\alpha. \tag{5.4}$$

By using the telescoping method, e.g. see [III, (3.6)], we will encounter two questions

(i) Is $\max_{\alpha\in[0,1)} |\widehat{f_A}(\alpha) - \widehat{f}_0(\alpha)|$ small ($o(N)$) ?

(ii) Do there exist $2 \le q < 3$ such that $\|\widehat{f_A}\|_q$ and $\|\widehat{f}_0\|_q$ are small ($\ll N^{1-1/q}$)?

If (i) holds, then, for the special case $\alpha = 0$, we notice that $|\widehat{f}_A(0) - \widehat{f}_0(0)| = o(N)$ implies

$$\sum_{n\le N} f_A(n) \gg N,$$

which will be called the **mean condition**. We will see later a stronger version of the **mean condition** where uniform distribution in arithmetic progressions is required. If (i) holds for $f_0 = \mathbf{1}_{[N]}$, we say f_A is "pseudorandom" or satisfies **pseudorandomness**. If (ii) is true, we say that f satisfies the **restriction estimate**. In most cases, L_2 norm does not satisfy the restriction estimate. For example, by Parserval's identity

$$\|\widehat{f_A}\|_2^2 = \|f_A\|_2^2 \asymp N \log N,$$

which has an extra $\log N$ factor compared to what we need (see Theorem 5.4 for our restriction estimate). Thus, one needs some new ideas to bound $\|\widehat{f_A}\|_q$ for some $2 < q < 3$. Fortunately, Bourgain [8] provided a very clever approach for handling $\|\widehat{f}_A\|_q$. We will give a brief explanation in Section 5.6.

Let us turn to discuss how to construct f_0 that satisfies (i) and (5.2). The idea is derived from harmonic analysis and we want to find a "good" kernel $K : [N] \to \mathbb{C}$ such that

(C1) $\widehat{K}(0) = 1$,

(C2) $|\widehat{f_A} - \widehat{f_A * K}| = |\widehat{f_A} - \widehat{f_A}\widehat{K}|$ is small.

(C3) There exist $\mathcal{A} \subset [N]$ with the size $|\mathcal{A}| \gg N$ such that $f_A * K \gg \mathbf{1}_{\mathcal{A}}$.

The convolution $f_A * K$ will be our f_0. To define K, we introduce Bohr sets defined by

$$B(\alpha, \epsilon) := \bigcap_i B(\alpha_i, \epsilon),$$

where

$$B(\alpha_i, \epsilon) = \{1 \le n \le N : \|n\alpha_i\| \le \epsilon\}.$$

The definition implies that integers in a Bohr set have additive structure.

In practice, we choose α_i to be $1/N$-spaced points in $[0, 1)$ such that $|f(\alpha_i)|$ are large. For readers who are familiar with the circle method, these intervals around α_i can be regarded as the "major arcs" while intervals far away from all α_i can be regarded as the "minor arcs". If we only required (C1) and (C2), then the good kernel K could be chosen to be

$$K'(n) = \frac{1}{|B(\alpha, \epsilon)|} \mathbf{1}_{B(\alpha,\epsilon)}(n).$$

The condition (C1) holds trivially. Next we show that (C2) also holds. The reason is that when α is in the "minor arcs" (i.e. far way from any α_i), $|\widehat{f_A}(\alpha)|$ is small. Otherwise, by the definition of the Bohr set, $\widehat{K'}(\alpha) \approx \widehat{K'}(0) = 1$, so $\widehat{f_A}(\alpha) - \widehat{f_A}\widehat{K}(\alpha) = \widehat{f_A}(1 - \widehat{K})(\alpha)$ is small. We leave details to the interested reader (or see [25]). In order to satisfy (C3), we need a smoother kernel K such that $f_A * K$ is bounded (smoother). Therefore, we choose $K = K' * K'$ to make the kernel K smoother. Then one can show that $f_A * K$ is bounded.

To apply Green's transference principle in studying Vinogradov's three primes theorem with Piatetski-Shapiro primes, we require a variant of the transference principle that was developed by Matomäki, Maynard and Shao [48].

***Theorem* 5.4** (Matomäki-Maynard-Shao's transference principle). *Let $\epsilon, \eta \in (0, 1)$. Let N be a positive integer and let $f_1, f_2, f_3 : [N] \to \mathbb{R}_{\geq 0}$ be functions, with each $f \in \{f_1, f_2, f_3\}$ satisfying the following assumptions:*

- ***(mean condition)*** *For each arithmetic progression $P \subset [N]$ with $|P| \geq \eta N$ we have $\mathbb{E}_{n \in P} f(n) \geq 1/3 + \epsilon$;*
- ***(pseudorandomness)*** *There exists a majorant $\nu : [N] \to \mathbb{R}_{\geq 0}$ with $f \leq \nu$ pointwise, such that $\|\widehat{\nu} - \widehat{\mathbf{1}}_{[N]}\|_\infty \leq \eta N$;*
- ***(restriction estimate)****We have $\|\widehat{f}\|_q \leq K N^{1-1/q}$ for some fixed $K \geq \eta$ and $q \in (2, 3)$.*

Then, for each $n \in [N/2, N]$, we have

$$f_1 * f_2 * f_3(n) \geq (c(\epsilon) - O_{\epsilon,K,q}(\eta))N^2,$$

where $c(\epsilon) > 0$ is a constant depending only on ϵ.

To define suitable f_i for $i \in \{1, 2, 3\}$, we can start with the normalized weighted function by

$$f_i'(n) := \begin{cases} \log n^c \cdot n^{1-\frac{1}{c}} \mathbf{1}_{\mathcal{P}}(n), & \text{if } n \in \mathbb{N}^c, \\ 0, & \text{otherwise}. \end{cases} \tag{5.5}$$

However, in order to fulfill the pseudorandomness and get a better c , we need the following complicated version of f_i for $i \in \{1,2,3\}$

$$f_i(n) := \begin{cases} \frac{\log N^c}{\alpha^+} \cdot \frac{\phi(W)}{W}(Wn+b_i)^{1-\frac{1}{c}}\mathbf{1}_{\mathcal{P}}(Wn+b_i), & \text{if } Wn+b_i \in \mathbb{N}^c, \\ 0, & \text{otherwise,} \end{cases} \tag{5.6}$$

where $w = \log\log\log N$ and $W = \prod_{p\le w} p$, and α^+ will be defined later. The reason we need to introduce w will be explained in the next section.

5.4 Pseudorandomness and W-trick

In order to establish the pseudorandomness, we restrict elements in $\mathcal{P}_c$ to the set $A = \{n \le N : Wn+b \in \mathcal{P}_c\}$, where w and W are as before. This technique is called the W-trick. The primary motivation to utilize the W-trick is, roughly speaking, to avoid certain "local" problems.

Let us consider the prime numbers as an example and to illustrate what are "local" problems. Without the W-trick, for instance, for $\nu(n) = \log n \cdot \mathbf{1}_{\mathcal{P}}(n)$, we have, by the prime number theorem,

$$\begin{aligned} \widehat{\nu}(1/2) - \widehat{\mathbf{1}}_{[N]}(1/2) &= \sum_{1\le p\le N} \log p \cdot e(p/2) - \sum_{1\le n\le N} e(n/2) \\ &= -\sum_{3\le p\le N} \log p + O(1) \\ &= -N + O\left(\frac{N}{\log^{100} N}\right). \end{aligned}$$

and thus ν is not pseudorandom. Moreover, for $q = \prod_{2\le p\le C} p$ and $(a,q)=1$ with any constant $C \ge 2$, we have

$$\widehat{\mathbf{1}}_{[N]}(a/q) = O(q),$$

and by the Siegel–Walfisz theorem,

$$\widehat{\nu}(a/q) = \sum_{1\le p\le N} \log p \cdot e(a/q) = \sum_{(b,q)=1} \sum_{\substack{1\le p\le N \\ p\equiv b \pmod q}} \log p \cdot e(pa/q)$$

$$= \sum_{(b,q)=1} e(ba/q) \sum_{\substack{1\le p\le N \\ p\equiv b \pmod q}} \log p = \frac{\mu(q)}{\phi(q)} N + O\left(\frac{N}{\log^{100} N}\right).$$

Now we have seen that "local" problems mean that $|\widehat{\nu}(\theta) - \widehat{\mathbf{1}}_{[N]}(\theta)| \ne o(N)$ when θ close to a/q with small q. However, for the W-tricked function

$$\nu(n) = \frac{\phi(W)}{W} \log(Wn+b)\mathbf{1}_{\mathcal{P}}(Wn+b),$$

one can use standard exponential sum estimates to show that $\|\widehat{\nu} - \widehat{\mathbf{1}}_{[N]}\|_\infty = o(N)$.

5.5 Mean condition and Harman's sieve method

5.5.1 Mean condition

To satisfy the mean condition, we need to normalize the indicator function of Piatetski-Shapiro primes. We choose f as in (5.6) and

$$\nu(n) := \begin{cases} c \cdot \frac{\log X}{\alpha^+} \cdot \frac{\phi(W)}{W}(Wn+b)^{1-\frac{1}{c}}\rho^+(Wn+b), & \text{if } Wn+b \in \mathbb{N}^c, \\ 0, & \text{otherwise,} \end{cases} \tag{5.7}$$

where $\rho^+ \geq \mathbf{1}_{\mathcal{P}}$ and the constant α^+ is chosen so that $\widehat{\nu}(0) = N(1+o(1))$. Clearly, ν is a majorant function of f. By utilizing the arithmetic information for the lower bound result for the number of Piatetski-Shapiro primes, if we choose appropriate lower bound sieve weight $\rho^- \leq \mathbf{1}_{\mathcal{P}}$, we have that

$$\begin{aligned}\sum_{n\in P} f(n) &= c\frac{\log X}{\alpha^+}\frac{\phi(W)}{W} \sum_{m\in(W\cdot P+b)\cap\mathbb{N}^{c_i}} m^{1-\frac{1}{c_i}}\mathbf{1}_{\mathcal{P}}(m) \\ &\geq c\frac{\log X}{\alpha^+}\frac{\phi(W)}{W} \sum_{m\in(W\cdot P+b)\cap\mathbb{N}^{c_i}} m^{1-\frac{1}{c_i}}\rho^-(m) \geq \frac{\alpha^-}{\alpha^+}|W\cdot P+b| = \frac{\alpha^-}{\alpha^+}|P|,\end{aligned}$$

where α^- is the lower bound sieve coefficient corresponding to ρ^-. Recalling the mean condition, we must be very careful in choosing lower bound and upper bound sieves such that $\frac{\alpha^-}{\alpha^+} > \frac{1}{3}$.

5.5.2 Harman's sieve method

We now explain how we choose ρ^+ and ρ^- using Harman's sieve method (for a comprehensive account of Harman's sieve, see [26]). Let

$$\mathcal{A} = \{n \sim X : n \in \mathbb{N}^c, n \equiv l \pmod{d}\},$$

and

$$\mathcal{B} = \{n \sim X : n \equiv l \pmod{d}\},$$

To successfully apply Harman's sieve method to find the lower bound sieve weights ρ^- and the upper bound sieve weights ρ^+ we employ the following Type I and II information for $a(n), b(n) \ll d_k(n)$ with some $k \geq 2$ and $g(n) = 1$ or $g(n) = e(\alpha n)$ for $\alpha \in [0,1)$.

Type I :

$$\sum_{\substack{dn\in\mathcal{A} \\ d\leq D}} \frac{1}{\gamma}a(d)(dn)^{1-\gamma}g(dn) = \sum_{\substack{dn\in\mathcal{B} \\ d\leq D}} a(d)g(dn) + O(X^{1-\delta})$$

Type II :

$$\sum_{\substack{mn\in\mathcal{A}\\ m\sim M}} \frac{1}{\gamma} a(m)b(n)(mn)^{1-\gamma} g(mn) = \sum_{\substack{mn\in\mathcal{B}\\ m\sim M}} a(m)b(n)g(mn) + O(X^{1-\delta}),$$

where $D \le X^{\theta_1}$ and $M \in [X^{1-\theta_2}, X^{\theta_2}] \cup [X^{\theta_3}, X^{\theta_4}] \cup [X^{1-\theta_4}, X^{1-\theta_3}]$ and with some suitable parameters $0 < \theta_i < 1$ depending on g and c for $i \in \{1,2,3,4\}$. The interested reader can refer to [III, Sections 5 and 6] for the explicit values of θ_i. In practice, for the lower bound case, we need $g = 1$ and for the upper bound case, we need $g(n) = e(n\alpha)$ for every fixed $\alpha \in [0, 1)$.

Now, we provide a simple example about how to apply Type I and II information to construct ρ^- and ρ^+ for $1 < c < 15/13$.

Recall the result of Liu and Rivat [46] who proved that for $1 < c < 15/13 = 1.1538...$,

$$\pi_c(x) \sim \frac{x}{\log x}.$$

If we assume that $1 < c < 15/13 = 1.1538...$, then Type I information holds for $D < X^{7/15}$ and Type II information holds for $X^{2/15} < M < X^{1/3}$. Thanks to the asymptotic formula of $\pi_c(x)$, we can take $\rho^-(n) = \mathbf{1}_{\mathcal{P}}(n)$. For the upper bound sieve, we choose $\rho^+(n) = \rho(n, x^{1/5}) \ge \mathbf{1}_{\mathcal{P}}(n)$. Now we have

$$c \sum_{\substack{n\le x\\ n\in\mathcal{A}}} n^{1-\frac{1}{c}} \rho(n, x^{1/5}) e(n\alpha) = c \sum_{d|\mathcal{P}(x^{1/5})} \mu(d) \sum_{\substack{nd\le x\\ n\in\mathcal{A}}} (dn)^{1-\frac{1}{c}} e(nd\alpha). \tag{5.8}$$

If $d \le X^{1/3}$, then the right-hand side of (5.8) satisfies Type I condition. Otherwise $d = p_1p_2\cdots p_k > X^{1/3}$ with p_i decreasing. Note that $p_i < X^{1/5}$ for all $i \in \{1, 2, \dots, k\}$, so there exist $1 < t < k$ such that $X^{2/15} = X^{1/3-1/5} < p_1p_2\cdots p_t < X^{1/3}$. Therefore we let $m = p_1p_2\cdots p_t$ be from the range of Type II information. Thus (5.8), working out some technical details, equals

$$\sum_{d|\mathcal{P}(x^{1/5})} \mu(d) \sum_{\substack{nd\le x\\ n\in\mathcal{B}}} e(nd\alpha) + O(X^{1-\delta}) = \sum_{\substack{n\le x\\ n\in\mathcal{B}}} \rho(n, x^{1/5}) e(n\alpha) + O(X^{1-\delta}),$$

for some $\delta > 0$. The corresponding coefficient $\alpha^+ \le \frac{5}{3}(1 + \log 2) < 3 = 3\alpha^-$, (see [26, (1.4.16)]) and thus the mean condition is satisfied

In order to get the better result (larger range of c for Vinogradov's three primes theorem with Piatetski-Shapiro primes) we need to do a more complicated Buchstab decomposition. The coefficients α^+ and α^- of upper bound and lower bound sieves come from the prime number theory and integrals of Buchstab's function (see [26, Chapter 1]). We suggest the interested reader to refer to [III, section 5 and 6] for more details.

5.6 Restriction estimate and Bourgain's strategy

As mentioned in Section 5.3, in certain cases, the L_2 norm may introduce an additional $\log N$. In the context of Piatetski-Shapiro primes, the L_2 norm even introduces a larger term, specifically $N^{1-\frac{1}{c}}$, which deviates significantly from our expectations. Let us turn to the reasons behind the challenges posed by the L_2 norm and the advantages of studying L_q norms, particularly for $2 < q < 3$.

Let us compare studying ternary Goldbach problem with studying Vinogradov's three primes theorem for Piatetski-Shapiro primes to see why we need to study L_q norms for $2 < q < 3$ in Piatetski-Shapiro primes case. In the ternary Goldbach problem, when applying the circle method, one breaks down the contributions from the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ writing

$$\begin{aligned}
&\int_0^1 \left(\sum_{p\le N} \log p \cdot e(p\alpha)\right)^3 e(-N\alpha)d\alpha \\
&= \int_{\mathfrak{M}} \left(\sum_{p\le N} \log p \cdot e(p\alpha)\right)^3 e(-N\alpha)d\alpha + \int_{\mathfrak{m}} \left(\sum_{p\le N} \log p \cdot e(p\alpha)\right)^3 e(-N\alpha)d\alpha.
\end{aligned}$$

The main term comes from the integral over major arcs and is

$$\int_{\mathfrak{M}} \left(\sum_{p\le N} \log p \cdot e(p\alpha)\right)^3 e(-N\alpha)d\alpha = \mathfrak{S}(N)N^2 + O\left(\frac{N^2}{\log N}\right),$$

where $\mathfrak{S}(N) \gg 1$ if N is a sufficiently large odd integer. Moving to the integral over the minor arcs, by exponential sum estimates, if $\alpha \in \mathfrak{m}$, then

$$\sum_{p\le N} \log p \cdot e(p\alpha) \ll \frac{N}{\log^{100} N}.$$

Hence, from the above and Parserval's identity

$$\begin{aligned}
&\int_{\mathfrak{m}} \left(\sum_{p\le N} \log p \cdot e(p\alpha)\right)^3 e(-N\alpha)d\alpha \\
&\ll \max_{\alpha\in\mathfrak{m}} \left|\sum_{p\le N} \log p \cdot e(p\alpha)\right| \int_0^1 \left|\sum_{p\le N} \log p \cdot e(p\alpha)\right|^2 d\alpha \\
&\ll \max_{\alpha\in\mathfrak{m}} \left|\sum_{p\le N} \log p \cdot e(p\alpha)\right| N\log N \\
&\ll \frac{N}{\log^{99} N}.
\end{aligned}$$

If we apply the same strategy to deal with Vinogradov's theorem for Piatetski-Shapiro primes, we consider the integral

$$\int_0^1 F(\alpha)^3 e(-N\alpha)d\alpha,$$

where

$$F(\alpha) = \sum_{\substack{p\leq N\\ p\in\mathbb{N}^c}} c\cdot p^{1-\frac{1}{c}}\log p\cdot e(p\alpha).$$

We also split the interval $[0,1)$ into similar major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$. For the major arc case, standard arguments yield that the major arcs contribution is $\gg N^2$. However, for the minor arc case, using Parseval's identity to bound it, we get

$$\begin{aligned}\int_{\mathfrak{m}} F(\alpha)^3 e(-N\alpha)d\alpha &\ll \max_{\alpha\in\mathfrak{m}}|F(\alpha)|\int_0^1 |F(\alpha)|^2\\ &\ll \max_{\alpha\in\mathfrak{m}}|F(\alpha)|N^{2-\frac{1}{c}}\log N.\end{aligned}$$

While we expect that the last term is bounded by $o(N^2)$, achieving this requires $\max_{\alpha\in\mathfrak{m}}|F(\alpha)| = o(N^{1/c}/\log N)$, which becomes challenging when c is somewhat greater than 1.

To bypass this difficulty we apply the restriction estimate. Suppose there exist some $2<q<3$ such that $\int_0^1 |F(\alpha)|^q d\alpha \ll N^{q-1}$. Then we have

$$\begin{aligned}\int_{\mathfrak{m}} F(\alpha)^3 e(-N\alpha)d\alpha &\ll \max_{\alpha\in\mathfrak{m}}|F(\alpha)|^{3-q}\int_0^1 |F(\alpha)|^q\\ &\ll \max_{\alpha\in\mathfrak{m}}|F(\alpha)|^{3-q}N^{q-1}.\end{aligned}$$

This requires only $\max_{\alpha\in\mathfrak{m}}|F(\alpha)| = o(N)$, which can be achieved when α is far away from rational points with small denominators, even if c is somewhat larger than 1.

Now we briefly discuss how to prove the restriction estimate when $1<c<73/64\approx 1.141$. Specifically, we need to show that there exist some $q_0\in(2,3)$ such that

$$\int_0^1 |F(\alpha)|^{q_0}d\alpha \ll N^{q_0-1}. \tag{5.9}$$

If we can show that there exist some $q\in(2,3)$ such that for all $g(n)\leq n^{1-\frac{1}{c}}\mathbf{1}_{n\in\mathbb{N}^c}$

$$\int_0^1 \left|\sum_{n\leq N} g(n)e(n\alpha)\right|^q d\alpha \ll N^{q-1}, \tag{5.10}$$

then we have

$$\int_0^1 |F(\alpha)|^q d\alpha \ll N^{q-1}(\log N)^q. \tag{5.11}$$

It is possible to remove the extra $(\log N)^q$ by Bourgain's strategy [8] which shows that if the behavior of $F(\alpha)$ is similar to $\sum_{p\le N} \log p \cdot e(p\alpha)$, namely, for any sufficiently large $A > 0$, one has

$$\max_{\alpha\in[0,1)} \left| F(\alpha) - \sum_{p\le N} \log p \cdot e(p\alpha) \right| = o\left(\frac{N}{\log^A N}\right), \tag{5.12}$$

then one can remove the extra factor $(\log N)^q$ from (5.11) if one replaces q by $q+\epsilon$ for any $\epsilon > 0$. For details of Bourgain's strategy, see [8] or [III, Section 4].

Kumchev [41] showed that (5.12) holds for $1 < c < 73/64 \approx 1.141$. Hence if we can show (5.10) for some $2 < q < 3$ then by Bourgain's strategy, (5.9) holds for $q_0 = q + \epsilon$.

Thus, the remaining task is to show (5.10). We write $G(\alpha) = \sum_{n\le N} g(n)e(n\alpha)$. By van der Corput's method, one can obtain that $|G(\alpha)| \le (1+o(1))N$ for $1 \le c < 2$ (see e.g. [III, section 4]), so by a dyadic splitting, we have that

$$\int_0^1 |G(\alpha)|^q d\alpha \ll \sum_{n\ge 0} \left(\frac{N}{2^{n-1}}\right)^q \mu\left(\left\{\alpha \in [0,1) : \frac{N}{2^n} < |G(\alpha)| \le \frac{N}{2^{n-1}}\right\}\right)$$

where μ is the Lebesgue measure. Now we only need to show that for some $0 < t < q$, we have for every $n \ge 0$,

$$\mu\left(\left\{\alpha \in [0,1) : |G(\alpha)| > 2^{-n}N\right\}\right) \ll \frac{1}{N2^{-tn}}. \tag{5.13}$$

We first study the case that g is pseudorandom, namely $|G(\alpha) - \widehat{\mathbf{1}_{[N]}}(\alpha)|$ is small for all $\alpha \in [0,1)$. In this case, Parseval's identity implies that

$$(2^{-n}N)^2 \mu\left(\left\{\alpha \in [0,1) : |\widehat{\mathbf{1}_{[N]}}(\alpha)| > 2^{-n}N\right\}\right) \le \int_0^1 |\widehat{\mathbf{1}_{[N]}}(\alpha)|^2 d\alpha = N$$

and consequently

$$\mu\left(\left\{\alpha \in [0,1) : |\widehat{\mathbf{1}_{[N]}}(\alpha)| > 2^{-n}N\right\}\right) \le \frac{1}{N2^{-2n}}.$$

Hence (5.13) holds. In [III, Section 4], we will see that if $g(n) = n^{1-\frac{1}{c}}\mathbf{1}_{n\in\mathbb{N}^c}$, then $\max_{\alpha\in[0,1)} |G(\alpha) - \widehat{\mathbf{1}_{[N]}}(\alpha)| = o(N^{\frac{3}{2}-\frac{1}{c}} \log N)$. By working out some technical details, we can prove (5.13) in the general case using the fact g has a pseudorandom majorant.

In order to show that (5.9) holds for a larger range of c, we will choose $F(\alpha)$ such that

$$\max_{\alpha\in[0,1)}\left|F(\alpha)-\sum_{n\leq N}\rho^{+}(n)\log n\cdot e(n\alpha)\right|=o\left(\frac{N}{\log^{A}N}\right),$$

for some suitable upper bound sieve weights ρ^{+}. For more details, the interested reader may refer to [III].

6 On divisor bounded multiplicative functions in short intervals

6.1 Multiplicative functions in short intervals

Recall that $\Lambda(n)$ and $\mu(n)$ are the von Mangoldt function and the Möbius function. In Chapter 3, we introduced results for primes and arithmetic functions in short intervals. Here we concentrate on results in "almost all" short intervals, which means that the results hold for all but at most $o(X)$ intervals $[x, x+h]$ with $x \in [X, 2X]$. By the zero density result due to Huxley [31], one can prove an almost all intervals result for primes. Specifically, if $X > 0$ is sufficiently large, and $h \geq X^{1/6+\epsilon}$ with any $\epsilon > 0$, then

$$\frac{1}{h} \sum_{x<n\leq x+h} \Lambda(n) - \frac{1}{X} \sum_{X<n\leq 2X} \Lambda(n) = o(1)$$

for all but $o(X)$ integers $x \in [X, 2X]$.

Ramachandra [68] applied a similar strategy to prove a similar result for the Möbius function. He showed that for any sufficiently large X and $h \geq X^{1/6+\epsilon}$ with $\epsilon > 0$, we have

$$\frac{1}{h} \sum_{x<n\leq x+h} \mu(n) - \frac{1}{X} \sum_{X<n\leq 2X} \mu(n) = o(1)$$

for all but $o(X)$ integers $x \in [X, 2X]$.

Assuming the Riemann hypothesis, the above two almost all results hold for $h > \log^A X$ for some $A > 0$ (by [74] and an unpublished work of Peng Gao).

Recently, Matomäki and Radziwiłł [49] made a breakthrough concerning short sums of 1-bounded multiplicative functions, significantly improving the previous results even beyond those established under the Riemann hypothesis.

***Theorem* 6.1** (Matomäki-Radziwiłł theorem). *Let $X \geq h \geq 2$ and $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function. Then for almost all $x \in [X, 2X]$,*

$$\frac{1}{h} \sum_{x<n\leq x+h} f(n) - \frac{1}{X} \sum_{X<n\leq 2X} f(n) = o(1), \tag{6.1}$$

provided $h \to \infty$ with $X \to \infty$.

To prove the Matomäki-Radziwiłł theorem, Matomäki and Radziwiłł utilized more information about multiplicativity, for example, Ramaré's identity.

Thanks to Matomäki-Radziwiłł methods, some previous results involving multiplicative functions can be improved. For example Matomäki and Teräväinen [51] proved the following theorem.

***Theorem* 6.2.** *Suppose that X is sufficiently large and $H \geq X^{0.55+\epsilon}$ for any $\epsilon > 0$. Then*

$$\sum_{X<n\leq X+H} \mu(n) = o(H).$$

This improved the length $H \geq X^{7/12+\epsilon}$ due to Motohashi [57] and Ramachandra [68].

6.2 Dirichlet divisor problem and divisor bounded multiplicative function in short intervals

The famous Dirichlet divisor problem is the conjecture that, for any $\epsilon > 0$,

$$\sum_{n\leq X} d(n) = X \log X + (2\gamma - 1)X + O(X^{1/4+\epsilon}).$$

By the hyperbola method, one can prove that for any $\epsilon > 0$,

$$\sum_{n\leq X} d(n) = X \log X + (2\gamma - 1)X + O(X^{1/2}). \tag{6.2}$$

The best known error term was given by Huxley [32], who proved that the error term is $O(X^{131/416+\epsilon})$. Naturally, the better the error term the better the implied result for short sums.

In [IV] we consider the sum of $d_k(n) = \sum_{n=m_1m_2\ldots m_k} 1$ for $k \geq 2$ in almost all short intervals and prove

***Theorem* 6.3.** *Let $\epsilon > 0$ and $k \geq 2$. If $h \geq (\log X)^{k\log k-k+1+\epsilon}$, then*

$$\frac{1}{h}\sum_{x<n\leq x+h} d_k(n) - \frac{1}{x}\sum_{x<n\leq 2x} d_k(n) = o(\log^{k-1} x) \tag{6.3}$$

for all but at most $o(X)$ integers $x \in [X, 2X]$.

Note that $\sum_{x<n\leq 2x} d_k(n) \asymp x\log^{k-1} x$, so Theorem 6.3 gives an asymptotic formula for almost all short intervals.

Mangerel's [47, Theorem 1.7] shows that if $h \geq h_0(\log X)^{(k-1)^2}$ and $h_0 \to \infty$ with $X \to \infty$, then

$$\int_X^{2X} \left| \frac{1}{h}\sum_{x<n\leq x+h} d_k(n) - \frac{1}{x}\sum_{x<n\leq 2x} d_k(n)\right|^2 dx = o(X\log^{2k-2} X).$$

In fact the length of short interval h in [47, Theorem 1.7] is sharp, and as a corollary in almost all short intervals sense, one can obtain that if $h \geq h_0(\log X)^{(k-1)^2}$, then (6.3) holds for all but at most $o(X)$ integers. Hence, Theorem 6.3 is an improvement of Mangerel's result in almost all short intervals sense.

The exponent of $\log X$ in Theorem 6.3 is essentially optimal, meaning that replacing ϵ with $-\epsilon$ renders this theorem incorrect. However, when comparing the length of short intervals in our result to those in the Matomäki-Radziwiłł theorem or Mangerel's result, there is some room for improvement. This could be achieved by optimising $(\log X)^\epsilon$ with some a function $h_0(X)$ that grows slower than $(\log X)^\epsilon$ and tends to infinity as $X \to \infty$.

6.3 Proof ideas

For convenience, I will use the divisor function case ($k = 2$) to illustrate the proof ideas.

6.3.1 Two restrictions

We will restrict n in our sum in two ways. The first one is to restrict to integers having at least one prime factor from certain ranges. The purpose of the first restriction is to help to create a bilinear structure as explained later.

Let $\epsilon_0 > 0$. We define

$$P_1 := \exp((\log\log X)^{1/2}), \quad Q_1 := (\log X)^{\epsilon_0} \tag{6.4}$$

$$P_2 := \exp((\log\log X)^{2}), \quad Q_2 := \exp((\log\log X)^{100}) \tag{6.5}$$

Let A denote the set of all $n \in (X, 2X]$ having at least one prime factor in each interval $[P_j, Q_j]$ for $j \in \{1, 2\}$. The set A is dense and we have

***Lemma* 6.1.** *Let $X > 0$ be sufficiently large. Then*

$$\sum_{\substack{X \leq n \leq 2X \\ n \notin A}} d(n) = o(X \log X).$$

Proof. See [IV, Lemma 2.2]. □

The second one is to restrict to integers that do not have too many prime factors. The purpose of the second restriction is to have a good bound for the second moment of the divisor function in this restricted set.

The famous Erdős-Kac theorem tells us that almost all integers in $[X, 2X]$ have $\Omega(n) = (1 + o(1)) \log\log X$, where $\Omega(n)$ is the total number of prime factors. However, if one consider the sum of the divisor function, the main contribution does not come from those integers having $\Omega(n) = (1+o(1))\log\log X$. This can be seen from

$$\frac{1}{X}\sum_{\substack{X\le n\le 2X\\ \Omega(n)=((1+o(1))\log\log X}} d(n) \le \frac{1}{X}X2^{(1+o(1))\log\log X} = \log^{(1+o(1))\log 2} X,$$

which is much smaller than the expected $\log X$. The following lemma tells us that the main contribution to the sum of divisor function comes from integers having $(2+o(1))\log\log X$ prime factors.

***Lemma* 6.2.** *Let $l(n) \in \{\omega(n), \Omega(n)\}$. For any sufficiently large X and small $\epsilon' > 0$, we have*

$$\frac{X}{(\log X)^{(2+\epsilon')\log 2-1}} \ll \sum_{\substack{X\le n\le 2X\\ |l(n)-2\log\log X|\le \epsilon'\log\log X}} 1 \ll \frac{X}{(\log X)^{(2-\epsilon')\log 2-1}}.$$

Proof. See [IV, Lemma 2.6]. □

From the above, we see that

$$\sum_{\substack{X\le n\le 2X\\ l(n)=(2+o(1))\log\log x}} 1 = \frac{X}{(\log X)^{(2\log 2-1+o(1))}},$$

so we get

$$\sum_{\substack{X\le n\le 2X\\ l(n)=(2+o(1))\log\log x}} d(n) = \frac{X}{(\log X)^{(2\log 2-1+o(1))}} 2^{(2+o(1))\log\log X} = X\log^{1+o(1)} X$$

corresponding our expectation except $o(1)$ in the exponent of $\log X$. Thus, intuitively, the principal contribution to the sum comes from integers with approximately $2(1+o(1))\log\log X$ prime factors. For any $\epsilon > 0$, let

$$B := \{n \in [X, 2X] : |\Omega(n) - 2\log\log X| \le \epsilon\log\log X\}.$$

Now we give the rigorous statement.

***Lemma* 6.3.** *Let $X > 0$ be sufficiently large. Then*

$$\sum_{\substack{X<n\le 2X\\ n\notin B}} d(n) \ll X\log X(\log X)^{-\frac{1}{100}\min\{1,\epsilon^2\}}.$$

Proof. See [IV, Lemma 2.3]. □

Remark. Restricting our focus on the set B is the key point in article [IV]. The advantage is to obtain a better upper bound after applying a large-sieve type estimate. In the next section, we will compare the long average to short average of the sum of the divisor function in the L_2 sense, then applying a large-sieve type estimate, after getting a Parserval's bound (see Lemma 6.4). The large-sieve type upper bound consists of a diagonal term and an off-diagonal term. The off-diagonal term will be easily controlled by Henriot's bound (see [30, Theorem 3]). However, in case of the diagonal term, i.e., $\sum_{X<n\leq 2X} d^2(n)$, if we directly apply Shiu's bound, as Mangerel did, we will get an upper bound $X(\log X)^{k^2-1}$ which is much larger than we expect. Fortunately, when we consider those integers in B, we have a pointwise upper bound $d(n) \leq (\log X)^{2\log 2+\epsilon}$ which means that we get a better upper bound

$$\sum_{\substack{X<n\leq 2X\\ n\in B}} d^2(n) \ll X(\log X)^{2\log 2+1+\epsilon}$$

for the diagonal term.

For convenience, we denote $S = A \cap B$ in the following and we will restrict our summation to the set S.

Recall (6.2) which implies that for $h \geq X^{1/2}$,

$$\frac{1}{h}\sum_{X<n\leq X+h} d(n) = (1+o(1))\log X.$$

Thus, by the above and Lemmas 6.1 and 6.3, we only need to prove that, for

$$(\log X)^{k\log k-k+1+\epsilon} \leq h \leq X^{1/2},$$

we have

$$\frac{1}{h}\sum_{\substack{x<n\leq x+h\\ n\in S}} d(n) - \frac{1}{X^{1/2}}\sum_{\substack{X<n\leq X+X^{1/2}\\ n\in S}} d(n) = o(\log X)$$

holds almost all $x \in [X, 2X]$.

6.3.2 Parserval's bound and the Matomäki-Radziwiłł method

In order to compare the short sum to the long sum of the divisor function on average, we study

$$\frac{1}{X}\int_X^{2X}\left|\frac{1}{h}\sum_{\substack{x<n\leq x+h\\ n\in S}} d(n) - \frac{1}{X^{1/2}}\sum_{\substack{x<n\leq x+X^{1/2}\\ n\in S}} d(n)\right|^2 dx. \tag{6.6}$$

One can now transfer the two discrete sums inside the integral of (6.6) to the integral of two Dirichlet Polynomials, by applying Perron's formula (see, e.g., [26, Lemma 1.1])

$$\sum_{x\le n\le y} d(n) = \lim_{T\to\infty}\int_{-T}^{T}\sum_{x/2<n<2y}\frac{d(n)}{n^{1+it}}\frac{y^{1+it}-x^{1+it}}{1+it}dt.$$

The principal contribution of the above integral comes from the integral around $t = 0$. Note that when $t\approx 0$, we have $y^{1+it}-x^{1+it}\approx y-x$. Therefore, if one applies Perron's formula to the averages of $\frac{1}{X^{1/2}}\sum_{x<n\le x+X^{1/2}} d(n)$ and $\frac{1}{h}\sum_{x<n\le x+h} d(n)$ separately, then the principal contributions (integrals around $t = 0$) of the integrals of the two Dirichlet polynomials are eliminated. Then the remaining task to bound (6.6) is to bound

$$\frac{1}{X}\int_X^{2X}\left|\int_{T_0}^{\infty}\sum_{x/2<n<2y}\frac{d(n)}{n^{1+it}}\frac{y^{1+it}-x^{1+it}}{(y-x)(1+it)}dt\right|^2 dx$$

for some $y\in\{x+h, x+X^{1/2}\}$. This can be bounded well by a "large sieve-type" argument. The above process can be formulated as the following lemma.

***Lemma* 6.4.**

$$\frac{1}{X}\int_X^{2X}\left|\frac{1}{h}\sum_{\substack{x<n\le x+h\\ n\in S}} d(n) - \frac{1}{X^{1/2}}\sum_{\substack{x<n\le x+X^{1/2}\\ n\in S}} d(n)\right|^2 dx.$$

$$\ll \int_{X^{1/6}\le|t|\le X/h}\left|\sum_{\substack{X<n\le 4X\\ n\in S}}\frac{d(n)}{n^{1+it}}\right|^2 dt + \max_{T\ge X/h}\frac{X}{hT}\int_T^{2T}\left|\sum_{\substack{X<n\le 4X\\ n\in S}}\frac{d(n)}{n^{1+it}}\right|^2 dt. \tag{6.7}$$

Proof. See [49, Lemma 14]. □

By looking at the integral of the Dirichlet polynomial, we may have several ideas in mind. Firstly, one could consider to apply the mean-value theorem ([IV, Lemma 4.1]) which is effective when Th/X is sufficiently large. The key challenge arises when T is close to X/h as the mean-value theorem is no longer sufficient. Fortunately, thanks to the Matomäki-Radiziwłł method, we are able to get a better upper bound beyond the limitation of using the mean-value theorem.

Dirichlet polynomials and Ramare's identity

We start from the following lemma giving a pointwise upper bound for a Dirichlet polynomial over primes.

***Lemma* 6.5.** *Let* $\epsilon > 0$*,* $A > 0$ *be given and* $X \geq 1$*. Assume that* $\exp((\log X)^{2/3+\epsilon}) \leq P \leq Q \leq X$*. Then for* $|t| \leq X$*,*

$$\sum_{P \leq p \leq Q} \frac{1}{p^{1+it}} \ll \frac{\log X}{1+|t|} + \exp\left(-\frac{\log P}{(\log X)^{2/3+\epsilon}}\right) + \frac{1}{P^{1/2}}. \tag{6.8}$$

Proof. See [53, Lemma 2]. □

To apply Lemma 6.5, we need the following decomposition called Ramaré's identity, which follows from the multiplicativity. We leave the proof to the interested reader.

***Lemma* 6.6.** *Suppose that* $f : \mathbb{N} \to \mathbb{C}$ *is a multiplicative function. Then, for any* $X \geq Q \geq P \geq 1$*,*

$$\begin{aligned}\sum_{n \leq X} f(n) = &\sum_{p \in [P,Q]} f(p) \sum_{m \leq X/p} \frac{f(m)}{1 + \sum_{q \in [P,Q] \setminus \{p\}} \mathbf{1}_{q|m}} \\ &+ \sum_{p \in [P,Q]} \sum_{m \leq X/p^2} \frac{f(p^2 m) - f(p) f(pm)}{1 + \sum_{q \in [P,Q] \setminus \{p\}} \mathbf{1}_{q|m}} + \sum_{\substack{n \leq X \\ p|n \Rightarrow p \notin [P,Q]}} f(n).\end{aligned}$$

where p, q *are primes.*

Now by Lemma 6.6 with slight modification, we have for $\mathcal{T} \subset [-T, T]$

$$\begin{aligned}&\int_{\mathcal{T}} \left| \sum_{\substack{n \in S \\ X < n \leq 4X}} \frac{d(n)}{n^{1+it}} \right|^2 dt \\ \ll &H \log\left(\frac{Q}{P}\right) \times \sum_{v \in \mathcal{I}} \int_{\mathcal{T}} |Q_{v,H}(1+it) R_{v,H}(1+it)|^2 \, dt + \text{ERROR},\end{aligned}$$

where

$$Q_{v,H}(s) := \sum_{\substack{P \leq p \leq Q \\ e^{v/H} < p \leq e^{(v+1)/H}}} \frac{1}{p^s},$$

$R_{v,H}(s)$ a Dirichlet polynomial whose explicit form is not important, the "ERROR" is acceptable and $\mathcal{I}$ is the interval $\lfloor H \log P \rfloor \leq v \leq H \log Q$. Thanks to Lemma 6.5, we may be able to evaluate the above integral by combining a pointwise upper bound of $Q_{v,H}(1+it)$ with an upper bound for the mean-value of $|R_{v,H}(1+it)|^2$.

Matomäki-Radziwiłł decomposition

Now we apply a Matomäki-Radziwiłł type decomposition to deal with the integral

$$\int_{X^{1/6}}^{X/h} |Q_{v,H}(1+it)R_{v,H}(1+it)|^2 dt.$$

Let P_1, Q_1, P_2, Q_2 be as in (6.4) and (6.5), and

$$[P_3, Q_3] = [\exp((\log X)^{9/10}), \exp((\log X)/(\log\log X)^{100})],$$

$\alpha_1 = \frac{1}{4} - \frac{1}{50}$, $\alpha_2 = \frac{1}{4} - \frac{1}{100}$ and $H = P_1^{1/6}$. For any $s \in \mathbb{C}$, let

$$Q_{v,j}(s) := \sum_{\substack{P_j \le p \le Q_j \\ e^{v/H} < p \le e^{(v+1)/H}}} \frac{1}{p^s}$$

We split $[X^{1/6}, X/h] = [T_0, T]$ into a disjoint union, where

$$\begin{aligned}
\mathcal{T}_1 &= \{t \in [T_0, T] : |Q_{v,1}(1+it)| \le e^{-\alpha_1 v/H} \text{ for all } v \in \mathcal{I}_1\} \\
\mathcal{T}_2 &= \{t \in [T_0, T] : |Q_{v,2}(1+it)| \le e^{-\alpha_2 v/H} \text{ for all } v \in \mathcal{I}_2\} \setminus \mathcal{T}_1 \\
\mathcal{T}_3 &= [T_0, T_1] \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)
\end{aligned}$$

We now briefly discuss how to use the Matomäki-Radziwiłł method to deal with

$$\int_{\mathcal{T}_j} |Q_{v,j}(1+it)R_{v,j}(1+it)|^2 dt$$

for $j \in \{1, 2, 3\}$.

- $\int_{\mathcal{T}_1}$: In this case, by the definition of $\mathcal{T}_1$, we have a uniform upper bound for $|Q_{v,1}(1+it)|$, so we can extract $|Q_{v,1}(1+it)|^2$ out from the integral and bound it by its uniform upper bound. Since $Q_1 < H$ it then suffices to apply the mean-value theorem to the remaining integral $\int_{\mathcal{T}_1} |R_{v,1}(1+it)|^2 dt$.

- $\int_{\mathcal{T}_2}$: In this case, by the definition of $\mathcal{T}_2$, we have a uniform upper bound for $|Q_{v,2}(1+it)|$. Additionally, for any $t \in \mathcal{T}_2$, there exist some $r \in \mathcal{I}_1$ fulfilling the pointwise lower bound $|Q_{r,1}(1+it)| > e^{-\alpha_1 r/H}$. Then we apply Matomäki-Radziwiłł amplification technique to bound $\int_{\mathcal{T}_2}$ by

$$e^{-2\alpha_2 v/H} \sum_{r \in \mathcal{I}_1} e^{2l\alpha_1 r/H} \int_{\mathcal{T}_{2,r}} |Q_{r,1}(1+it)|^{2l} \times |R_{v,2}(1+it)|^2 dt,$$

  where $\mathcal{T}_{2,r} = \{t \in \mathcal{T}_2 : |Q_{r,1}(1+it)| > e^{-\alpha_1 r/H}\}$ and l is chosen so that the coefficients of the Dirichlet polynomial $Q_{r,1}(1+it)^l R_{v,2}(1+it)$ are supported around $n \asymp X$. Now, the upper bound from the mean-value theorem will not waste a lot.

- $\int_{\mathcal{T}_3}$: In this case, for every $t \in \mathcal{T}_3$, we have a pointwise lower bound for $Q_{v,2}(1+it)$ for some $v \in \mathcal{I}_2$. Hence, we can apply a large value result for Dirichlet polynomials, for example [49, Lemma 8], to show that $\mathcal{T}_3$ has small measure. Now we bound the integral by a discrete sum, namely

$$\int_{\mathcal{T}_3} |Q_{v,3}(1+it)R_{v,3}(1+it)|^2 dt \leq \sum_{t\in\mathcal{T}_3'} |Q_{v,3}(1+it)R_{v,3}(1+it)|^2,$$

  where $\mathcal{T}_3'$ is a set of well-spaced points in $\mathcal{T}_3$. By Lemma 6.5, $|Q_{v,3}(1+it)| \ll \frac{1}{\log^A X}$ for any $A > 0$, so we can extract $|Q_{v,3}(1+it)|^2$ out from above sum and bound it uniformly. Then we use a discrete mean-value theorem (see e.g. [33, Theorem 9.6]) to bound $\sum_{t\in\mathcal{T}_3'} |R_{v,3}(1+it)|^2$.

For the general case of the d_k bounded multiplicative functions, we need to use a Halász type result and more complicated arguments. The interested reader may refer to [IV] for more details.

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