TURUN YLIOPISTO
UNIVERSITY OF TURKU

ON THE TOPOLOGICAL ENTROPY AND LYAPUNOV EXPONENTS OF CELLULAR AUTOMATA

Decision Problems, Dynamical Properties and Generalizations

Toni Hotanen

TURUN YLIOPISTON JULKAISUJA – ANNALES UNIVERSITATIS TURKUENSIS
SARJA – SER. AI OSA – TOM. 722 | ASTRONOMICA – CHEMICA – PHYSICA – MATHEMATICA | TURKU 2024

TURUN YLIOPISTO
UNIVERSITY OF TURKU

ON THE TOPOLOGICAL ENTROPY AND LYAPUNOV EXPONENTS OF CELLULAR AUTOMATA

Decision Problems, Dynamical Properties and Generalizations

Toni Hotanen

TURUN YLIOPISTON JULKAISUJA – ANNALES UNIVERSITATIS TURKUENSIS
SARJA – SER. AI OSA – TOM. 722 | ASTRONOMICA – CHEMICA – PHYSICA – MATHEMATICA | TURKU 2024

University of Turku

Faculty of Science
Department of Mathematics and Statistics
Mathematics
Doctoral Programme in Exact Sciences (EXACTUS)

Supervised by

Professor, Jarkko Kari
Department of Mathematics and Statistics
University of Turku
Turku, Finland

Docent, Ville Salo
Department of Mathematics and Statistics
University of Turku
Turku, Finland

Reviewed by

Professor, Nicolas Ollinger
Université d'Orléans

Dr., Silvio Capobianco
Tallinn University of Technology

Opponent

Dr., Julien Cassaigne
Centre National Recherche Scientifique (CNRS)

ISBN 978-951-29-9806-7 (PRINT)
ISBN 978-951-29-9807-4 (PDF)
ISSN 0082-7002 (PRINT)
ISSN 2343-3175 (ONLINE)
Painosalama, Turku, Finland, 2024

To Amelia & Natalia

UNIVERSITY OF TURKU
Faculty of Science
Department of Mathematics and Statistics
Mathematics
HOTANEN, TONI: On the Topological Entropy and Lyapunov Exponents of Cellular Automata
Doctoral dissertation, 101 pp.
Doctoral Programme in Exact Sciences (EXACTUS)
August 2024

ABSTRACT

A dynamical system is a pair made out of a space of points and a function that determines how the points move in the space. Usually some extra properties are required from the space and the function to make them susceptible for studying. Topological dynamical systems require the space to be a compact metric space, meaning we can measure distances between the points and all sequences of points contain subsequences that converge to a single point. A function is required to be continuous meaning close enough points remain close to each other after one iteration of the function.

Cellular automata are examples of such systems. The space, called the configuration space, is made out of a regular lattice of symbols. Usually the lattice gets its structure from a group and the set of symbols is finite. The elements of the group are called cells and the symbols are called states. The continuous function, called the global rule, is made out of a neighbourhood vector and a local rule. The global rule applies the local rule for each cell independently and simultaneously and its value depends on the value of the cells in the neighbourhood of each cell.

Topological entropy is a measure of complexity of a given topological dynamical system. Simple systems tend to often have low or even zero entropy and by contrast complex systems often have high entropy. Two systems are called conjugate if they are equivalent in some sense. For example they have the same orbits and they share many dynamical properties. Conjugate systems have the same topological entropy, which makes it an useful value if we want to show that two given systems are not conjugate. The topological entropy has its measure-theoretic counterpart which we will also study. The entropies are related to each other by a variational principle.

Another measure of complexity are Lyapunov exponents. They can be thought of as the speed of the propagation of differences in a given system. The connections between Lyapunov exponents and various dynamical properties have been widely studied. In this dissertation we give an answer to the conjecture which states that a sensitive cellular automaton must have a configuration with a non-zero Lyapunov exponent. We construct a cellular automaton, which has no such configurations. The Lyapunov exponents are also related to measure-theoretic entropy. One can for instance calculate an upper bound for the measure-theoretic entropy of a given cellular automaton by first calculating the Lyapunov exponents. The Lyapunov exponents are only defined for one-dimensional cellular automata. In this dissertation we generalize the Lyapunov exponents for cellular automata over finitely generated groups and the measure-theoretic entropy for cellular automata over amenable groups and show that an analogous connection exists between them in the more general case.

A decision problem is a question with two possible answers: Yes or no. For example: "Is a given natural number a prime number?" and "Is a given route between two cities the shortest possible route?" are examples of decision problems. A decision problem is called decidable if there exists an algorithm (one can think of it as a computer program) that always gives the right answer to the problem for every possible input. The problem is called undecidable if no such algorithm exists. In this dissertation we study several decision problems related to reversible Turing machines, reversible cellular automata and group cellular automata. Namely these problems ask for example: "Is the topological entropy of a given system zero?" and "Are the Lyapunov exponents of a given system zero?". Some related decision problems are also considered.
KEYWORDS: cellular automaton, Turing machine, decision problem, decidable, undecidable, reversible, entropy, topological, measure-theoretic, Lyapunov exponents, sensitive, generalization, amenable, equivariant, endomorphism, automorphism

TURUN YLIOPISTO
Matemaattis-luonnontieteellinen tiedekunta
Matematiikan ja tilastotieteen laitos
Matematiikka
HOTANEN, TONI: On the Topological Entropy and Lyapunov Exponents of Cellular Automata
Väitöskirja, 101 s.
Eksaktien tieteiden tohtoriohjelma (EXACTUS)
Elokuu 2024

TIIVISTELMÄ

Dynaaminen systeemi on pari, joka muodostuu pisteistä koostuvasta avaruudesta sekä funktiosta, joka määrittelee miten kyseiset pisteet liikkuvat avaruudessa. Usein sekä avaruudelta että funktiolta vaaditaan joitakin ominaisuuksia, jotta niitä pystytään tutkimaan paremmin. Topologisten dynaamisten systeemien tapauksessa avaruuden täytyy olla kompakti metrinen avaruus, joka tarkoittaa sitä, että pystymme mittaamaan pisteiden välistä etäisyyttä ja jokaisesta pisteiden jonosta löytyy alijono, joka suppenee yksittäistä pistettä kohti. Funktion vaaditaan olevan jatkuva, joka tarkoittaa sitä, että tarpeeksi lähellä olevat pisteet kuvautuvat lähelle toisiaan kun funktiota toistetaan kerran.

Soluautomaatit ovat esimerkkejä tällaisista systeemeistä. Avaruus, jota kutsutaan konfiguraatioavaruudeksi, muodostuu säännöllisestä symbolien hilasta. Hila saa yleensä rakenteensa ryhmältä ja symboleja on äärellinen määrä. Ryhmän alkioita kutsutaan soluiksi ja symboleja kutsutaan tiloiksi. Jatkuva funktio, jota kutsutaan globaaliksi säännöksi, muodostuu naapurustovektorista ja lokaalista säännöstä. Globaali sääntö käyttää lokaalia sääntöä joka solulle itsenäisesti ja samanaikaisesti ja sen arvo määräytyy jokaiselle solulle niiden naapurustojen solujen arvosta.

Topologinen entropia on kompleksisuuden mitta topologisille dynaamisille systeemeille. Yksinkertaisilla systeemeillä on usein matala tai jopa nolla entropia, kun taas monimutkaisilla systeemeillä on usein korkea entropia. Kaksi systeemiä ovat konjugaatteja, jos ne ovat tietyssä mielessä ekvivalentteja. Esimerkiksi niillä on samat kiertoradat ja niillä on monta samaa dynaamista ominaisuutta. Konjugaateilla systeemeillä on sama topologinen entropia ja siitä syystä kyseinen arvo on hyödyllinen jos halutaan näyttää, että kaksi annettua systeemiä eivät ole konjugaatteja. Topologisella entropialla on mittateoreettinen vastine, jota me myös tutkimme. Entropiat ovat yhteydessä toisiinsa variaatioperiaatteen nojalla.

Muita kompleksisuuden mittoja ovat Lyapunovin eksponentit. Niitä voi ajatella muutosten leviämisen nopeuksina annetussa systeemissä. Lyapunovin eksponenttien yhteyksiä erilaisiin dynaamisiin ominaisuuksiin on tutkittu laajasti. Tässä väitöskirjassa vastaamme konjektuuriin, joka väittää, että herkillä soluautomaateilla on aina olemassa jokin konfiguraatio, jonka Lyapunovin eksponentit eivät ole nollia. Me konstruoimme soluautomaatin, jolla tällaisia konfiguraatioita ei ole olemassa. Lyapunovin eksponentit liittyvät mittateoreettiseen entropiaan. Mittateoreettiselle entropialle voidaan esimerkiksi laskea yläraja laskemalla ensin Lyapunovin eksponentit. Lyapunovin eksponentit ovat määritelty vain yksiulotteisille soluautomaateille.

Tässä väitöskirjassa yleistämme Lyapunovin eksponentit soluautomaateille yli äärellisesti generoitujen ryhmien ja mittateoreettisen entropian "myöntyvien" ryhmien yli. Tämän lisäksi näytämme, että niiden välillä on vastaavanlainen yhteys yleisemmässäkin tapauksessa.

Päätösongelma on kysymys, jolle on kaksi mahdollista vastausta: Kyllä tai ei. Esimerkiksi: "Onko annettu luonnollinen luku alkuluku?" ja "Onko annettu reitti kahden kaupungin välillä lyhyin mahdollinen reitti?" ovat esimerkkejä päätösongelmista. Päätösongelma on ratkeava, jos on olemassa algoritmi (joka voidaan ajatella tietokoneohjelmana), joka antaa aina oikean vastauksen ongelmaan sen jokaisella syötteellä. Ongelma on ratkeamaton, jos sellaista algoritmia ei ole olemassa. Väitöskirjassa tutkimme joitakin päätösongelmia, jotka liittyvät kääntyviin Turingin koneisiin, kääntyviin soluautomatteihin ja ryhmäsoluautomaatteihin. Tarkemmin sanottuna ongelmiin kuuluvat esimerkiksi: "Onko annetun systeemin topologinen entropia nolla?" ja "Onko annetun systeemin Lyapunovin eksponentit nollat?". Myös joitakin näihin liittyviä ongelmia tarkastellaan.
ASIASANAT: soluautomaatti, Turingin kone, päätösongelma, ratkeava, ratkeamaton, kääntyvä, entropia, topologinen, mittateoreettinen, Lyapunovin eksponentit, herkkä, yleistys, myöntyvä, ekvivarianttti, endomorfismi, automorfismi

Acknowledgements

Standing on the shoulders of giants is an apt metaphor to express my feelings throughout the journey of my PhD studies. Not only have I benefited from all the written knowledge out there by many beautiful minds, but in addition, I have been fortunate to participate in many stimulating conversations as well. As such there are many people to thank for making this dissertation possible.

First I would like to express my sincerest gratitude to both my supervisors, Professor Jarkko Kari and Docent Ville Salo. Both of them possess a wealth of knowledge and their guidance has been invaluable. Their remarks have improved all my works, this dissertation being no exception. Furthermore, it was most likely Jarkko's courses surrounding the topics of this dissertation that got me interested and made me want to pursue this degree in the first place. Therefore without him, none of these pages would ever have been written.

I am deeply grateful to both of the reviewers of this dissertation, Professor Nicolas Ollinger and Doctor Silvio Capobianco for taking the time to read this work and provide valuable comments and suggestions. I would also like to thank Doctor Julien Cassaigne for being my opponent. I am honoured.

My thanks to the head of the Department of Mathematics and Statistics, Professor Iiro Honkala for providing a great place to work at. Thanks to University of Turku and all the professors, lecturers and colleagues that I have ever interacted with, you have all been tremendous. I am thankful for the financial support received from the emmy.network foundation and Academy of Finland project number 354965.

Lastly, but most importantly. My sincerest thanks to all my family, extended family and friends. I especially cherish the big family gatherings at Christmas in Rauma. Special thanks to my wife Natalia and my daughter Amelia for their love and bringing so much joy into my life. I love you more than anything!

August 2024
Toni Hotanen

Table of Contents

List of Original Publications

This monograph is based on research reported in the following original publications:

I Toni Hotanen. Everywhere Zero Pointwise Lyapunov Exponents for Sensitive Cellular Automata. AUTOMATA 2020: Cellular Automata and Discrete Complex Systems, 2020; 26th IFIP WG 1.5 International Workshop, AUTOMATA 2020, Stockholm, Sweden, August 10–12, 2020, Proceedings: pages 71–85.

II Toni Hotanen. Generalized measure-theoretic entropy and Lyapunov exponents for G-equivariant functions. Proceedings of the Sixth Russian-Finnish Symposium on Discrete Mathematics, 2021; pages 71–82.

III Toni Hotanen. Undecidability of the Topological Entropy of Reversible Cellular Automata and Related Problems. UCNC 2023: Unconventional Computation and Natural Computation, 2023; 20th International Conference, UCNC 2023, Jacksonville, FL, USA, March 13–17, 2023, Proceedings: pages 108–123.

IV Toni Hotanen. Decision Problems Related to Topological Entropy for Specific Classes of Cellular Automata and Turing Machines. Submitted in 2024.

V Toni Hotanen. Generalized Measure-Theoretic Entropy and Lyapunov Exponents for Equivariant Dynamical Systems. Submitted in 2024.

1 Introduction

This thesis is focused on dynamical and computational results related to both the measure-theoretic and topological entropy and Lyapunov exponents. The dynamical systems we have studied consist of Turing machines, cellular automata and more general equivariant systems. The computational results consider reversible systems and the dynamical results consider more general systems. Before diving into the results any further let us begin by first introducing cellular automata, where most of our interest lies in.

Cellular automata are discrete dynamical systems. Their space consists of configurations, which are regular lattices of cells, each cell being assigned a symbol. The set of all possible symbols where the cells take values from is finite. These symbols are also referred to as states. The regular lattice usually comes from a specific mathematical structure, which very often is a group. In the most classical setting, the group is the additive set of integers and in such case we are talking about one-dimensional cellular automata. Configurations of one-dimensional cellular automata can be visually represented by a bi-infinite horizontal row of coloured squares, where each color represents a specific symbol. The dynamics of the cellular automata arise from a finitely described local rule, which maps a finite sequence of symbols to a single symbol. The sequence of symbols is read for each cell based on the content of the neighbouring cells, which are specified by a neighbourhood vector. The global rule applies the local rule at each cell independently and simultaneously. A space-time diagram of a one-dimensional cellular automaton can be represented as an infinite grid of coloured squares, where each horizontal row represents a configuration. For two consecutive rows, the row below of another one is the configuration where the global rule has been applied to the configuration on the row above it. This is repeated on each row and hence one can study the orbit of a single configuration looking at its space-time diagram. Of course in reality we can visually represent only a finite section of configurations and their space-time diagrams. Even so, the visualizations can help us to understand the systems better.

Let us look at a specific example. Consider a one-dimensional cellular automaton, where each configuration consists of cells that are either at a state 0 or a state 1. Let us pick a local rule that maps the sequences 111, 110, 101 and 000 to the state 0 and maps the sequences 100, 011, 010 and 001 to the state 1. The neighbourhood vector consists of -1, 0 and 1, so each of the cells look at the state they are in and the states of the nearest neighbours on the left and the right side in order from left to right. For example if the cells at the indices 0, 1 and 2 are at the states 0, 1 and 1 respectively, then at the next time-step, i.e. when the global rule is applied once the index 1 will be at state 1 because 011 maps to 1. This cellular automaton is one of the 256 elementary cellular automata (ECA). It is called Rule 30, because the sequence of outputs 00011110 represents 30 in binary base. The outputs in the sequence are ordered by the inputs viewed as binary representations of integers from 7 to 0. Let us represent 0s as white squares and 1s as black squares so we can give a visual example. The Figure 1 represents a part of the space-time diagram of a configuration that has a single cell at state 1.

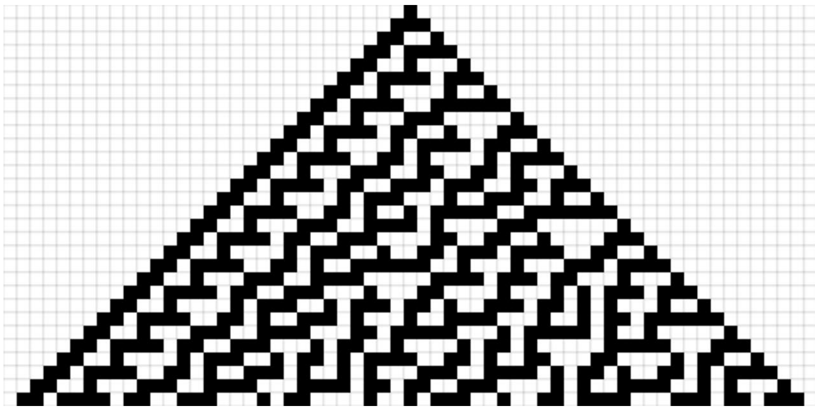

Figure 1. A space-time diagram of the ECA Rule 30 for a configuration with a single cell in state 1 and the rest in state 0.

If we study the space-time diagram in Figure 1 we can notice several things. First of all the diameter of the set of the cells in state 1 increases at each time-step. This is to say that the information that originated from a single cell seems to propagate at constant speed towards both directions. On the other hand the space-time diagram of a configuration where each cell is at state 0, would consist entirely of white squares. This is because the configuration is a constant of Rule 30, which follows from the fact that the local rule maps 000 to 0. We do not need a visualization of this, but one can imagine how the space-time diagram looks like. Then if we compare these two space-time diagrams we notice that a very simple change to a configuration can have a tremendous effect to its orbit. This tells us of two dynamical phenomena. The Lyapunov exponents, which tell us how fast a difference can propagate towards origin from the left or the right side when we compare the orbits of specific configurations to an orbit of the given configuration, of the ”all zeros” -configuration are positive. Concordantly no matter how close we look, i.e. how large radius of zeroes we pick, we can always find a configuration such that its orbit contains a configuration such that the cell at index zero is at state 1. This is clear from Figure 1, as we can pick a configuration where the single cell at state 1 is located at any coordinate that we want. This hints us towards a notion of sensitivity. Sensitivity means that for each configuration one can always find another configuration arbitrarily close to it such that their orbits will be very far from each other at some time-step. In fact Rule 30 is sensitive. This can be shown by the fact the local rule is left permutive. This means that if two configurations differ only at index i, their images under the global rule will differ at index $i + 1$. Therefore for any configuration one can pick another one simply by changing the content of a single cell at an appropriately chosen negative index and the orbits of those two configurations will differ at the origin at some time-step.

As the notions of Lyapunov exponents and sensitivity are both related to a propagation of differences, it is natural to study how correlated they are and in which systems. For example does sensitivity imply a positive lower bound for Lyapunov exponents? Or does the existence of a positive pointwise Lyapunov exponent imply something vis-à-vis sensitivity? Similarly it is interesting to study the relations between the Lyapunov exponents and other dynamical properties as well. The properties we are particularly interested in include for example equicontinuity, which means that the orbits of close enough configurations stay close to each other at each time-step. Another interesting property is transitivity, which means that for any pair of open sets A and B, one can find a time-step such that the orbit of A intersects with the set B. We also consider expansivity, which means that orbits of any two different configurations will be far from each other at some time-step. It was shown by Michele Finelli, Giovanni Manzini and Luciano Margara in [19] that the set of the shift-invariant Lyapunov exponents of expansive cellular automata is bounded from below by a constant value. The average Lyapunov exponents of equicontinuous cellular automata were shown always to be zero in [79] by Pierre Tisseur. Additionally, although not explicitly stated, one can see from the results of [15] by Michele D'Amico, Giovanni Manzini and Luciano Margara that the global Lyapunov exponents (which in the case of linear cellular automata are equal to the shift-invariant Lyapunov exponents for any configuration) of sensitive linear cellular automata are always positive. The latter result was generalized to the class of group cellular automata by the author in [33]. In general this connection does not hold. It was conjectured in [9] by Xavier Bressaud and Pierre Tisseur and the same conjecture was later restated in [55] by Petr Kůrka (Conjectures 3 and 11 respectively) that sensitivity always implies the existence of a configuration with a non-zero pointwise Lyapunov exponent. This conjecture was proven false in [30] by the author of which chapter 4.3 is based on.

Lyapunov exponents can be also thought of as a measure of complexity of a given system. To justify this viewpoint, we can see that if we take subsystems by decreasing the amount of configurations, then the Lyapunov exponents can not increase but they can decrease. Another related notion of complexity is the topological entropy.

Similarly, the topological entropy of a subsystem, is bounded from above by the topological entropy of the original system. Hence one might assume that complex cellular automata have higher entropy and Lyapunov exponents than simpler ones, which is true to some extent as we already witnessed in the previous paragraph in the case of Lyapunov exponents. Same is true for entropy. For example surjective equicontinuous cellular automata have zero entropy as was shown in [79]. And in [15] it was shown that the topological entropy of expansive cellular automata is always positive and, in fact, its exact value is given. The topological entropy of a dynamical system is an important invariant under conjugacy, meaning conjugate dynamical systems have the same topological entropy. Ergo we can immediately tell non-conjugate systems apart if we know that they have different entropies. The converse of this does not hold however. Lyapunov exponents and entropy are closely related. For example in the case of surjective one-dimensional cellular automata, the topological entropy is bounded from above by the product of the sum of the global Lyapunov exponents and the logarithm of the size of the set of states as was shown in [79].

Computability theory is a branch of mathematics and computer science that studies whether or not certain problems can be solved with an effective procedure or, in another words, an algorithm. Alternatively one can think of an algorithm less abstractly as a computer program. The types of questions that can be asked for example are: Can we build a computer program that takes an arbitrary cellular automaton as an input and outputs its topological entropy? Or can we at least build a computer program that tells if the topological entropy is zero or not? The latter problem is an example of a decision problem. In the general form a decision problem is a question about a set of objects where the answer is always binary: Yes or no. If a decision problem can be solved by an algorithm, we say that the problem is decidable and if no such algorithm can possibly exist then it is undecidable. The most famous decision problem is called the Halting problem. It asks if a given program ever halts its processing with a given input. This problem was shown to be undecidable by Alan Turing in [81]. Many decision problems can be shown to be undecidable by a reduction to the Halting problem. This means that for such problems, one can show that if an algorithm exists that solves the problem, then it would imply an existence of an algorithm that solves the Halting problem, which is a contradiction.

Many problems in mathematics can be solved with an algorithm. For example we can build an algorithm that sorts a finite set of rational numbers and we can tell if a given number from said set is the smallest or the largest number of that set. We can build an algorithm that calculates the determinant of any given rational square matrix. When given a finite amount of roads and cities, we can build an algorithm that finds the shortest route visiting a set of given destination cities, i.e. the Travelling Salesman Problem. Here we notice that when we said an effective procedure, we did not imply anything about the computational complexity of the algorithm. In this context we do not seek practical algorithms, so the algorithms can take as much time and space as they need, i.e. they can be in P, NP or EXPSPACE, etc and it does not make any difference to us. We just simply care if an algorithm exists or not.

While many mathematical problems can be solved algorithmically, we note that this is often not the case in the context of cellular automata. Perhaps the first decision problems that were shown to be undecidable by Karel Culik II and Sheng Yu in [14] concerned the membership problems with respect to a set of categories given by a classification schema, i.e. does a given cellular automaton belong into a certain category. Since then the decision problems regarding cellular automata have been widely studied. The questions about dynamical properties of cellular automata turn out often, if not always, to be undecidable in the most general framework. For example the problem that asks if a given cellular automaton is nilpotent was proven undecidable independently in [44] by Jarkko Kari and in [1] by Stål Aanderaa and Harry Lewis. In [17] by Bruno Durand, Enrico Formenti, and Georges Varouchas it was shown that it is undecidable if a given cellular automaton is equicontinuous and likewise for sensitivity. It was shown that the conjugacy of two given one-dimensional cellular automata is undecidable in [38] by Joonatan Jalonen and Jarkko Kari. In [35] by Lyman P. Hurd, Jarkko Kari, and Karel Culik it was shown that the topological entropy can not be estimated to a given precision. In the same paper it was also shown that one can not decide if the topological entropy is zero or not. The underlying group might sometimes affect the decidability of a problem. For example the problems that ask if a given cellular automaton is injective or surjective are decidable for one-dimensional cellular as shown in [2] by Serafino Amoroso and Yale N. Patt, but undecidable for multidimensional cellular automata as shown in [43] and [45] by Jarkko Kari.

Since everything seems undecidable in the most general framework we can restrict ourselves to specific classes of cellular automata and try to find such classes where problems become decidable. For example it can be shown that many questions can be answered algorithmically when restricted to the class of linear cellular automata. It was shown in [12] by Gianpiero Cattaneo, Enrico Formenti, Giovanni Manzini, and Luciano Margara, [61] and [62] by Giovanni Manzini and Luciano Margara that the dynamical properties of equicontinuity, sensitivity and transitivity of multidimensional linear cellular automata and the expansivity of one-dimensional linear cellular automata can all be effectively deduced from the local rule. Furthermore one can decide if a given multidimensional linear cellular automaton is surjective or injective as was shown in [37] by Masanobu Itô, Nobuyasu Ôsato and Masakazu Nasu. In [15] an exact formula was given for the topological entropy and shift-invariant Lyapunov exponents for a given one-dimensional linear cellular automaton. These can again be calculated directly from the local rule. It was also shown in [15] that the topological entropy of a given multidimensional linear cellular automaton is zero or infinite in higher dimensions than one. And since it was already shown by [62] that multidimensional linear cellular automata are either sensitive or equicontinuous it follows that one can decide if the topological entropy is zero in such setting. Recently the decision problems have been studied in the more general framework of group cellular automata. It was shown in [6] by Pierre Béaur and Jarkko Kari that surjectivity, injectivity, nilpotency, periodicity, eventual periodicity, equicontinuity and sensitivity are all decidable properties for a given multidimensional group cellular automaton. It was shown in [33] that one can decide if the topological entropy or the values of both Lyapunov exponents of a given one-dimensional group cellular automaton are zero or not. The chapter 4.2.2 is based on [33].

On the other hand we can show that some problems remain undecidable even in the more restricted classes. For example the class of reversible cellular automata has been well studied. The decision problems regarding sensitivity, transitivity and mixingness were proven undecidable in [59] by Ville Lukkarila. It was shown that the conjugacy of two given reversible one-dimensional cellular automata is undecidable in [74] by Ville Salo. It was shown in [52] by Johan Kopra that the Lyapunov exponents of a given one-dimensional cellular automaton can not be estimated up to a given precision in the class of reversible cellular automata. In the same paper it was left as an open question whether one can decide if the both global Lyapunov exponents are zero or not. This question was answered in [32] by the author where it was shown that the problem is undecidable. The analogous question concerning if the topological entropy for a given reversible cellular automaton was zero was asked in [35]. This was answered also in [32] where it was shown that the problem is undecidable. The paper [33] is an extended and improved version of [32]. Chapter 4 is partly based on [32] and [33].

Many decision problems in cellular automata have been proven undecidable by reductions to decision problems concerning Turing machines. A Turing machine is a mathematical model of computation. Any algorithm can be implemented as a Turing machine. Turing machines contain two main parts. A computational unit, called the Turing machine head and a tape, which is defined in exactly the same way as a configuration of a (one-dimensional) cellular automaton. In this context the set of symbols from which the cells of the configuration can take values from is called the tape alphabet. The Turing machine head stores a location, which is the cell on the tape it is currently located at and it also stores a state from a finite set of states. The machine then reads the symbol of the cell it is currently located at and the state it is in and determines what it does next based on those two values. It can either move to a new location or change the content of the cell it is located at to a new symbol (or both at the same step, but we do not allow these to happen concurrently in this work). Typically the set of states is split into the initial state, accepting states and possibly rejecting states and the tape alphabet would contain a unique blank symbol. Then it would be interesting to know which language, i.e. a set of finite words, does the Turing machine recognize. This language consists of words such that when input to the Turing machine, the machine halts in an accepting state. The input word is written on the tape in positions from cells 0 to a non-negative n surrounded by the blank symbol in the rest of the tape. The machine would then start the computation with the Turing machine head starting at cell 0 in the initial state. If the Turing machine head eventually reaches an accepting state, the computation stops and the word is accepted. In fact an algorithm to a decision problem could be formalized as a Turing machine. The language it recognizes would be the set of encodings of instances for which a given statement is true for. A decision problem would then be decidable if such machine, which additionally halts with every input, exists.

In this work we will not be focused on this type of computational aspects of Turing machines. Instead we want to view Turing machines as dynamical systems and so we do not need to make the distinction between the different elements of the tape alphabet and the different states, which also slightly simplifies their definition. The study of Turing machines as dynamical systems was started by Cristopher Moore in [64] and Petr Kůrka in [56]. We will mostly focus on the decision problems related to some of the dynamical properties of reversible Turing machines. Previously it has been shown that it is undecidable if a given complete reversible Turing machine is periodic in [46] by Jarkko Kari and Nicolas Ollinger. In the same paper it was also shown that the immortality and reachability are undecidable properties. And it was also shown that it is undecidable if a given complete deterministic Turing machine or a given reversible Turing machine admits a periodic configuration. This last result was improved in [11] by Julien Cassaigne, Nicolas Ollinger, and Rodrigo Torres-Avilés where it was first of all shown that a complete aperiodic reversible Turing machine exists (their construction was named the SMART machine) answering positively to a conjecture in [46] and furthermore it was shown that it is undecidable if a given complete reversible Turing machine admits a periodic configuration answering positively to a second part of the same conjecture. It was shown in [23] by Anahí Gajardo, Nicolas Ollinger, and Rodrigo Torres-Avilés that minimality and transitivity are undecidable properties for complete reversible Turing machines. It was shown in [28] by Pierre Guillon and Ville Salo that it is undecidable if a given reversible Turing machine is distorted. It was shown in [21] by Anahí Gajardo, Nicolas Ollinger, and Rodrigo Torres-Avilés that the decision problem asking if the topological entropy of a given complete reversible Turing machine is zero or not is undecidable. Interestingly it was shown in [39] by Emmanuel Jeandel that there exists an algorithm that estimates the topological entropy and speed of a given Turing machine to a given precision. The questions if the speed of a given Turing machine is zero and the existence of a strictly weakly periodic point were shown to be undecidable in [32], which chapter 3 is based on. These results were independently proven in [80] by Rodrigo Torres-Avilés using a different construction and reduction.

Turing machines can be visualized in a similar manner to cellular automata. A

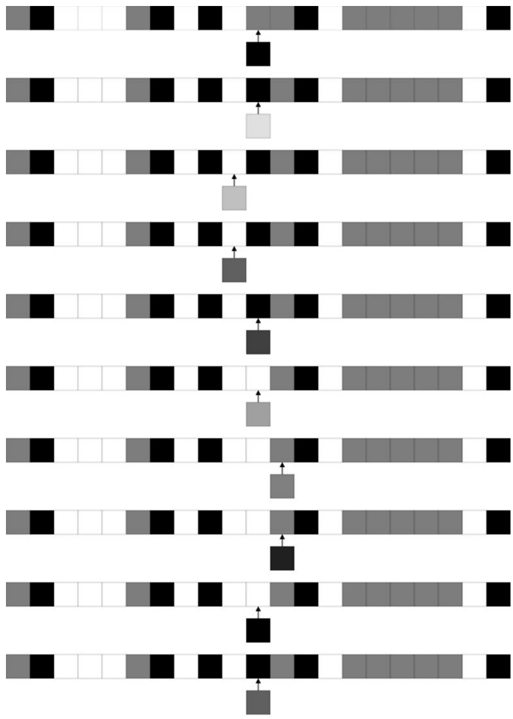

Figure 2. A space-time diagram of a random configuration of the complete reversible aperiodic Turing machine known as the SMART machine

tape is a single row of squares, that can be coloured or labelled. The Turing machine head is represented as a coloured or labelled square pointing to the cell where it is located at. In Figure 2 we have depicted a space-time diagram of a random configuration of the SMART machine (To be more precise, the original SMART machine consists of four states and the alphabet size is 3, the version of the figure uses the same alphabet, but has eight states. The machines are equivalent, but the original one allows simultaneous writing and moving).

Thus far we have focused on one-dimensional cellular automata (and Turing machines). More generally we can consider cellular automata over any given groups. That is to the say that the configurations are mappings from the elements of the group to a finite set of states and the neighbourhood vector contains elements of the group. Otherwise the definitions remain the same. The structure of the given group affects the dynamics and the computational properties of the cellular automata. For instance, it was shown in [76] by Mark A. Shereshevsky that there are no expansive cellular automata over the groups $\mathbb{Z}^d$, when $d > 1$. And as we stated earlier, surjectivity and injectivity are decidable properties over multidimensional cellular automata if and only if the dimension is one. It is interesting to find such classes of groups for which a certain property holds, but does not for the groups outside of the class. One particularly interesting class of groups is that of amenable groups, meaning the set of such groups that admit a left-invariant and finitely additive measure. Many equivalent definitions exist as well and we will use one of such in this work. Garden-of-Eden theorem states that a cellular automata is surjective if and only if it is pre-injective (which means that configurations that differ in finitely many cells must map to different configurations) and was first proven for one-dimensional cellular automata in [65] and [69] by Edward F. Moore and John R. Myhill respectively. What makes amenable groups particularly interesting is the fact that they are exactly the groups such that the Garden-of-Eden theorem holds: It was shown in [13] by Tullio G. Ceccherini-Silberstein, Antonio Machi, and Fabio Scarabotti that the Garden-of-Eden theorem holds for amenable groups. It was shown in [4] by Laurent Bartholdi that amenability is equivalent with the fact that surjectivity implies pre-injectivity, which means that the Garden-of-Eden theorem fails to hold for non-amenable groups.

One unfortunate fact about cellular automata over more complicated groups is that we can not present their space-time diagrams as conveniently as in the case of one-dimensional cellular automata. But we can visualize the configurations and space-time diagrams by Cayley graphs with coloured nodes. In the graphs we have nodes depicted as circles and edges between those nodes depicted as lines or arrows. The circles will be coloured based on what state the cell is at, just as in the case of the one-dimensional cellular automata. For example in Figure 3 we can see a part of a Cayley graph of a random configuration of a cellular automaton over the Heisenberg group. As one can notice it might be difficult to decipher how the nodes are connected to each other. Often it is easier to examine these kind of graphs with an interactive 3-D model. Nevertheless we will make due with the 2-D space given to us and instead of visuals we shall in any case focus on mathematical formalism.

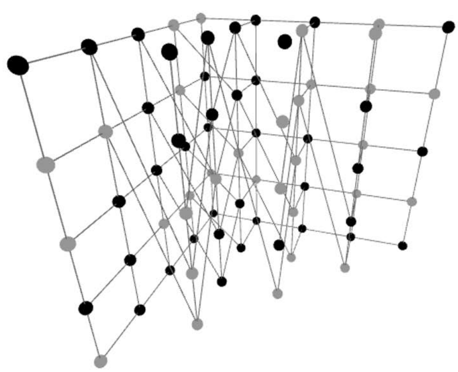

Figure 3. A part of a coloured Cayley graph of a random configuration of a cellular automaton over the Heisenberg group.

We can generalize cellular automata further by considering functions that commute with a given group action, i.e. equivariant dynamical systems. And yet another way to generalize our framework is by equipping our dynamical systems with a measure and considering them as measure-preserving dynamical systems. This is useful for example if we want to study the average behaviour of the system.

Topological entropy has its analogue in the setting of measure-preserving dynamical systems. The measure-theoretic entropy indicates how much uncertainty the system inhibits. And as is the case with the topological entropy, the measure-theoretic entropy is higher for more complex systems as it of course should be easier to predict the time-evolution of simpler systems. Like its topological analogue, it is also invariant under measure-theoretic conjugacies. This in fact has historical significance. For instance the question by John von Neumann from 1930s of whether the two Bernoulli shifts with distributions $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ were isomorphic was an open problem for a few decades. This was because there were no reliable tools to test if two given systems were isomorphic or not. Andrey Kolmogorov defined the measure-theoretic entropy and showed that if two different Bernoulli shifts have different entropies, then they can not be isomorphic. The converse of the problem, i.e. the question of whether two Bernoulli shifts with same entropies were necessarily isomorphic gained a lot of attention in the following years. Donald Ornstein gave the positive answer to the question in [71]. For an interesting historical account on this development see for example [47] by Anatole Katok. For one-dimensional cellular automata the measure-theoretic and topological entropies are always finite. But for cellular automata over more complex groups, the entropies are often zero or infinite. It even remained an open question for quite some time if there existed a multidimensional cellular automaton with a finite non-zero entropy when the dimension is higher than one. According to [67] by Gary Morris and Thomas Ward it was even conjectured by Shereshevsky that the topological entropies for multidimensional cellular automata (with dimension higher than one) are always either zero or infinite. The conjecture was proven false by Tom Meyerovitch in [63]. This indicates that such cellular automata might be somewhat rare. And for multidimensional linear cellular automata the entropy is always zero or infinite, when the dimension is higher than one. As such the entropies are not useful if we want to tell non-conjugate systems apart. Lyapunov exponents are only defined for one-dimensional cellular automata. It was left as an open question to generalize the notion of Lyapunov exponents for multidimensional cellular automata in [75], where the one-dimensional Lyapunov exponents were first defined by Mark A. Shereshevsky. In [7] François Blanchard and Pierre Tisseur generalized Lyapunov exponents for cellular automata over $\mathbb{Z}^2$. In [31] by the author Lyapunov exponents were generalized for cellular automata over arbitrary groups and the measure-theoretic entropy was generalized for equivariant dynamical systems over amenable groups. The extended and improved version of this paper is [34] by the author. Chapter 5 is based on [31] and [34].

2 Preliminaries

With the preliminaries introduced in this chapter our aim is to make this thesis as self-contained as possible. Some knowledge of set theory, general topology, metric spaces and real analysis and perhaps some mathematical maturity are still assumed from the reader. For the readers unfamiliar with these topics we recommend for example the following works: [48] by John L Kelley, [78], [77] by Terence Tao and [42] by Irving Kaplansky. One familiar with the topics of this chapter may skip over it entirely, although the first subchapter is recommended as some notations differ from the standard conventions.

2.1 A Few Notational Conventions

We use the standard interval notations of real numbers to denote intervals that contain only integers. Hence $\{n, n+1, \ldots m-1, m\} = [n, m] = (n-1, m] = [n, m+1) = (n-1, m+1)$. On the other hand if we are dealing with real numbers, then we denote $[x, y]_{\mathbb{R}} = \{z \in \mathbb{R} \mid x \leq z \leq y\}$. The intervals $(x, y]_{\mathbb{R}}$ and $[x, y)_{\mathbb{R}}$ are analogously defined. The reason we do it this way and not the other way, i.e. denote $[n, m]_{\mathbb{Z}}$ for integer intervals and keep the standard notation for real intervals, is simply because we need the integer intervals far more frequently. We denote $\mathbb{Z}_+ = [1, \infty)$ for the set of positive integers, $\mathbb{Z}_- = (-\infty, -1]$ for the set of negative integers and $\mathbb{N} = [0, \infty)$ for the set of natural numbers. We may also denote $\mathbb{R}_+ = [0, \infty)_{\mathbb{R}}$ for the set of non-negative real numbers.

The *cardinality* of a set X is denoted as $|X|$. Hence $|X| = n \in \mathbb{N}$ means that the set X contains n elements. Of course the cardinality of a set may also be infinite and in such case we would denote $|X| = \infty$. The set of finite subsets of a set X is denoted as $\mathrm{Fin}(X)$. The *empty set* is the unique set of cardinality zero and is denoted as $\emptyset$.

A *disjoint union* is denoted as $\uplus$, that is $A \uplus B$ is the union of two disjoint sets A and B. Let I be a set of indices. Then more generally if $A_i \cap A_j = \emptyset$ for each $i \in I$ and $j \in I \setminus \{i\}$. Then $\biguplus_{i \in I} A_i$ is the disjoint union of all the sets $A_i \in I$.

We use the notation X^Y for the set of mappings from Y to X, i.e. $X^Y = \{f : Y \to X\}$. We might also denote $x(y) = x_y$ for each $y \in Y$ and $x \in X^Y$. A set Y is a subset of a set X denoted as $Y \subseteq X$ if for each $x \in Y$ we have that $x \in X$. Let $f : X \to Y$, $A \subseteq Y$ and $B \subseteq X$, then we denote the *image* $f(B) = \{f(x) \in$

$Y \mid x \in B\}$ and the *pre-image* $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. For a collection α of subsets of Y, we denote $f^{-1}(\alpha) = \{f^{-1}(A) \subseteq X \mid A \in \alpha\}$. For a subset $A \subseteq X$ we use the notation $f|_A$ for the *restriction* of the function f to the set A, i.e. $f|_A : A \to Y$ is such that $f|_A(x) = f(x)$ for each $x \in A$.

Let α and β be two collections of sets, then the *join* $\alpha \vee \beta$ of α and β is defined as $\alpha \vee \beta = \{A \cap B \mid A \in \alpha \text{ and } B \in \beta\}$. Inductively we define a join of n collections of sets $\alpha_0, \alpha_1 \ldots, \alpha_{n-1}$ as $\bigvee_{i=0}^{n-1} \alpha_i = \bigvee_{i \in I} \alpha_i = \alpha_{n-1} \vee \bigvee_{i \in I \setminus \{n-1\}} \alpha_i$, where $I = [0, n-1]$. More generally I can be any finite set of indices. Given a function $f : X \to X$, we denote $\alpha^n = \bigvee_{i=0}^{n-1} f^{-i}(\alpha)$. And given a group action on a set X, we will denote $\alpha_g = g^{-1}(\alpha)$ and more generally $\alpha_F = \bigvee_{g \in F} \alpha_g$ for $F \subseteq G$.

A *partition* α of a set X is such a collection of pairwise disjoint subsets of X, that the union of all the elements in the partition equals X. That is, if $A \in \alpha$ and $B \in \alpha$, then $A = B$ or $A \cap B = \emptyset$ and $\biguplus_{A \in \alpha} A = X$.

A *relation* is a subset $R \subseteq X \times X$, where X is a set. We will use the notation aRb if $(a, b) \in R$. We will denote the complement of R as R^c, i.e. $R^c = (X \times X) \setminus R$.

We use the ball notation $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$ for an *ϵ-ball* centered at point x belonging to some metric space X, where $d : X \times X \to \mathbb{R}_+$ is a metric.

If $f : X \to Y$ and $g : Y \to Z$ are function. We use the notation $g \circ f : X \to Z$ for the *composition* of the functions f and g.

2.2 Groups

Let $(G, *)$ be a tuple, where G is a set and $*$ is a mapping $* : G \times G \to G$ and let us introduce the following axioms:

G1: $\forall a \in G, b \in G$ and $c \in G : (a * b) * c = a * (b * c)$.
G2: $\exists e \in G : \forall a \in G : a * e = e * a = a$.
G3: $\forall a \in G : \exists a^{-1} \in G : a * a^{-1} = a^{-1} * a = e$.
G4: $\forall a \in G$ and $b \in G : a * b = b * a$.

Then if the axiom G1 holds, a tuple $(G, *)$ is a *semigroup*. A semigroup that has a (necessarily unique) *neutral element*, i.e. for which G2 holds, is called a *monoid*. If G3 holds for a monoid, then each element has an inverse element and such monoid is called a *group*. Furthermore if the order of applying the group operation does not matter, i.e. G4 holds, a group is called an *abelian group*. If the operation of any of the aforementioned *algebraic structures* is clear from context, then the operation notation might be omitted so that $a * b = ab$ and $(G, *) = G$.

Let G be an algebraic structure, then if $A \subseteq G$ and $B \subseteq G$ we denote as AB the set $\{ab \mid a \in A \text{ and } b \in B\}$. If in this notation either of the sets is a singleton set we use simply the notation a instead of $\{a\}$.

Let G be a group and $H \subseteq G$. Then H is a *subgroup* of G if it is a group with the same operation. This is equivalent with the condition that a subset H is a subgroup if it is non-empty and if $ab \in H$ and $a^{-1} \in H$ for each $a \in H$ and $b \in H$. We denote $H \leq G$ if H is a subgroup of G. A subgroup N of G is a *normal subgroup* of G if $aN = Na$ for each $a \in G$. We denote $N \trianglelefteq G$ if N is a normal subgroup of G.

Let $H \subseteq G$. Then the elements in the set $G/H = \{gH \mid g \in G\}$ are called *(left) cosets* of G. The set of cosets G/H forms a partition of G. If $N \trianglelefteq G$, then the set of cosets G/N forms a group under the operation $*$ that maps the elements gN and hN to the element $gN * hN = ghN$. Any group G/N is called a *quotient group* of the group G.

Let $(G, *)$ and $(H, \cdot)$ be two groups. Then a *(group) homomorphism* is a mapping $\varphi : G \to H$ such that $\varphi(a * b) = \varphi(a) \cdot \varphi(b)$. A homomorphism is called a *monomorphism* if it is injective, an *epimorphism* if it surjective and an *isomorphism* if it is bijective. As an example the *natural epimorphism* is a mapping $\varphi : G \to G/N$ such that $\varphi(g) = gN$.

Let G be a group and X a set, then a mapping $\varphi : G \times X \to X$ is a *(left) group action* if $\varphi(e, x) = x$ and $\varphi(a, \varphi(b, x)) = \varphi(ab, x)$ for each $a \in G$, $b \in G$ and $x \in X$. If we fix an element $a \in G$, then we denote by φ_a, the mapping such that $\varphi_a(x) = \varphi(a, x)$. This mapping is a bijection because $\varphi_{a^{-1}} \circ \varphi_a(x) = \varphi(a^{-1}a, x) = \varphi(e, x) = x$. If the group action is clear from the context, we might also simply denote $a.x = ax = \varphi(a, x)$. Similarly we might make no distinction between a group element a and the mapping φ_a and simply denote $a = \varphi_a$ if this does not lead to ambiguities. A group action is *continuous* if the function φ_a is continuous for each $a \in G$. We will use the notation $Gx = \{gx \mid g \in G\}$ for the *orbit* of each point $x \in X$.

A set $S \subseteq G$ is a *(symmetric) set of generators* of a group G if $S = S^{-1}$ and for every $g \in G$ there exists a finite sequence $(s_0, s_1, \ldots s_n)$ such that $g = s_0 s_1 \cdots s_n$. If S is a set of generators of G, we say that G is *generated by* S and denote $\langle S \rangle = G$. If S is finite, we say that G is *finitely generated*. Groups are equipped with the *word metric* $d : G \times G \to G$ with respect to a given generating set S, where $d(g, h) = n$ if $g^{-1}h = s_1 s_2 \ldots s_n$ is the shortest finite sequence of elements (i.e. the shortest word) of S that expresses $g^{-1}h$. We will denote $|g| = d(1, g)$ and $B_n^G = \{g \in G \mid |g| \leq n\}$, where we omit the superscript if the group is clear from the context.

2.3 Graphs

A *multi(di)graph* is a triple (V, E, ν), where V is a set of *vertices*, E is a set of *edges* and $\nu : E \to V \times V$ is a mapping. If $e \in E$ and $\nu(e) = (v, v')$, then v is called the *source* of the edge and v' is called the *target* of the edge. Let S_V and S_E be two sets. A mapping $\phi : V \to S_V$ is called a *vertex labelling* and the set S_V is called the set of *vertex labels*. Similarly a mapping $\varphi : E \to S_E$ is called an *edge labelling* and the set S_E is called the set of *edge labels*. In all of our use cases the vertex labelling will be an identity map if even defined.

Graphs are often presented graphically; vertices are depicted as circles or rectangles and edges are depicted as arrows pointing from the source to the target, vertex labels are written inside the circles and the edge labels are written next to the arrows.

As an example a *Cayley graph* of a given group G with respect to a given a set of generators S of G is the graph (V, E, ν), where $V = G$ and for each $g \in G$, $g' \in G$ and $a \in S$ such that $ga = g'$ there exists a unique $e \in E$ such that $\nu(e) = (g, g')$. If furthermore an edge labelling is present, it would be defined as $\varphi(e) = a$.

2.4 Words and Languages

An *alphabet* Σ is a finite set of *symbols*. A *word* of length n over an alphabet Σ is any finite sequence $w = (w_0, w_1, \ldots, w_{n-1}) = w_0 w_1 \cdots w_{n-1}$ from the set $\Sigma^{[0,n)} = \Sigma^n$ and $|w| = n$ is the *length* of the word w. The *empty word* is denoted as ϵ and it is the unique word of length zero. A set of all finite words i.e. $\bigcup_{n \in \mathbb{N}} \Sigma^n$ is denoted as Σ^* and a set of all finite non-empty words $\Sigma^* \setminus \{\epsilon\}$ is denoted as Σ^+.

A *concatenation* $\cdot : \Sigma^* \times \Sigma^* \to \Sigma^*$ is a mapping defined in a way such that $u \cdot v = u_0 u_1 \ldots u_n v_0 v_1 \ldots v_m$, where $u = u_0 u_1 \ldots u_n$ and $v = v_0 v_1 \ldots v_m$. We will adapt the shorthand notation uv for the concatenation of any two words. Although we do not particularly make use of this fact, we note that a pair $(\Sigma^*, \cdot)$ is a monoid.

Elements from the sets $\Sigma^{\mathbb{N}}$, $\Sigma^{\mathbb{Z}_-}$ and $\Sigma^{\mathbb{Z}}$ are called *right-infinite*, *left-infinite* and *bi-infinite* words, respectively. Furthermore we define a set $\Sigma^\Omega = \Sigma^+ \cup \Sigma^{\mathbb{N}} \cup \Sigma^{\mathbb{Z}_-} \cup \Sigma^{\mathbb{Z}}$. A concatenation can be generalized for elements $u \in \Sigma^\Omega$ and $v \in \Sigma^\Omega$ when u is a finite or left-infinite word and v is a finite or right-infinite word.

The *reversal* w^R of a word $w \in \Sigma^\Omega$ is defined as follows: If $w = w_0 w_1 \cdots w_{n-1}$, then $w^R = w_{n-1} w_{n-2} \cdots w_0$. If $w \in \Sigma^{\mathbb{N}}$, then $w^R \in \Sigma^{\mathbb{Z}_-}$ and $w_i^R = w_{-i-1}$. If $w \in \Sigma^{\mathbb{Z}_-}$, then $w^R \in \Sigma^{\mathbb{N}}$ and $w_i^R = w_{-i-1}$. If $w \in \Sigma^{\mathbb{Z}}$, then $w^R \in \Sigma^{\mathbb{Z}}$ and $w_i^R = w_{-i}$.

Let $\Sigma = \bigcup_{i=1}^{n} \prod_{j=1}^{i} \Sigma_j$, where Σ_j are finite sets of symbols and $n \in \mathbb{Z}_+$. Suppose that $m \in [1, n]$ and $w \in \Sigma^\Omega$ is such that $w_k \in \bigcup_{i=m}^{n} \prod_{j=1}^{i} \Sigma_j$ for each k in the domain of w and let S_m denote the set of such words. Then a *projection* $\pi_m : S_m \to \Sigma_m^\Omega$ is defined in a way such that $\pi_m(w)_k = (w_k)_m$.

Let $u \in \Sigma^\Omega$ and $w \in \Sigma^\Omega$, we will denote $u \sqsubset w$ if there exists such $j \in \mathbb{Z}$, that $u_i = w_{i+j}$ for each i in the domain of u, and say that u is a *subword* of w.

If Σ and Γ are two alphabets, we will denote the set $\{uv \mid u \in \Sigma^\alpha, v \in \Gamma^\beta\}$ as $\Sigma^\alpha \Gamma^\beta$. where $\alpha \in \{\mathbb{Z}_-, *, +\}$ and $\beta \in \{\mathbb{N}, *, +\}$. In this notation, if $\Sigma = \{a\}$, we will omit the brackets.

If $w \in \Sigma^*$, we will use the notation w^∞ for the right-infinite word $ww\cdots$ and $^\infty w$ for the left-infinite word $\cdots ww$. If $A \subseteq \Sigma$ and $w \in \Sigma^\Omega$, then $w_A = |\{i \mid w_i \in A\}|$. If $A = \{a\}$ then we omit the brackets from the notation and denote $w_a = w_{\{a\}}$.

A subset $\mathcal{L} \subseteq \Sigma^*$ is called a *language*. A language $\mathcal{L}$ is a *finite language* if $|\mathcal{L}| \in \mathbb{N}$. For any $X \subseteq \Sigma^\Omega$ we define the language of X as the set

$$\mathcal{L}(X) = \{u \in \Sigma^* \mid u \sqsubset w \in X\}.$$

2.5 Dynamical Systems

Main topic of this dissertation is the study of dynamical systems. While some properties which we will be considering are specific to one-dimensional cellular automata or Turing machines, others can be introduced in the more general framework of topological or measure-preserving dynamical systems and we too will be doing so whenever it makes sense to do so. Cellular automata and Turing machines can be viewed as either of these systems as we will see later on.

2.5.1 Topological Dynamical Systems

Let X be a set. A *metric* is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following axioms:

M1: $d(x, y) = 0$ if and only if $x = y$.
M2: $d(x, y) = d(y, x)$.
M3: $d(x, y) + d(y, z) \leq d(x, z)$.

A set X equipped with a metric d is called a *metric space* and it is denoted as (X, d). If the specific metric is not of interest or if it is clear from the context we omit the metric from the notation and simply say that X is a metric space. A metric space is *compact* if every sequence of X has a converging subsequence.

Let α be a collection of subsets of X, then α is a *cover* of $Y \subseteq X$ if $Y \subseteq \bigcup_{A \in \alpha} A$. A cover is a *finite cover* if it has finitely many elements and it is an *open cover* if all of its elements are open sets. If α is a cover of Y, then a cover of Y, β, is a *subcover* of α if $\beta \subseteq \alpha$. In compact spaces every open cover of a closed set has a finite subcover. If α is an open cover of X, then we denote by $N(\alpha)$ the cardinality of the smallest subcover and furthermore we define $H(\alpha) = \ln(N(\alpha))$.

Definition 2.5.1. Let X be a metric space. Then for a subset $Y \subseteq X$ the diameter of Y is defined as $\text{diam}(Y) = \sup\{d(x, y) \mid x \in Y \text{ and } y \in Y\}$. For a family α of subsets of X the diameter of α is defined as $\text{diam}(\alpha) = \sup\{\text{diam}(A) \mid A \in \alpha\}$.

Definition 2.5.2. Let X be a metric space. Let α be a cover of X. Then α is *topologically G-generating* if $\lim_{n\to\infty} \text{diam}(\alpha_{B_n}) = 0$ with respect to a group action on

X. Similarly α is *topologically f-generating* with respect to a continuous function $f : X \to X$ if $\lim_{n\to\infty} \text{diam}(\alpha^n) = 0$. In either case we might call α simply *generating* if doing so is not ambiguous from the context.

In the above definition we used the shorthand $\alpha_{B_n} = \alpha_{B_n^G}$ as we will often do if the group is clear from the context.

The following lemma can be found in many books about general topology and metric spaces. One such source for example is in [68] by James R. Munkres.

Lemma 2.5.3. [68][Lebesgue's Number Lemma] Let X be a compact metric space. Then for each open cover α of X, there exists $\epsilon > 0$ such that for each $Y \subseteq X$ such that diam$(Y) < \epsilon$, there exists $A \in \alpha$ such that $Y \subseteq A$. Any such number ϵ is called a *Lebesgue number* of the cover α.

Definition 2.5.4. A *(topological) dynamical system* is a pair (X, f), where X is a compact metric space and f is either a continuous function $f\colon X \to X$ or a continuous group action $f\colon G \times X \to X$. In the latter case we will often denote the dynamical system as (X, G) instead of (X, f) if the specific group action is clear from the context or not of specific interest. By default (X, f) will always refer to the dynamical system where f is a function, unless otherwise specified.

Let X and X' be two compact metric spaces. Let $T' : X \to X'$ be a continuous function. Then T' is called a *homeomorphism* if it is bijective (and its inverse is continuous, but that follows from the compactness of X). Let (X, f_0) and (X', f_1) be two dynamical systems such that $T \circ f_0 = f_1 \circ T$ (or in the case that f_0 and f_1 are group actions, then $T \circ (f_0)_g = (f_1)_g \circ T$ for each $g \in G$). Then T is called a *factor map* if it is surjective and in such case (X', f_1) is a *factor* of (X, f_0). If T is injective, then it is called an *embedding* and then (X, f_0) is a *subsystem* of (X', f_1). If T is a homeomorphism, then it is called a *conjugacy* and we say that (X, f_0) are *conjugate* (X', f_1).

Definition 2.5.5. Let (X, f) be a dynamical system. If $\varphi : G \times X \to X$ is a group action such that $g \circ f = f \circ g$, for each $g \in G$, where $\varphi_g = g$, then (X, f) is *G-equivariant*. We also say that (X, f) is *equivariant* if we have not specified the group or the group action.

One property that we will consider is periodicity. Existence of certain types of periodic points is related to speed positiveness in Turing machines. We will discuss this in more details in the following chapter. We define the periodicity of points only, but similarly one can consider periodicity of the whole system.

Definition 2.5.6. Let (X, f) be a dynamical system. A point $x \in X$ is called *periodic* if there exists $n \in \mathbb{Z}_+$ such that $f^n(x) = x$.

Analogously we can define periodicity for group actions.

Definition 2.5.7. Let (X, G) be a dynamical system. A point $x \in X$ is called *periodic* if there exists a non-identity element $g \in G$ such that $gx = x$.

For equivariant dynamical systems (X, f) we can consider a weaker form of periodicity. We call a point $x \in X$ *weakly periodic* if there exists $n \in \mathbb{Z}_+$ and $g \in G$ such that $f^n(x) = gx$. We will only consider such periodicity when $G = \mathbb{Z}$.

The following four definitions are related to stability of a system. The first property states that orbits of points that are close enough to a certain point stay close to the orbits of said point. This means for example that if we are given an estimate of such a point with enough precision then we can reliably estimate the orbit of the actual point up to a given precision. Systems whose each point has such property can be considered stable. In contrast the last two properties of the four can be associated with unstable systems.

Definition 2.5.8. Let (X, f) be a dynamical system. A point $x \in X$ is *equicontinuous*, if

$$\forall \epsilon > 0 \colon \exists \delta > 0 \colon \forall y \in B_\delta(x) \colon \forall n \geq 0 \colon f^n(y) \in B_\epsilon(f^n(x)).$$

A dynamical system (X, f) is called *almost equicontinuous* if the set of equicontinuous points is residual (countable intersection of open dense sets) in X. And it is called *equicontinuous* if all of its points are equicontinuous.

Definition 2.5.9. A dynamical system (X, f) is *uniformly equicontinuous*, if

$$\forall \epsilon > 0 \colon \exists \delta > 0 \colon \forall x \in X \colon \forall y \in B_\delta(x) \colon \forall n \geq 0 \colon f^n(y) \in B_\epsilon(f^n(x)).$$

Definition 2.5.10. A dynamical system (X, f) is *sensitive* if

$$\exists \epsilon > 0 \colon \forall \delta > 0 \colon \forall x \in X \colon \exists y \in B_\delta(x) \colon \exists n \in \mathbb{N} \colon f^n(y) \notin B_\epsilon(f^n(x)).$$

Any value $\epsilon > 0$ that satisfies the above statement is called a *sensitivity constant*.

Sensitive systems can be thought of as unstable. Sensitivity means that we can always fluctuate each point with arbitrarily tiny differences in such a way that those differences will propagate to large ones as time moves forward. Sensitivity and equicontinuity are closely related. One can for example notice that sensitive systems have no equicontinuous points. The converse is not necessarily true in general, but there exists classes of systems such that a given system is always either sensitive or equicontinuous as we will see later on. Another especially strong form of sensitivity is expansivity.

Definition 2.5.11. A dynamical system (X, f) is *positively expansive* if

$$\exists \epsilon > 0 \colon \forall x \in X, y \in X, x \neq y \colon \exists n \in \mathbb{N} \colon f^n(y) \notin B_\epsilon(f^n(x)).$$

We do not consider transitive systems in this dissertation, but we will refer to an interesting open problem related to our results. In the framework of cellular automata the property is also directly connected to sensitivity and positive expansivity as every transitive cellular automaton is sensitive (and surjective) and every positively expansive cellular automaton is transitive.

Definition 2.5.12. A dynamical system (X, f) is *transitive* if for each pair of non-empty open subsets $A \in X$ and $B \in X$ there exists $n \in \mathbb{N}$ such that $f^n(A) \cap B \neq \emptyset$.

The next interesting property will become very useful when we study sensitive group cellular automata. The property is sometimes also referred to as *pseudo-orbit tracing property*, which might be more descriptive. The property means that we can estimate the pseudo-orbits with actual orbits up to any given precision.

Definition 2.5.13. Let (X, f) be a dynamical system. A finite sequence (x_i) of points of X is called a δ*-chain* from x_0 to x_n if $d(f(x_i), x_{i+1}) < \delta$ for each $i < n$. A point $x \in X$ is said to ϵ*-shadow* a finite sequence (x_i) if $d(f^i(x), x_i) < \epsilon$ for each $i \leq n$. A dynamical system (X, f) has the *shadowing property* if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-chain is ϵ-shadowed by some point.

Finally we define the most important property in our dissertation. Recall the function H which was defined earlier in the chapter as the logarithm of the cardinality of the smallest subcover of each given open cover.

Definition 2.5.14. Let (X, f) be a dynamical system and α be a finite open cover of X. The *topological entropy with respect to partition* α is defined as

$$h(\alpha, f) = \lim_{n\to\infty} \frac{H(\alpha^n)}{n} = \lim_{n\to\infty} \frac{H(\bigvee_{i=0}^{n} f^{-i}(\alpha))}{n}.$$

Let $\mathcal{C}(X)$ be the set of finite open covers of X, then the *topological entropy* of (X, f) is defined as

$$h_f = h(f) = \sup_{\alpha \in \mathcal{C}(X)} h(\alpha, f).$$

The following two theorems and their proofs can be found in many introductory textbooks about dynamical systems,see for example [54] by Petr Kůrka.

***Theorem* 2.5.15.** *[54] Let (X, f) be a dynamical system and α be a finite open cover of X. If α is generating, then*

$$h(\alpha, f) = h(f).$$

***Theorem* 2.5.16.** *[54] Let (X, f_0) and (X', f_1) be two dynamical systems. If (X, f_0) is a subsystem of (X', f_1), then $h(f_0) \leq h(f_1)$. If (X', f_1) is a factor of (X', f_0), then $h(f_1) \leq h(f_0)$. If (X, f_0) and (X', f_1) are conjugate then $h(f_0) = h(f_1)$.*

2.5.2 Measure-Preserving Dynamical Systems

Let X be a set. A collection $\mathcal{A}$ of subsets of X is a *σ-algebra* of X if the following axioms hold:

S1: $\emptyset \in \mathcal{A}$.
S2: $X \setminus A \in \mathcal{A}$ for each $A \in \mathcal{A}$.
S3: $\bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$, when each $A_i \in \mathcal{A}$.

If $\mathcal{Y}$ is a collection of subsets of X, then the smallest σ-algebra containing $\mathcal{Y}$ is called the σ-algebra generated by $\mathcal{Y}$, which is denoted as $\sigma(\mathcal{Y})$ and $\mathcal{Y}$ is called a *generator* of $\sigma(\mathcal{Y})$. The *Borel* σ-algebra of a topological space X is the smallest σ-algebra that contains every open subset of X or equivalently the σ-algebra generated by the set of the open sets of X. If a topological space X is second countable, meaning its topology has a countable basis, then for any basis $\mathcal{U}$ of X's topology it holds that $\sigma(\mathcal{U})$ is the Borel σ-algebra of X.

A *measure* μ is a function $\mu : \mathcal{B} \to [0, \infty]_{\mathbb{R}}$ such that $\mu(\emptyset) = 0$ and it is *countably additive*, i.e. $\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$, where $\mathcal{B}$ is a σ-algebra, I is a countable index set and $A_i \cap A_j = \emptyset$ for each distinct $i \in I$ and $j \in I$. The sets $A \in \mathcal{B}$ are called *measurable*. A measure μ is a *probability measure* if its codomain is the interval $[0, 1]_{\mathbb{R}}$ and $\mu(X) = 1$.

We introduce three more axioms for a collection $\mathcal{A}$ of subsets of X:

S2': For each $A \in \mathcal{A}$ and $B \in \mathcal{A}$ there exists
a set of pairwise disjoint sets $A_i \in \mathcal{A}$ such that $A \setminus B = \bigcup_{i=0}^{n} A_i$.
S3': $\bigcup_{i=0}^{n} A_i \in \mathcal{A}$, when each $A_i \in \mathcal{A}$.
S3": $\bigcap_{i=0}^{n} A_i \in \mathcal{A}$, when each $A_i \in \mathcal{A}$.

A collection $\mathcal{Y}$ of subsets of X is called a *semiring* of X if the axioms S1, S2' and S3" hold and it is called a *ring* if the axioms S1, S2 and S3' hold. A probability measure can be defined for the semirings and rings in the same way as for the σ-algebras, except that countable additivity is required only when $\bigcup_{i \in I} A_i \in \mathcal{Y}$. Our only reason in bringing up semirings and rings is the next extension theorem, proof of which can be found for example in [51] by Achim Klenke.

***Theorem* 2.5.17.** *[51][Carathéodory's Extension Theorem] Let $\mathcal{Y}$ be a (semi)ring of X and let $\mu_0 : \mathcal{Y} \to [0, 1]_{\mathbb{R}}$ be a probability measure. Then there exists a unique probability measure $\mu : \sigma(\mathcal{Y}) \to [0, 1]_{\mathbb{R}}$ such that $\mu|_{\mathcal{Y}} = \mu_0$.*

A triple $(X, \mathcal{B}, \mu)$ is a *measure space*, where X is a set, $\mathcal{B}$ is its σ-algebra and $\mu : \mathcal{B} \to [0, \infty]_{\mathbb{R}}$ is an arbitrary measure and furthermore it is a *probability space* if μ is a probability measure. Let $(X_i, \mathcal{B}_i, \mu_i)$ be two measure spaces, where $i \in \{0, 1\}$. A function $f : X_0 \to X_1$ is called *measurable*, if $f^{-1}(A) \in \mathcal{B}_0$ for each $A \in \mathcal{B}_1$. A measurable function is called *measure-preserving* if furthermore $\mu_0(f^{-1}(A)) = \mu_1(A)$ for every $A \in \mathcal{B}_1$. If $(X, \mathcal{B}, \mu)$ is a probability space and $f : X \to X$ is a measure-preserving function, then the measure μ is called *f-invariant*. Let $(X, \mathcal{B}, \mu)$ be a probability space and let G be a group acting on X. We call μ *G-invariant* if the function $g : X \to X$ is measure-preserving for each $g \in G$ and we call such G measure-preserving. Let μ be f-invariant. If whenever $A \in \mathcal{B}$ and $f^{-1}(A) = A$ we have that $\mu(A) = 0$ or $\mu(A) = 1$, then μ is *f-ergodic*. We might also say that μ is ergodic with respect to f in such case.

The proof of the following handy property can be found for example in [82] by Peter Walters.

***Theorem* 2.5.18.** *[82] Let $(X_i, \mathcal{B}_i, \mu_i)$ be two probability spaces, where $i \in \{0, 1\}$. Let $f : X_0 \to X_1$ be a function. Let $\mathcal{Y}_1$ be such a semiring that $\sigma(\mathcal{Y}_1) = \mathcal{B}_1$. Then f is measurable and measure-preserving if and only if $f^{-1}(A) \in \mathcal{B}_0$ and $\mu_0(f^{-1}(A)) = \mu_1(A)$ for each $A \in \mathcal{Y}_1$.*

Definition 2.5.19. A *measure-preserving dynamical system* is a tuple $(X, \mathcal{B}, \mu, f)$, where $(X, \mathcal{B}, \mu)$ is a probability space and f is either a measure-preserving function or a measure-preserving group action.

In the notation of the above definition we might replace the group action by the group instead, if the action does not need to be specified or is otherwise clear from the context. If we are using f in the notation it will always refer to measure-preserving function unless otherwise specified.

Suppose that $(X_i, \mathcal{B}_i, \mu_i, f_i)$ are two measure-preserving dynamical systems, for each $i \in \{0, 1\}$. Suppose that there exists measurable sets $M_i \in \mathcal{B}_i$ such that $\mu_i(M_i) = 1$ and $f_i(M_i) \subseteq M_i$ for both $i \in \{0, 1\}$. Let $\varphi : M_0 \to M_1$ be measure-preserving such that $\varphi \circ f_0 = f_1 \circ \varphi$. If φ is surjective then it is a *(measure-theoretic) factor map* and $(X_1, \mathcal{B}_1, \mu_1, f_1)$ is a *(measure-theoretic) factor* of $(X_0, \mathcal{B}_0, \mu_0, f_0)$. If φ is bijective and its inverse is a factor map then it is a *(measure-theoretic) isomorphism* and $(X_0, \mathcal{B}_0, \mu_0, f_0)$ and $(X_1, \mathcal{B}_1, \mu_1, f_1)$ are *(measure-theoretically) isomorphic*.

If α and β are such partitions of the set X that for each $A \in \alpha$ there exists $B \in \beta$ such that $A \subseteq B$, then α is a *refinement* of β. A partition is a *finite partition*, if it contains finitely many elements and it is a *measurable partition* if each of its element is measurable. Let α be a partition. For each $x \in X$ we will denote $\alpha(x) = A$, where $A \in \alpha$ is such that $x \in A$. Let $\mathcal{B}$ be a σ-algebra of a set X. A sequence (α_i) of partitions of X is a *generating partition* of $\mathcal{B}$ if for each $i \in \mathbb{N}$, α_{i+1} is a refinement of α_i and $\sigma(\bigcup_{i\in\mathbb{N}} \alpha_i) = \mathcal{B}$.

Definition 2.5.20. Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving dynamical system. The *measure-theoretic entropy* of a partition α is defined as

$$H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A)\ln(\mu(A))$$

and the *conditional measure-theoretic entropy* of a finite partition β, given α is defined as

$$H_\mu(\beta|\alpha) = \sum_{A\in\alpha}\sum_{B\in\beta} -\mu(A\cap B)\ln(\frac{\mu(A\cap B)}{\mu(A)}).$$

For convenience we define $0\ln 0 = 0$. The *measure-theoretic entropy of a system* with respect to a partition α is defined as

$$h_\mu(f,\alpha) = \lim_{n\to\infty}\frac{H_\mu(\alpha^n)}{n}.$$

Finally the *measure-theoretic entropy of a system* is

$$h_\mu(f) = \sup\{h_\mu(f,\alpha) \mid \alpha \text{ is a finite and measurable partition of } X\}.$$

The facts contained in the following theorems and lemmas and their proofs can be found in many introductory texts to ergodic theory, for example in [82] by Peter Walters.

***Theorem* 2.5.21.** *[82] Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving dynamical system and α be a partition. The limit $h_\mu(f,\alpha) = \lim_{n\to\infty}\frac{H_\mu(\alpha^n)}{n}$ exists.*

Lemma 2.5.22. [82] The following properties hold for the (conditional) measure-theoretic entropy:

$$\begin{aligned}
&H_\mu(\alpha\vee\beta) = H_\mu(\alpha) + H_\mu(\beta|\alpha)\\
&H_\mu(\alpha\vee\beta|\gamma) \le H_\mu(\alpha|\gamma) + H_\mu(\beta|\alpha\vee\gamma)\\
&H_\mu(\beta|\alpha) \le H_\mu(\beta)\\
&H_\mu(\beta) \le H_\mu(\alpha) \text{ if } \alpha \text{ is a refinement of } \beta\\
&H_\mu(\beta|\alpha) \le H_\mu(\beta'|\alpha) \text{ if } \beta' \text{ is a refinement of } \beta\\
&H_\mu(\beta|\alpha') \le H_\mu(\beta|\alpha) \text{ if } \alpha' \text{ is a refinement of } \alpha\\
&h_\mu(\beta) \le h_\mu(\alpha) + H_\mu(\beta|\alpha)\\
&h_\mu(\alpha) = h_\mu(\alpha^n) \text{ for each } n\in\mathbb{N}
\end{aligned}$$

Lemma 2.5.23. [82] Let X be a set and $\mathcal{B}$ be a σ-algebra. Let (α_i) be a generating sequence of partitions. Then for any partition β we have that for every ϵ there exist such n_ϵ that

$$H_\mu(\beta|\alpha_n) < \epsilon,$$

whenever $n > n_\epsilon$

Theorem 2.5.24. *[82] Let $(X_i, \mathcal{B}_i, \mu_i, f_i)$ be two measure-preserving dynamical systems, for each $i \in \{0, 1\}$. If the system $(X_1, \mathcal{B}_1, \mu_1, f_1)$ is a measure-theoretic factor of the system $(X_0, \mathcal{B}_0, \mu_0, f_0)$ then $h_\mu(f_0) \geq h_\mu(f_1)$. If the two systems are isomorphic then $h_\mu(f_0) = h_\mu(f_1)$.*

Topological and measure-theoretic entropies are connected to each other via a variational principle. The theorem was proven in two parts. In [25] L. Wayne Goodwyn proved that the topological entropy is an upper bound for the supremum of the set of measure-theoretic entropies. And the converse was proven independently in [16] and [24] by Efim I. Dinaburg and Tim N. T. Goodman.

Theorem 2.5.25. *[16; 24; 25][Variational Principle] Let (X, f) be a dynamical system. Let $\mathcal{A}$ be the Borel σ-algebra of X. Let $\mathcal{M}$ the set of all probability measures over the domain $\mathcal{A}$ preserved under f. Then $h(f) = \sup\limits_{\mu \in \mathcal{M}} h_\mu(f)$.*

If (X, f) is a dynamical system, a measure $\mu \in \mathcal{M}$ is called a *measure of maximal entropy* if $h_\mu(f) = h(f)$.

2.6 Subshifts

A (one-dimensional) *full shift* is a dynamical system $(\Sigma^{\mathbb{M}}, \sigma)$, where $\mathbb{M} \in \{\mathbb{Z}, \mathbb{N}\}$, Σ is a finite set of *symbols* and σ, called the *shift*, is the mapping defined by $\sigma(x)_i = x_{i+1}$. The metric d_σ of the space $\Sigma^{\mathbb{M}}$ is defined as $d_\sigma(x, y) = 2^{-\inf\{|i| \in \mathbb{N} | x_i \neq y_i\}}$. This metric induces the prodiscrete topology, i.e. the product topology of the discrete topology on Σ, on the space $\Sigma^{\mathbb{M}}$. One can confirm that the space is compact and the function σ is continuous. A *subshift* (X, σ) is a dynamical system where X is any closed subset of a full shift such that $\sigma(X) \subseteq X$. The space itself will also be referred to as either a full shift or a subshift.

The set of words of length n which appear in a subshift X is denoted as $\mathcal{L}_n(X)$, i.e. $\mathcal{L}_n(X) = \{u \in \Sigma^n \mid \exists x \in X : u \sqsubseteq x\}$. The language $\mathcal{L}(X)$ of a given subshift X is the set $\mathcal{L}(X) = \bigcup\limits_{n=0}^{\infty} \mathcal{L}_n(X)$.

Each subshift can be defined by a set of *forbidden words* $\mathcal{F}$. In this case we denote

$$X_{\mathcal{F}} = \{x \in \Sigma^{\mathbb{M}} \mid \forall u \in \mathcal{F} \colon u \not\sqsubseteq x\}.$$

A subshift $X_{\mathcal{F}}$ is called a *subshift of finite type (SFT)* if $\mathcal{F}$ is finite.

A subshift (X, σ) is called *irreducible* if for each pair $u \in \mathcal{L}(X)$ and $v \in \mathcal{L}(X)$ there exists such $w \in \mathcal{L}(X)$ that $uwv \in \mathcal{L}(X)$. For irreducible SFTs there exists a unique measure of maximal entropy called the Parry measure by William Parry in [73].

The next theorem is well known and follows by noticing that a partition α that contains all elements of the form $\{x \in X \mid x_0 = a\}$ for each $a \in \Sigma$ is a generating partition. A proof can be found for example in [54].

***Theorem* 2.6.1.** *[54] Let (X, σ) be a subshift. Then $h(\sigma) = \lim_{n\to\infty} \frac{|\mathcal{L}_n(X)|}{n}$.*

More generally we can consider shifts as a group action. Let G be a countable group. A *full shift* is a dynamical system (Σ^G, σ), where analogously Σ is a finite set of symbols and the group action σ is called the shift and it is defined in such a way that $(\sigma(g,x))_h = (gx)_h = x_{g^{-1}h}$. The metric d_σ of the space Σ^G is defined as $d_\sigma(x,y) = 2^{-\inf\{|g|\in\mathbb{N}|x_g\neq y_g\}}$. Recall that here $|g|$ denotes the distance to the identity element of the group with respect to the word metric. A *subshift* (X, σ) is a dynamical system where X is any closed subset of a full shift such that $G(X) \subseteq X$. Again we refer to the space itself as a full shift or subshift.

Let $F \subseteq G$ be finite and let $p \in \Sigma^F$. A *cylinder* is a set $D_p = \{x \in X \mid x|_F = p\}$. Denote by $\mathcal{C}(\Sigma^G)$ the set of all cylinders of Σ^G together with the empty set, i.e.

$$\mathcal{C}(\Sigma^G) = \{D_p \mid p \in \Sigma^F, F \subseteq G \text{ and } |F| < \infty\} \cup \{\emptyset\}.$$

The set $\mathcal{C}(\Sigma^G)$ is countable and it is a clopen basis of the topology of Σ^G so the space is second countable. Therefore $\sigma(\mathcal{C}(\Sigma^G))$ is the Borel σ-algebra of Σ^G.

The set $\mathcal{C}(\Sigma^G)$ is a semiring. To see this clearly $\emptyset \in \mathcal{C}(\Sigma^G)$ so S1 holds. Let $F \subseteq G$ and $F' \subseteq G$ be finite. Let $p \in \Sigma^F$ and $p' \in \Sigma^{F'}$ and suppose that $D_p \cap D_{p'} \neq \emptyset$. Let $F'' = F \cup F'$ and let $p'' \in \Sigma^{F''}$ be such that $p''|_F = p$ and $p''|_{F'} = p'$, such p'' exists because $D_p \cap D_{p'}$ is non-empty. Then $D_p \cap D_{p'} = D_{p''}$. Hence inductively we have that S3" holds. We have that $\Sigma^G \setminus D'_p = \bigcup_{q\in\Sigma^{F'}\setminus\{p'\}} D_q$. Then $D_p \setminus D'_p = \bigcup_{q\in\Sigma^{F'}\setminus\{p'\}} (D_q \cap D_p)$, where each $D_q \cap D_p$ is in the set $\mathcal{C}(\Sigma^G)$ by S3". This shows that S2' holds. Then by Carathéodory's Extension Theorem 2.5.17, we have that any measure defined on the set of cylinders extends uniquely to the Borel σ-algebra of Σ^G.

We can consider shifts as measure-preserving dynamical systems if we equip them with a G-invariant measure.

Example 2.6.2. Let $\mathcal{B}$ be the Borel σ-algebra of Σ^G. Consider a set of symbols $\Sigma = [0, n-1]$ and associate them with a set of probabilities $\{q_0, q_1, \ldots q_{n-1}\}$ such that $\sum_{i=0}^{n-1} q_i = 1$. Define $\mu : \mathcal{B} \to [0,1]_\mathbb{R}$ on the cylinder sets in such a way that $\mu(D_p) = \prod_{g\in F} q_{p_g}$. By Carathéodory's Extension Theorem 2.5.17 μ is then uniquely defined on the set $\sigma(\mathcal{C}(\Sigma^G))$, and because the set of cylinders is a countable basis, we have that $\sigma(\mathcal{C}(\Sigma^G)) = \mathcal{B}$. Furthermore we have that $g^{-1}D_p = \{c \in \Sigma^G \mid c|_{g^{-1}F} = p\}$ for every $p \in \Sigma^F$ and $g \in G$ and hence $\mu(D_p) = \prod_{g\in F} q_{p_g} = \mu(g^{-1}D_p)$

for each $g \in G$. Then by Theorem 2.5.18 we have that μ is G-invariant. Therefore we have that $(X, \mathcal{B}, \mu, G)$ is a measure-preserving dynamical system. It is called the *Bernoulli shift*.

We do not consider Bernoulli shifts further in this work. Nevertheless they are interesting examples of measure-preserving dynamical systems and as we stated in the introduction, the study of them motivated the definition of the measure-theoretic entropy.

2.7 Cellular Automata

In this subchapter we will give two equivalent definitions for cellular automata. We might use either of them, depending on whichever is convenient for our purposes.

Definition 2.7.1. A *cellular automaton* is a dynamical system (X, f), where $X \subseteq \Sigma^G$ is a subshift and f is a continuous G-equivariant function, i.e. f commutes with the shift, i.e. $f \circ g = g \circ f$ for each $g \in G$.

We might also denote the cylinders by $P_F(x) = \{y \in X \mid x_g = y_g \ \forall g \in F\}$, where $F \subseteq G$ and $x \in X$, if we want to highlight a specific cylinder where a specific configuration belongs to. Then if $\alpha = \{P_{1_G}(x) \mid x \in X\}$, we have that $\alpha_F = \{P_{F^{-1}}(x) \mid x \in X\}$. This is because $\alpha_g = \{g^{-1}(P_{1_G}(x)) \mid x \in X\}$ and since $x_{1_G} = (g(g^{-1}x))_{1_G} = g^{-1}x_{g^{-1}}$, we have that $g^{-1}(P_{1_G}(x)) = P_{g^{-1}}(g^{-1}x)$ and so $\alpha_g = \{P_{g^{-1}}(g^{-1}x) \mid x \in X\} = \{P_{g^{-1}}(x) \mid x \in X\}$.

In the context of cellular automata, the subshift $X \subseteq \Sigma^G$ is called the *configuration space* and the elements $c \in X$ are called *configurations*. The symbols of Σ are called *states*.

A finite vector $V = (g_0, g_1, \ldots g_{n-1})$, where $g_i \in G$ for each $i \in [0, n)$ is called a *neighbourhood* vector. We will also by slight abuse of notation sometimes consider V as a set. If $r \in \mathbb{N}$ is smallest such that $V \subseteq B_r^G$, then V is a *radius-r neighbourhood* and r is the *radius* of the neighbourhood. A mapping $w : \Sigma^n \to \Sigma$ is called a *local rule*. A *global rule* $f : X \to X$ is defined in a way such that $f(x)_h = w(x_{hg_0}, x_{hg_1}, \ldots x_{hg_{n-1}})$ for each $h \in G$. We may also refer to the global rule f as a cellular automaton.

Definition 2.7.2. A tuple (X, V, w, G) is a *cellular automaton*, where X is a configuration space, V is a neighbourhood, w is a local rule and G is a group.

The proof of the following theorem can be found in [29] by Gustav A. Hedlund, where he also credits Morton L. Curtis and Roger Lyndon for the theorem.

***Theorem* 2.7.3.** *[29] [Curtis-Hedlund-Lyndon] The Definitions 2.7.1 and 2.7.2 are equivalent, i.e. a function $f : X \to X$ is a global rule if and only if f is continuous and G-equivariant.*

A cellular automaton $f : X \to X$ is *reversible* if it is bijective and the inverse mapping is also a cellular automaton. In fact each bijective cellular automaton is automatically reversible.

As a side remark we mention that in literature cellular automata are also referred to as *endomorphisms* of the subshifts and bijective cellular automata are referred to as *automorphisms* of the subshifts. When a subshift is fixed the set of endomorphisms forms a monoid and the set of automorphisms forms a group under the operation of composition. In this work we do not study the properties of these algebraic objects.

A *quiescent state* is a state satisfying $w(q, q, \dots q) = q$. A configuration c such that $|\{g \in G \mid c_g \neq q\}| < \infty$ is called q-finite or just finite if q is clear from the context.

Definition 2.7.4. A cellular automaton (X, V, w, G) is called *g_i-permutive* if the functions $\varphi_c : \Sigma \to \Sigma$ defined as $\varphi_c(x) = w(c_{g_1}, c_{g_2}, \dots c_{g_{i-1}}, x, c_{g_{i+1}} \dots c_{g_k})$ are permutations for each $c \in \Sigma^G$.

Definition 2.7.5. Let $\mathcal{A} = (X, V, w, G)$ be a cellular automaton. Let N be a normal subgroup such that $G/N \cong \mathbb{Z}$. Let $g \in G$ be such that $\langle gN \rangle = G/N$. Suppose that $\mathcal{A}$ is h-permutive. Let $I = \{k \mid g' \in V \text{ and } g' \in g^k N\}$. Suppose that $hN \neq g'N$ for each $g' \in V \setminus \{h\}$ and that $h \in g^k N$. If $k = \min I$, then $\mathcal{A}$ is *left permutive* (*with respect to* h, g and N). If $k = \max I$, then $\mathcal{A}$ is *right permutive* (*with respect to* h, g and N).

2.7.1 One-Dimensional Cellular Automata

Many of our results consider one-dimensional cellular automata and some properties are only defined for them. We may define them either as $(\Sigma^{\mathbb{Z}}, f)$, (X, f), $\mathcal{A} = (\Sigma, V, w)$, $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ or $f : X \to X$, where Σ is a finite set of states, f is a continuous mapping, which commutes with the shift, V is a neighbourhood vector, w is a local rule and X is a subshift.

We will call a configuration $c \in \Sigma^{\mathbb{Z}}$ *weakly periodic* if there exists such $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, that $f^n(c) = \sigma^m(c)$. A weakly periodic configuration is called *strictly weakly periodic* if it is not periodic. A finite strictly weakly periodic configuration is called a *glider*.

Definition 2.7.6. Let $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be a CA. Define $\tau_k : \Sigma^{\mathbb{Z}} \to (\Sigma^{2k+1})^{\mathbb{N}}$ such that $\tau_k(c)_i = f^i(c)_{[-k,k]}$. The mapping τ_k and the space $\Sigma_k(f) = \tau_k(\Sigma^{\mathbb{Z}})$ are called the *k-trace shift* or alternatively the *k-th column shift*.

Note that a trace shift is always a subshift over $\mathbb{N}$. We might sometimes denote $P_k(j) = |\mathcal{L}_j(\Sigma_k(f))|$ for the number of words of length j appearing in the trace shift.

The following theorem is well known and shows that the topological entropy can be calculated as the limit of the entropies of the column shifts as the column gets wider. We have included one source where the proof can be found, albeit it is not the first one as it was already used for example in [35].

***Theorem* 2.7.7.** *[54] Let $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be a CA. Then $h_f = \lim_{k\to\infty} h_{\tau_k}$, where $h_{\tau_k} = \lim_{j\to\infty} \frac{\ln(P_k(j))}{j}$.*

It is easy to see that $P_{k'}(j) \geq P_k(j)$ for any $k' \geq k$ and hence for any k we have that $h_{\tau_k} \leq h_f$.

The following was proven by Petr Kůrka in [53]. It gives a sufficient condition for when a one-dimensional cellular automaton has the shadowing property.

***Theorem* 2.7.8.** *[53] Let $(\Sigma^{\mathbb{Z}}, f)$ be a one-dimensional cellular automaton. If $\Sigma_k(f, X)$ is a SFT for each $k \in \mathbb{N}$, then $(\Sigma^{\mathbb{Z}}, f)$ has the shadowing property.*

Lyapunov exponents are a measure for the rate of (exponential) divergence of infinitesimally close trajectories in dynamical systems. They were first introduced in 1892 by Alexandr M. Lyapunov in his doctoral thesis titled: The General Problem of the Stability of Motion, English translation of which can be found in [60]. Since then they have been widely studied in the context of differentiable dynamical systems. Their importance in the study of non-linear dynamical systems, for example, can be found stated in [26] by Walter Greiner. In the context of cellular automata, the first formal definition of Lyapunov exponents for one-dimensional cellular automata is due to Mark A. Shereshevsky in [75], which he defines as shift-invariant values of the left and right perturbation speeds. In fact he proves an inequality connecting the measure-theoretical entropy of a cellular automaton and its shift-invariant Lyapunov exponents. Pierre Tisseur altered the definitions slightly and considers average Lyapunov exponents in [79], where he establishes a similar connection. Finally the pointwise Lyapunov exponents were defined in [9] by Xavier Bressaud and Pierre Tisseur.

Definition 2.7.9. Let $(\Sigma^{\mathbb{Z}}, f)$ be a one-dimensional cellular automaton, with a global rule $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$. For every $c \in \Sigma^{\mathbb{Z}}$, we define

$$W_m^+(c) = \{c' \in \Sigma^{\mathbb{Z}} \mid \forall i \geq m, c'_i = c_i\}$$

and

$$W_m^-(c) = \{c' \in \Sigma^{\mathbb{Z}} \mid \forall i \leq m, c'_i = c_i\}.$$

Furthermore we define

$$I_n^+(c) = \min\{m \in \mathbb{N} \mid f^i(W_{-m}^+(c)) \subseteq W_0^+(f^i(c)), \forall i \leq n\}$$

and

$$I_n^-(c) = \min\{m \in \mathbb{N} \mid f^i(W_m^-(c)) \subseteq W_0^-(f^i(c)), \forall i \leq n\}$$

Finally we define the *pointwise Lyapunov exponents* as

$$\lambda^+(c) = \liminf_{n\to\infty} \frac{I_n^+(c)}{n}$$

and

$$\lambda^-(c) = \liminf_{n\to\infty} \frac{I_n^-(c)}{n}$$

and the *global Lyapunov exponents* as

$$\lambda^+ = \lim_{n\to\infty} \max_{c\in\Sigma^{\mathbb{Z}}} \frac{I_n^+(c)}{n}$$

and

$$\lambda^- = \lim_{n\to\infty} \max_{c\in\Sigma^{\mathbb{Z}}} \frac{I_n^-(c)}{n}.$$

The Lyapunov exponents can be thought of as how fast a difference can propagate across configurations of a given cellular automaton.

The *shift-invariant Lyapunov exponents* are defined only for such points $c \in \Sigma^{\mathbb{Z}}$ that the following limits exist:

$$\lambda_\sigma^+(c) = \lim_{n\to\infty} \max_{i\in\mathbb{Z}} \frac{I_n^+(\sigma^i(c))}{n}$$

and

$$\lambda_\sigma^-(c) = \lim_{n\to\infty} \max_{i\in\mathbb{Z}} \frac{I_n^-(\sigma^i(c))}{n}$$

It was established in [75] that if μ is ergodic with respect to f then $\lambda_\sigma^+(c) = \lambda_\mu^+$ and $\lambda_\sigma^-(c) = \lambda_\mu^-$ almost everywhere, where $\lambda_\mu^+ \in \mathbb{R}_+$ and $\lambda_\mu^- \in \mathbb{R}_+$.

The first inequality was established by Mark A. Shereshevsky in [75]. To understand what it says we need to define the *local entropy* with respect to a point. Let $(X, \mathbb{B}, \mu, f)$ be a measure-preserving dynamical system. Define for each $x \in X$, $\epsilon \in (0,\infty)_{\mathbb{R}}$ and $n \in \mathbb{Z}_+$ the set $B_n(f,x,\epsilon) = \{y \in X \mid \forall i \in [0, n-1] : d(f^i(x), f^i(y)) \leq \epsilon\}$. Then $h_\mu(f,x) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} -\frac{\ln(\mu(B_n(f,x,\epsilon)))}{n}$. As a side note, it was established by Michael Brin and Anatole Katok in [10] that $h_\mu(f) = \int_{x\in X} h_\mu(f,x)\, d\mu$.

Theorem 2.7.10. *[75] Let $(\Sigma^{\mathbb{Z}}, \mathbb{B}, \mu, f)$ be a measure-preserving dynamical system, where $\mathbb{B}$ is the Borel σ-algebra of $\Sigma^{\mathbb{Z}}$ and $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is a cellular automaton. Suppose that μ is σ-invariant, where $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is the shift. Then*

$$h_\mu(f) \leq \int_{c\in\Sigma^{\mathbb{Z}}} h_\mu(\sigma, c)(\lambda_\sigma^+(c) + \lambda_\sigma^-(c))\, d\mu.$$

If additionally μ is ergodic with respect to f, then

$$h_\mu(f) \leq h_\mu(\sigma)(\lambda_\mu^+ + \lambda_\mu^-).$$

In addition the *average Lyapunov exponents* were considered by Pierre Tisseur in [79].

$$I_{n,\mu}^+ = \int_{c \in X} I_n^+(c)\, d\mu$$

and

$$I_{n,\mu}^- = \int_{c \in X} I_n^-(c)\, d\mu.$$

Then furthermore

$$I_\mu^+ = \liminf_{n \to \infty} \frac{I_{n,\mu}^+}{n}$$

and

$$I_\mu^- = \liminf_{n \to \infty} \frac{I_{n,\mu}^-}{n}.$$

In the same article the following upper bound is established, with the main difference with Theorem 2.7.10 being that μ is expected to be ergodic with respect to the shift and not the cellular automaton. It is also established that if μ is ergodic with respect to σ then $\lambda_\sigma^+(c) = \lambda_\mu^+$ and $\lambda_\sigma^-(c) = \lambda_\mu^-$ almost everywhere, where $\lambda_\mu^+ \in \mathbb{R}_+$ and $\lambda_\mu^- \in \mathbb{R}_+$. And furthermore in such case $I_\mu^+ \leq \lambda_\mu^+$ and $I_\mu^- \leq \lambda_\mu^-$.

***Theorem* 2.7.11.** *[79] Let $(\Sigma^\mathbb{Z}, \mathbb{B}, \mu, f)$ be a measure-preserving dynamical system, where $\mathbb{B}$ is the Borel σ-algebra of $\Sigma^\mathbb{Z}$ and $f : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z}$ is a cellular automaton. Suppose that μ is σ-invariant and ergodic with respect to σ, where $\sigma : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z}$ is the shift. Then*

$$h_\mu(f) \leq h_\mu(\sigma)(I_\mu^+ + I_\mu^-).$$

Using the knowledge that the uniform measure is ergodic with respect to the shift and that surjective cellular automata preserve the uniform measure the following upper bound is established for the topological entropy in [79].

***Theorem* 2.7.12.** *[79] Let $(\Sigma^\mathbb{Z}, f)$ be a surjective cellular automaton. Let μ be the uniform measure on the Borel σ-algebra of $\Sigma^\mathbb{Z}$. Then*

$$h_f \leq (\lambda_\mu^+ + \lambda_\mu^-) \ln(\Sigma).$$

We also have that $\lambda_\sigma^+(c) \leq \lambda^+$ and $\lambda_\sigma^-(c) \leq \lambda^-$ for each $c \in \Sigma^\mathbb{Z}$ and so $\lambda_\mu^+ \leq \lambda^+$ and $\lambda_\mu^- \leq \lambda^-$ for any f-invariant measure μ. From the Variational Principle 2.5.25 then follows the purely topological inequality.

Corollary 2.7.13. Let $(\Sigma^\mathbb{Z}, f)$ be a surjective cellular automaton. Then $h_f \leq (\lambda^+ + \lambda^-) \ln(\Sigma)$.

3 Turing Machines

3.1 Definition of a Turing Machine and Periodicities

A *Turing machine* (TM for short) is a 3-tuple $\mathcal{M} = (Q, \Gamma, \delta)$, where $Q = Q_w \uplus Q_m$ is a finite set of *states*, Γ is a finite set of *symbols* and $\delta = \delta_w \cup \delta_m$ is a set of *instructions*, where $\delta_w \subseteq Q_w \times \Gamma \times \Gamma \times Q$ is a set of *write instructions* and $\delta_m \subseteq Q_m \times \Delta \times Q$ is a set of *move instructions*, where $\Delta = \{-1, 0, 1\}$. Furthermore the following two conditions must hold: (1) If $(q, d, r) \in \delta$ and $(q, d', r') \in \delta$ then $d = d'$ and $r = r'$. (2) If $(q, a, b, r) \in \delta$ and $(q, a, b', r') \in \delta$ then $b = b'$ and $r = r'$. These conditions imply that we are only considering *deterministic* Turing machines.

A Turing machine is *reversible* if $\mathcal{M}^{-1} = (Q, \Gamma, \delta^{-1})$ is a Turing machine, where $Q = Q_{w'} \cup Q_{m'}$, $(r, -d, q) \in \delta^{-1}$ if $(q, d, r) \in \delta$ and $(r, b, a, q) \in \delta^{-1}$ if $(q, a, b, r) \in \delta$. Notice that the partitioning to move and write states usually changes here, i.e. we usually have that $Q_m \neq Q_{m'}$ and $Q_w \neq Q_{w'}$. We will call $\mathcal{M}^{-1}$ the inverse machine of $\mathcal{M}$.

A *configuration* is a 3-tuple (w, i, q), where $w \in \Gamma^{\mathbb{Z}}$ is the *tape*, $i \in \mathbb{Z}$ is the location of the *Turing machine head* and $q \in Q$. If a write instruction is applied to the configuration we will write $(w, i, q) \vdash (w', i, r)$ if $(q, w_i, w'_i, r) \in \delta_w$, where $w' \in \Gamma^{\mathbb{Z}}$ is such that $w'_k = w_k$ for each $k \neq i$. If a move instruction is applied to the configuration we will write $(w, i, q) \vdash (w, i + d, r)$ if $(q, d, r) \in \delta_m$. Inductively we define $\vdash^n$, where $\vdash$ is applied n times. Furthermore we will write $(w, i, q) \vdash^+ (w', j, r)$ if there exists such $n \in \mathbb{Z}_+$, that $(w, i, q) \vdash^n (w', j, r)$ holds. If we need to specify in which Turing machine the computation happens in we might add the Turing machine as a subscript and denote $\vdash_{\mathcal{M}}$. We might also occasionally write $\vdash (w, i, q) = (w', j, r)$, when $(w, i, q) \vdash (w', j, r)$.

A Turing machine is *complete* if for each configuration (w, i, q), there exists (w', j, r) such that $(w, i, q) \vdash (w', j, r)$.

We will call a configuration (w, i, q) *periodic* if $(w, i, q) \vdash^+ (w, i, q)$ and *weakly periodic* if there exists such $j \in \mathbb{Z}$, that $(w, i, q) \vdash^+ (\sigma^j(w), i - j, q)$. Furthermore we will call a configuration *strictly weakly periodic* if it is weakly periodic, but not periodic. We will call a Turing machine *periodic* if all its configurations are periodic, and *aperiodic* if none of its configurations are weakly periodic.

More generally we can consider Turing machines as 4-tuples $\mathcal{M} = (Q, \Gamma, \delta, G)$, where G is a group. The set of states Q and the set of symbols Γ is defined as before and so is the set of write instructions δ_w. The set of move instructions is also analogous, except the set Δ is a finite subset of G. The set of configurations is then $\Gamma^G \times G \times Q$. Our need for the more general versions is minor and, with the exception of the case $G = \mathbb{Z}$, is limited to Turing machines such that $G = \mathbb{Z}_m$, i.e. the additive group of integers modulo m and $\Delta = \{-1, 0, 1\}$. Notice that we can convert any Turing machine over $\mathbb{Z}$ to one over $\mathbb{Z}_m$ by simply copying everything as is, expect taking the movement modulo m and considering finite configurations. The same exact computation is performed in both of the machines while the Turing machine head does not move from the cell 0 to the cell $m - 1$ or vice versa.

Many of our constructions require aperiodic reversible Turing machines and some even complete ones. It was not at all obvious that such machines even exist. It was even conjectured by Petr Kůrka in [56] that aperiodic Turing machines did not exist. This conjecture was proven false in [8] by Vincent D. Blondel, Julien Cassaigne, and Codrin Nichitiu, where a complete aperiodic Turing machine was constructed. Later a complete reversible aperiodic Turing machine was constructed in [11] by Julien Cassaigne, Nicolas Ollinger, and Rodrigo Torres-Avilés.

***Theorem* 3.1.1.** *[11] There exists a complete reversible aperiodic Turing machine.*

3.2 Graph Representation of Turing Machines

Graphs offer a convenient way to represent Turing machines. Let $\Delta' = \{-, 0, +\}$. Given a machine $\mathcal{M} = (Q, \Gamma, \delta)$ we can define a multigraph (V, E, ν) with an edge labelling $\varphi : E \to \Delta' \cup \Gamma|\Gamma$, where $V = Q$ and $e \in E$ if and only if $\nu(e) = (q, r)$ and either 1) $\varphi(e) = \alpha(d)$ if $(q, d, r) \in \delta_m$, where $\alpha(1) = +$, $\alpha(-1) = -$ and $\alpha(0) = 0$; or 2) $\varphi(e) = a|b$ if $(q, a, b, r) \in \delta_w$. A graph representation of the edges associated to move and write instruction can be seen in Figures 4 and 5 respectively.

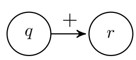

Figure 4. A graph representation of an edge and its labelling associated to a move instruction $(q, 1, r)$.

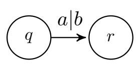

Figure 5. A graph representation of an edge and its labelling associated to a write instruction (q, a, b, r).

Example 3.2.1. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine, where $Q = \{q, r, s\}$, $\Gamma = \{a, b\}$ and $\delta = \{(q, b, a, r), (s, a, b, q), (r, 1, s)\}$. This Turing machine moves the symbol b in the right direction if the machine head starts at the location of b in the state q and the content of the tape elsewhere is a. The graph representation of this Turing machine can be seen at Figure 6.

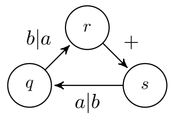

Figure 6. A graph representation of a Turing machine with two write states and one move state.

A version that disallows concurrent write and move instructions of the SMART machine constructed in [11] can be seen in Figure 7.

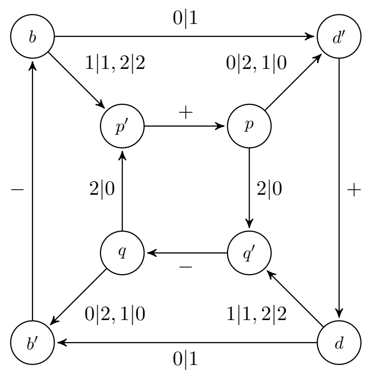

Figure 7. A graph representation of the SMART machine.

3.3 Construction Techniques for Turing Machines

In this subchapter we will recall some useful methods for constructing new Turing machines from existing ones.

Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine. We will call a TM $\mathcal{M}' = (Q', \Gamma, \delta')$ a *copy* of $\mathcal{M}$ if there exists a bijection $\varphi : Q \to Q'$ such that $(\varphi(q), d, \varphi(r)) \in \delta'_m$ if and only if $(q, d, r) \in \delta_m$ and $(\varphi(q), a, b, \varphi(r)) \in \delta'_w$ if and only if $(q, a, b, r) \in \delta_w$. We call Q' the *copied state set* and $\varphi(q)$ the *copied state* of q. It is of course a trivial process to make copies of existing Turing machines. To simplify the notation, provided that it is clear from context, we might denote the state sets of multiple TMs by the same set despite the state sets being disjoint.

A TM $\mathcal{M} = (Q, \Gamma, \delta)$ is an *union of n Turing machines* $\mathcal{M}_i = (Q_i, \Gamma, \delta_i)$, where $i \in \{0, 1, ..., n-1\}$, if $Q = \biguplus_{i=0}^{n-1} Q_i$ and $\delta = \bigcup_{i=0}^{n-1} \delta_i$. When constructing larger Turing machines by taking unions of them, one might want to be able to move between the sets of states of the different machines. Next we introduce special set of states and state-symbol pairs where such transitions can naturally occur.

Definition 3.3.1. If $q \in Q_w$ and $a \in \Gamma$ are such that $(q, a, b, r) \notin \delta$ for each pair $b \in \Gamma$ and $r \in Q$, then we call (q, a) an *error pair*. If $q \in Q_m$ and $(q, d, r) \notin \delta$ for each $d \in \{-1, 0, 1\}$ and $r \in Q$, we call q an *error state*.

Notice that a Turing machine is a complete Turing machine if and only if there are no error pairs and no error states.

Definition 3.3.2. Let $\mathcal{M}$ be a reversible TM. Then a pair (q, a) is a *defective pair* if (q, a) is an error pair of the inverse machine $\mathcal{M}^{-1}$. Similarly we call a state r a *defective state* if it is an error state of the inverse machine $\mathcal{M}^{-1}$.

In Example 3.2.1 (q, a) and (s, b) are error pairs and (r, b) and (q, a) are defective pairs. An error state would have no outgoing edges and a defective state would have no incoming edges in the graph representation.

When taking unions of Turing machines one can add transitions from error pairs of one machine to defective pairs of another one and similarly with error states and defective states. We will see an especially useful example of this construction method in Definition 3.3.3. The technique was developed in [46] by Jarkko Kari and Nicolas Ollinger to prove the undecidability of the periodicity problem for reversible and complete Turing machines. It is also applied extensively in [22] and [21] by Anahí Gajardo, Nicolas Ollinger and Rodrigo Torres-Avilés in the proofs of undecidability of the transitivity problem, the minimality problem and the zero entropy problem, for example. We will also apply it to prove undecidability of a problem considering strictly weakly periodic points in Theorem 3.6.2.

Definition 3.3.3. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a reversible Turing machine. Let $\mathcal{M}^+ = (Q^+, \Gamma, \delta^+)$ and $\mathcal{M}^- = (Q^-, \Gamma, \delta^-)$ be copies of $\mathcal{M}$ and its inverse machine respectively. For each $q \in Q$ we denote as $q^x \in Q^x$ the copied states of q, where $x \in \{+, -\}$. Let $\delta' = \{(q^x, a, a, q^y) \mid (q^x, a) \text{ is an error pair of } Q_w^x \text{ and } x \neq y\}$ and $\delta'' = \{(q^x, 0, q^y) \mid q^x \text{ is an error state of } Q_m^x \text{ and } x \neq y\}$. Let $\mathcal{M}^0 = (Q^0, \Gamma, \delta^0)$ be the union of $\mathcal{M}^+$ and $\mathcal{M}^-$. Define a TM $\mathcal{M}' = (Q^0, \Gamma, \delta^1)$, where $\delta^1 = \delta^0 \cup \delta' \cup \delta''$. The TM $\mathcal{M}'$ is referred as a TM constructed from $\mathcal{M}$ by *reversing the computation*.

It might happen that an error state is a write state of the inverse machine and the construction of the above definition does not work, but in such case we simply change such error states from move states to write states and we have an equivalent Turing machine, for which reversing the computation is well defined. Then with this modification in the previous definition the error pairs (q^x, a) of $\mathcal{M}^x$ are the defective pairs (q^y, a) of $\mathcal{M}^y$, when $x \neq y$ and analogously a similar statement is true for the error and defective states. It is easy to see that a machine constructed via reversing the computation is a reversible and complete Turing machine.

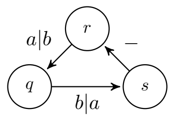

Figure 8. A graph representation of the inverse machine of the Turing machine of Example 3.2.1.

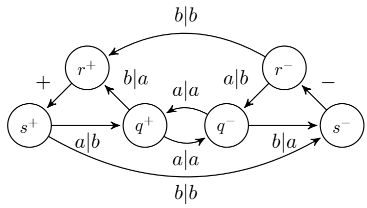

Figure 9. A graph representation of the Turing machine constructed from the Turing machine of Example 3.2.1 by reversing the computation.

Example 3.3.4. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be the Turing machine from Example 3.2.1. We will construct the Turing machine $\mathcal{M}'$ by reversing the computation as in Definition 3.3.3. The graph representation of its inverse machine can be seen in Figure 8.

Using the notation of Definition 3.3.3 we get a copy of the original machine as $\mathcal{M}^+ = (Q^+, \Gamma, \delta^+)$, where $Q^+ = \{q^+, r^+, s^+\}$, $\Gamma = \{a, b\}$ and

$$\delta^+ = \{(q^+, b, a, r^+), (s^+, a, b, q^+), (r^+, 1, s^+)\}.$$

And similarly we get a copy of the inverse machines as $\mathcal{M}^- = (Q^-, \Gamma, \delta^-)$, where $Q^- = \{q^-, r^-, s^-\}$, $\Gamma = \{a, b\}$ and

$$\delta^- = \{(r^-, a, b, q^-), (q^-, b, a, s^-), (s^-, -1, r^-)\}.$$

Connecting the error pairs with defective pairs will give us a complete and reversible Turing machine. These connections are contained in the sets δ' and δ''. We have that

$$\delta' = \{(q^+, a, a, q^-), (s^+, b, b, s^-), (r^-, b, b, r^+), (q^-, a, a, q^+)\}$$

and $\delta'' = \emptyset$.

The constructed machine is $\mathcal{M}' = (Q^0, \Gamma, \delta^1)$, where $Q^0 = Q^+ \cup Q^-$ and $\delta^1 = \delta' \cup \delta^+ \cup \delta^-$. The graph representation of this machine can be seen in Figure 9.

3.4 Turing Machines as Dynamical Systems

Petr Kůrka introduced two ways of defining complete Turing machines as dynamical systems in [56]. Both of them are straightforward constructions from the standard definition. We simply adjust the configuration space slightly to achieve a compact metric space and then we define a continuous function that remains faithful to the transition rule.

The first system is called *Turing machine with moving tape* or *TMT* for short. In TMT, the location of the Turing machine head is fixed to the origin and the tape moves instead of the Turing machine head. For example if the machine reads a right move, the tape moves left, i.e. the content at each cell gets shifted left by one cell. More specifically, the space is defined as $X = \Gamma^{\mathbb{Z}} \times Q$ and the function $f : X \to X$ works as follows: If $(q, d, r) \in \delta_m$ then $f(w, q) = (\sigma^d(w), r)$ for each $(w, q) \in \Gamma^{\mathbb{Z}} \times Q_m$. The write instruction reads the tape content at origin and rewrites it according to the instructions, i.e. for each $(q, a, b, r) \in \delta_w$ we have that $f(w, q) = (w', r)$, where $w_0 = a$, $w'_0 = b$, and $w_i = w'_i$ for each $i \neq 0$. The distance $d : X \to \mathbb{R}$ is defined as $d((w, q), (w', q')) = 2$ if $q \neq q'$ and $d_\sigma(w, w')$ if $q = q'$.

The second system is called *Turing machine with moving head* or *TMH* for short. The function of this system works more like a computation of a traditional Turing machine. The space is defined as $X = \{w \in ((Q \times \Gamma) \cup \Gamma)^{\mathbb{Z}} \mid w_{Q\times\Gamma} = 1\} \cup \Gamma^{\mathbb{Z}}$ equipped with the distance d_σ. The function $f : X \to X$ is defined as follows: If $w \in \Gamma^{\mathbb{Z}}$, then $f(w) = w$. Otherwise if $w_j = (q, a)$ and $(q, a, b, r) \in \delta_w$, then $f(w) = w'$, where $w'_j = (r, b)$ and $w_i = w'_i$ for each $i \neq j$. Finally if $w_j = (q, a)$ and $(q, d, r) \in \delta_m$, then $f(w) = w'$, where either 1) $w'_{j+d} = (r, w_{j+d})$, $w'_j = a$ and $w_i = w'_i$ for each $i \notin \{j, j + d\}$ if $d \neq 0$ or 2) $w'_j = (r, a)$ and $w_i = w'_i$ for each $i \neq j$ if $d = 0$.

One can check that the spaces are indeed compact metric spaces and the functions are continuous in their respective spaces.

3.5 Speed of Turing Machines

We first define a function that tracks the location of the Turing machine head given some initial configuration and a time step.

Definition 3.5.1. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine. Let X be the configuration space of the TM. Define $f_T : X \times \mathbb{N} \to \mathbb{Z}$ as $f_T((w, i, q), n) = j$ if $(w, i, q) \vdash^n (w', j, r)$ for some w' and r.

Using the tracking function we define a set of visited locations given some initial configuration and a time step.

Definition 3.5.2. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine. Let X be the configuration space of the TM. Define $f_V : X \times \mathbb{N} \to \text{Fin}(\mathbb{N})$ as $f_V(x, n) = \{f_T(x, j) \mid j \leq n\}$.

Finally we can calculate the maximum amount of visited locations by any computation by a given time and define the notion of speed.

Definition 3.5.3. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine. Let X be the configuration space of the TM. Define the *movement bound* $f_M : \mathbb{N} \to \mathbb{N}$ as $f_M(n) = \max_{x \in X} |f_V(x, n)|$. The *speed of the TM* $\mathcal{M}$ is defined as $f_S(\mathcal{M}) = \lim_{n \to \infty} \frac{f_M(n)}{n}$.

***Theorem* 3.5.4.** *[39] Let $\mathcal{M}$ be a Turing machine, and f_M be its movement bound. If $\mathcal{M}$ is aperiodic, then f_M is sublinear.*

The above Theorem by Emmanuel Jeandel implies that any aperiodic Turing machine $\mathcal{M}$ has zero speed, i.e. $f_S(\mathcal{M}) = 0$. We will also make use of the following upper bound for the movement bound by Pierre Guillon and Ville Salo.

***Theorem* 3.5.5.** *[28] Let $\mathcal{M} = (Q, \Gamma, \delta)$ be Turing machine, and f_M its movement bound. If $\mathcal{M}$ is aperiodic, then $f_M = \mathcal{O}(\frac{n}{\ln n})$.*

***Theorem* 3.5.6.** *[39] Let $\mathcal{M}$ be a Turing machine, then $f_S(\mathcal{M}) > 0$ if and only if there exists a strictly weakly periodic configuration.*

The following two theorems establish a connection between the speed and the topological entropy of Turing machines.

***Theorem* 3.5.7.** *[39] Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine and denote by $h_{\mathcal{M}}$ its entropy. Then $f_S(\mathcal{M}) \geq \frac{h_{\mathcal{M}}}{\ln |\Gamma|}$.*

***Theorem* 3.5.8.** *[39] Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine and denote by $h_{\mathcal{M}}$ its entropy and A a non-empty set of symbols. Then we can effectively construct a Turing machine $\mathcal{M}^A = (Q', \Gamma \times A, \delta')$ such that $\frac{h_{\mathcal{M}^A}}{\ln |A|} \geq f_S(\mathcal{M})$ and $f_S(\mathcal{M}) = f_S(\mathcal{M}^A)$.*

3.6 Decision Problems for Reversible Turing Machines

The decision problems that we are interested in are as follows:

ARTM REACHABILITY: Given an aperiodic and reversible Turing machine and two states q_α and q_ω, decide if q_ω is reachable from q_α.

RCTM STRICTLY WEAKLY PERIODIC CONFIGURATION: Given a reversible and complete Turing machine, decide if there exists a strictly weakly periodic configuration.

RCTM ZERO SPEED: Given a reversible and complete Turing machine, decide if its speed is zero.

RCTM ZERO ENTROPY: Given a reversible and complete Turing machine, decide if its entropy is zero.

The first and fourth have already been proven to be undecidable in [46] by Jarkko Kari and Nicolas Ollinger and [21] by Anahí Gajardo, Nicolas Ollinger and Rodrigo Torres-Avilés respectively. The second and the third will be proved to be undecidable in the following. Their undecidability has been independently proved in [80] by Rodrigo Torres-Avilés. We use the reduction from the first decision problem to the second and third and give an alternative proof that shows that the problems are undecidable. Our construction method introduces a novel way to move finite simulation areas of the tape, which makes it suitable for further reductions to problems concerning reversible cellular automata. There is no apparent way to make similar reductions, at least to all of the problems we consider, from the construction of [80] in a straightforward way. The construction method we develop in this chapter might also be of further interest in the framework of Turing machines. We also show how the undecidability of the fourth problem follows easily from the undecidability of the third problem, although this result is not new either, it gives an alternate proof all the same.

***Theorem* 3.6.1.** *[46]* ARTM REACHABILITY *is undecidable.*

***Theorem* 3.6.2.** RCTM STRICTLY WEAKLY PERIODIC CONFIGURATION *is undecidable.*

Proof. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be an aperiodic and reversible Turing machine. We will prove the theorem via reduction from the ARTM REACHABILITY problem, which is known to be undecidable by Theorem 3.6.1. To this end, for a given two states q_α and q_ω of Q, we will construct a Turing machine $\mathcal{M}_{wp}$ such that q_ω is reachable from q_α in $\mathcal{M}$ if and only if $\mathcal{M}_{wp}$ has a strictly weakly periodic configuration.

Without loss of generality, we can assume that q_α is a defective state and q_ω is an error state. The reason for that is that if we can reach q_ω from q_α during finitely many steps, then there exists a last time that the computation sees the state q_α and hence we can just begin the computation from that point. Furthermore we can assume that when starting from q_α, the first two instructions are a move to the right $(q_\alpha, +, q'_\alpha)$ and a move to the left $(q'_\alpha, -, q''_\alpha)$. If that is not the case, we add these moves to δ and the states q'_α and q''_α to Q. After that if $q_\alpha \in Q_m$, we replace (q_α, d, r_0) with (q''_α, d, r_0) and if $q_\alpha \in Q_w$, we replace each $(q_\alpha, a, b, r) \in \delta$ with (q''_α, a, b, r). We can also assume that there exists a special symbol $\#$ such that $(q, \#)$ is both an error and a defective pair for each $q \in Q_w$, as if not, we could simply add such symbol. We will denote $\Gamma' = \Gamma \setminus \{\#\}$.

We will construct three copies of $\mathcal{M}$ and three copies of its inverse machine $\mathcal{M}^-$ to achieve six new TMs $\mathcal{M}^x_y = (Q^x_y, \Gamma, \delta^x_y)$, for given pairs of subscripts and superscripts that we will introduce in the following paragraphs. The superscript x is either $+$ or $-$ depending on whether the machine is a copy of the original machine or its inverse, respectively. If we need to specify from which machine the state is, we will add the name of the Turing machine as a subscript. We will describe how to modify each copy to suit our needs. We will say that we will replace an instruction $(q, d, r) \in \delta_y^x$ with a sub-routine as described by a transition graph depicted in a given figure. What we will mean by this is that we will add all such states to the new machine's state set Q_y^x, which are depicted in the same color in the transition graph as the states q and r. Additionally we will then remove (q, d, r) from δ_y^x and add to it all the instructions that are between nodes of the same color.

First we will construct a TM $\mathcal{M}_R^+ = (Q_R^+, \Gamma, \delta_R^+)$ by taking a copy of $\mathcal{M}$ and replacing each instruction $(q, +, r) \in \delta_R^+$, where $q \neq q_\alpha$ with the sub-routine in Figure 10. We will also replace the instruction $(q_\alpha, +, q'_\alpha) \in \delta_R^+$ with the sub-routine in Figure 11.

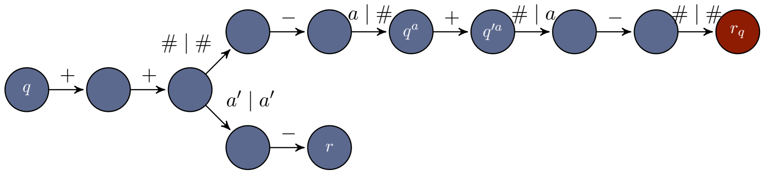

Figure 10. A transition graph that represents a sub-routine, which replaces each instruction $(q, +, r)$ of $\mathcal{M}_R^+$, where $q \neq q_\alpha$. Here $a \in \Gamma'$ and $a' \in \Gamma'$. The states q^a and q'^a are unique for each $a \in \Gamma'$. The last pair of nodes on the top-right corner represents a transition from $\mathcal{M}_R^+$ to $\mathcal{M}_R^-$.

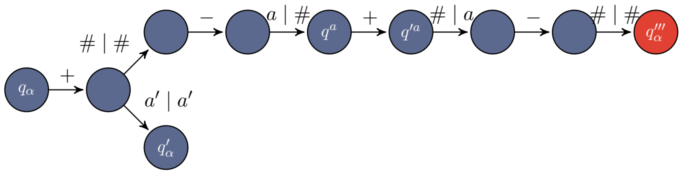

Figure 11. A transition graph that represents a sub-routine, which replaces the instruction $(q_\alpha, +, q'_\alpha)$ of $\mathcal{M}_R^+$. . Here $a \in \Gamma'$ and $a' \in \Gamma'$. The states q^a and q'^a are unique for each $a \in \Gamma'$. The last pair of nodes on the top-right corner represents the transition from $\mathcal{M}_R^+$ to $\mathcal{M}_L^-$.

Then we will construct a machine $\mathcal{M}_R^- = (Q_R^-, \Gamma, \delta_R^-)$ by taking a copy of the inverse machine $\mathcal{M}^-$ and by replacing each instruction $(r, -, q) \in \delta_R^-$ with the sub-routine in the left side of Figure 12.

Figure 12. A transition graph on the left side represents a sub-routine, which replaces each instruction $(r, -, q)$ of $\mathcal{M}_R^-$, where $r \neq q'_\alpha$. The transition graph on the right side represents the transition from $\mathcal{M}_R^-$ to $\mathcal{M}_R^+$ at the state q_α. Here $a \in \Gamma'$.

The machines $\mathcal{M}_R^+$ and $\mathcal{M}_R^-$ are constructed in such a way, that under certain conditions, when starting a computation from the state q_α, the machine $\mathcal{M}_R^+$ will try to find the first $\#$ symbol from the right side of the computation area while iterating as the original machine and if it finds one the symbol $\#$ is moved one step left and the computation transitions to that of the machine $\mathcal{M}_R^-$, which rewinds the computation back to the starting position at the state q_α.

The third TM $\mathcal{M}_L^+ = (Q_L^+, \Gamma, \delta_L^+)$ is constructed by copying the original machine $\mathcal{M}$ and by replacing each instruction $(q, -, r) \in \delta_L^+$ with the sub-routine in Figure 13.

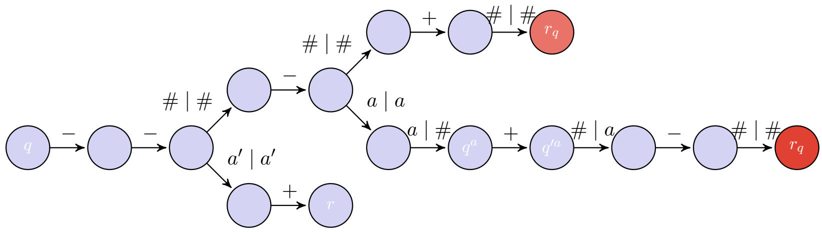

Figure 13. A transition graph that represents the sub-routine, which replaces the instruction $(q, -, r)$ of $\mathcal{M}_L^+$. Here $a \in \Gamma'$ and $a' \in \Gamma'$. The states q^a and q'^a are unique for each $a \in \Gamma$. The last pair of nodes on the top-right corner represents a transition from $\mathcal{M}_L^+$ to $\mathcal{M}_F^-$ and the last pair of nodes on the middle represents a transition from $\mathcal{M}_L^+$ to $\mathcal{M}_L^-$.

The fourth TM $\mathcal{M}_L^- = (Q_L^-, \Gamma, \delta_L^-)$ is constructed by taking a copy of the inverse machine $\mathcal{M}$ and by replacing each instruction $(r, +, q) \in \delta_L^-$ with the sub-routine in the top-left corner of Figure 14 and additionally $(q'_\alpha, -, q_\alpha) \in \delta_L^-$ is replaced with the sub-routine in the bottom of Figure 14.

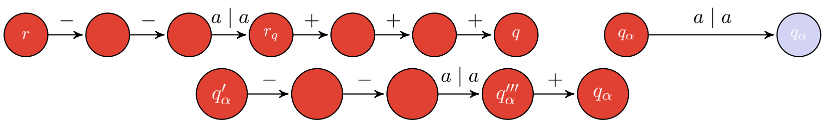

Figure 14. The transition graph on the bottom represents the sub-routine, which replaces the instruction $(q'_\alpha, -, q_\alpha)$ of $\mathcal{M}_L^-$. The transition graph on the top-left corner represents the sub-routine, which replaces the instruction $(r, +, q)$ of $\mathcal{M}_L^-$. The transition graph on the top-right corner represents the transition from $\mathcal{M}_L^-$ to $\mathcal{M}_L^+$ at the state q_α. Here $a \in \Gamma'$.

The machines $\mathcal{M}_L^+$ and $\mathcal{M}_L^-$ behave analogously to the machines with the subscript R, except this time the symbol $\#$ is searched from the left side of the computation area.

The fifth TM $\mathcal{M}_F^+ = (Q_F^+, \Gamma, \delta_F^+)$ is just the exact copy of the original machine $\mathcal{M}$.

And finally the sixth TM $\mathcal{M}_F^- = (Q_F^-, \Gamma, \delta_F^-)$ is constructed by taking a copy of the inverse machine $\mathcal{M}^-$ and by replacing each instruction $(r, +, q) \in \delta_F^-$ with the sub-routine in Figure 15.

The machines $\mathcal{M}_F^+$ and $\mathcal{M}_F^-$ are built to ensure that the machine reaches the state q_ω from q_α if the original machines does and then again rewinds the computation

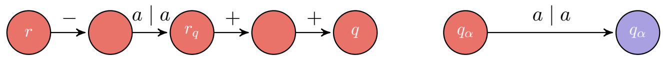

Figure 15. The transition graph on the left side represents a sub-routine, which replaces each instruction $(r, +, q)$ of $\mathcal{M}_F^-$. The transition graph on the right side represents the transition from $\mathcal{M}_F^-$ to $\mathcal{M}_F^+$ at the state q_α. Here $a \in \Gamma'$.

back to the starting position at the state q_α.

Let $\mathcal{M}'$ be the union of our six Turing machines constructed so far. We will add transitions between the state sets of the different machines by adding the instructions from Figures 10 - 16, where the pairs of nodes are depicted by two different colors. Our final construction is the Turing machine $\mathcal{M}_{wp}$, which is constructed from $\mathcal{M}'$ and the inverse machine $\mathcal{M}'^-$ by the method of reversing the computation. A high-level overview of the machine $\mathcal{M}_{wp}$ can be found from Figure 17, where the purpose of each of the six machine is briefly explained.

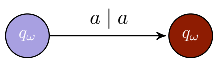

Figure 16. A transition graph that represents the only transition from $\mathcal{M}_F^+$ to $\mathcal{M}_R^-$. Here $a \in \Gamma'$.

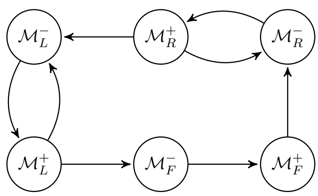

Figure 17. A graph depiction of the high-level overview of how the machine $\mathcal{M}_{wp}$ works. The machine $\mathcal{M}_R^+$ is tasked to search for the symbol $\#$ from the right side of the initial position. If the symbol is found, it is moved one step left. If the initial position was crossed during this move the computation transitions to the machine $\mathcal{M}_L^-$ and if not then to the machine $\mathcal{M}_R^-$. Similarly the machine $\mathcal{M}_L^+$ is tasked to search for the symbol $\#$ from the left side of the initial position. If the symbol is found, it is moved one step left. If the symbol $\#$ was moved next to another one the computation transitions to the machine $\mathcal{M}_F^-$ and if not then to the machine $\mathcal{M}_L^-$. All of the machines $\mathcal{M}_x^-$, where $x \in \{R, L, F\}$, are tasked with reversing the computation and if they reach state q_α the computation transitions to the machine $\mathcal{M}_x^+$. Finally the computation transitions from the machine $\mathcal{M}_F^+$ to the machine $\mathcal{M}_R^-$ if and only if it reaches the state q_ω.

Notice that the constructed machine $\mathcal{M}_{wp}$ is complete and reversible. This follows because the original machine was reversible and furthermore one can easily check that during our modifications the reversibility remains. The only parts where the reversibility could be lost is at the transitions between different machines. One can check that the state r_q or the state q_α''' is entered either from a state belonging to a copy of the inverse machine if a symbol other than $\#$ is read and from a state belonging to a copy of the original machine if the $\#$ symbol is read. The Turing machine does not alter the tape content in any of these moves so the reversibility remains. Other transitions happen at either the state q_α or at the state q_ω, but they are not an issue where reversibility could be lost since they are a defective state and an error state respectively in the original machine.

Next we will prove, that the machine $\mathcal{M}_{wp}$, which we constructed above, has a strictly weakly periodic point if and only if the state q_ω is reachable from the state q_α in the original machine $\mathcal{M}$.

We will first assume that q_ω is reachable from q_α. Since the computation is finite, there exists such $n \in \mathbb{N}$, that exactly $n+1$ indices of the tape are visited during the computation. By shifting, if necessary, we can assume that the visited indices are in the interval $[0,n]$. Let $x \in \Gamma^{\mathbb{Z}}$, $y \in \Gamma^{\mathbb{Z}}$, $i_\alpha \in [0,n]$ and $i_\omega \in [0,n]$ be such that $(x, i_\alpha, q_\alpha) \vdash^+ (y, i_\omega, q_\omega)$ in $\mathcal{M}$ and let $w = x_{[0,n]}$. For each $i \in [0,n]$ let $t_i > 2$ be the smallest integer such that $(x, i_\alpha, q_\alpha) \vdash^{t_i} (y', i, q)$ for some pair $y' \in \Gamma^{\mathbb{Z}}$ and $q \in Q$.

For any word $u \in \Gamma^{n+1}$ we adapt a shorthand $u^{\#i,j} = {}^{-\infty}\#u_{[0,i]}\#u_{[i+1,n]}\#^{\infty}$, where $u^{\#i,j}_{[j,j+n+1]} = u_{[0,i]}\#u_{[i+1,n]}$ and if $j = 0$ we denote $u^{\#i,j} = u^{\#i}$. Notice first that $u^{\#i,j} = \sigma^{-j}(u^{\#i})$. Our goal is then to show that $(w^{\#n}, i_\alpha, q_\alpha) \vdash^+ (w^{\#n,1}, i_\alpha + 1, q_\alpha)$ in $\mathcal{M}_{wp}$, i.e. the word $w_{[0,n]}$ surrounded by $\#$ symbols has moved one step right, which would mean that $w^{\#n}$ is a weakly periodic configuration.

From the way we constructed $\mathcal{M}_R^+$ follows that if $(x, i, q) \vdash_{\mathcal{M}} (x', i', r)$ and $x_j \neq \#$ for each $j \in [i, i+2]$, then $(x, i, q) \vdash^k_{\mathcal{M}_R^+} (x', i', r)$, where $k \in \{1, 2, 4\}$ depends on the state q. We have that $k = 1$, if $q \in Q_w$ or $(q, d, r) \in \delta$, for some r where $d \neq 1$ as those instructions were not altered in $\mathcal{M}_R^+$. We have $k = 2$ if $q = q_\alpha$ as in this case the instructions from the bottom of Figure 11 are applied. And $k = 4$ if $(q, +, r) \in \delta$ for some r as in this case the instructions from the bottom of Figure 10 are applied.

We will then show that $(w^{\#i}, i_\alpha, q_\alpha) \vdash^+ (w^{\#i-1}, i_\alpha, q_\alpha)$ holds when the machine is using the states and instructions of $\mathcal{M}_R^+$ and $q_\alpha \in Q_R^+$ and $i \in (i_\alpha, n]$.

From the assumption that we do not have $\#$ at index $i_\alpha + 1$ it follows that $(w^{\#i}, i_\alpha, q_\alpha) \vdash^3 (w^{\#i}, i_\alpha, q''_\alpha)$. Let $t = t_i$. Then for each $k < t$, there exists such n_k, that if $(x, i_\alpha, q_\alpha) \vdash^k_{\mathcal{M}} (x', j, q)$ then $(w^{\#i}, i_\alpha, q_\alpha) \vdash^{n_k}_{\mathcal{M}_R^+} (w'^{\#i}, j, q)$, where $w' = x'_{[0,n]}$. This is because either the machine head is at a location of the indices $[0, i-2]$, and there are no $\#$ symbols in the indices $[0, i]$ or the machine head is at the index $i - 1$ and the state dictates no right moves by the definition of t.

Now let $x' \in \Gamma^{\mathbb{Z}}$ and $q \in Q$ be such that $(x, i_\alpha, q_\alpha) \vdash^{t-1}_{\mathcal{M}} (x', i-1, q)$ and $(w^{\#i}, i_\alpha, q_\alpha) \vdash^{n_{t-1}}_{\mathcal{M}_R^+} (w'^{\#i}, i-1, q)$, where $w' = x'_{[0,n]}$. As $(x', i-1, q) \vdash_{\mathcal{M}} (x', i, r)$, by definition of t and since the symbol at the index $i+1$ is $\#$, the machine $\mathcal{M}_R^+$ uses the instructions from the upper path of Figure 10. The machine takes 2 steps right, witnesses the symbol $\#$ at index $i+1$, moves one step left, stores the tape symbol and exchanges it to $\#$, moves right and replaces the symbol $\#$ at index with the symbol previously at index i, and then moves left and reads $\#$. After all this we have $(w'\#^i, i-1, q) \vdash^9_{\mathcal{M}_R^+} (w'\#^{i-1}, i, r_q)$ and then one step after $(w'\#^{i-1}, i, r_q) \vdash_{\mathcal{M}_R^-} (w'\#^{i-1}, i-1, q)$ as in Figure 12.

As the content at each index in $[0, i-1]$ differs from $\#$ and by definition of t and the assumption that $i_\alpha < i$ the computation stayed inside the indices $[0, i]$, therefore the computation using the states from $\mathcal{M}_R^-$ never sees the symbol $\#$ when moving from right to left as this would require a left move from index $i+1$. Hence the changes in Figure 12 do not affect the computation compared to the original machine, other than potentially adding some extra steps. This means that $(w'\#^{i-1}, i-1, q) \vdash^+_{\mathcal{M}_R^-} (w\#^{i-1}, i_\alpha, q_\alpha)$.

Assume then that $i = i_\alpha$. Then $(w\#^i, i_\alpha, q_{\alpha,\mathcal{M}_R^+}) \vdash^8_{\mathcal{M}_R^+} (w\#^{i-1}, i_\alpha, q'''_\alpha) \vdash^2_{\mathcal{M}_L^-} (w\#^{i-1}, i_\alpha+1, q_{\alpha,\mathcal{M}_L^+})$. This can be seen from the instructions of Figures 11 and 14.

If $i = i_\alpha - 1$, then the computation starts with $(w\#^i, i_\alpha+1, q_\alpha) \vdash_{\mathcal{M}_L^+} (w\#^i, i_\alpha+2, q'_\alpha)$ followed by a left move and as there is a symbol $\#$ at index i_α the middle path from Figure 13 is followed and so $(w\#^i, i_\alpha+2, q'_\alpha) \vdash^{10}_{\mathcal{M}_L^+} (w\#^{i-1}, i_\alpha-1, r_q)$, where $r = q''_\alpha$ and $q = q'_\alpha$. After that the instructions from top left of Figure 14 are followed first and then after the instructions in bottom of the same Figure and finally the transition in the top right of the same Figure and thus $(w\#^{i-1}, i_\alpha-1, r_q) \vdash^3_{\mathcal{M}_L^-} (w\#^{i-1}, i_\alpha+2, q'_\alpha) \vdash^5_{\mathcal{M}_L^-} (w\#^{i-1}, i_\alpha+1, q_{\alpha,\mathcal{M}_L^+})$, where $r = q''_\alpha$ and $q = q'_\alpha$.

If $i \in [0, i_\alpha - 1)$, then let $t = t_{i+1}$. Then as the computations of the original machine stays at the indices $[i+2, n]$ for the first $t-1$ steps for each $k < t$ there exists such n_k, that if $(x, i_\alpha, q_\alpha) \vdash^k_{\mathcal{M}} (x', j, q)$ then $(w\#^i, i_\alpha+1, q_\alpha) \vdash^{n_k}_{\mathcal{M}_L^+} (w'\#^i, j+1, q)$, where $w' = x'_{[0,n]}$. This is because the machine $\mathcal{M}_L^+$ behaves as the original with the exception of the left moves, but again those only cause some extra steps unless a left move happens at index $i+2$ of the original machine, but by definition no such moves happen before step t. So let $(w\#^i, i_\alpha+1, q_\alpha) \vdash^{n_{t-1}}_{\mathcal{M}_L^+} (w'\#^i, i+3, q)$. Then by definition of t a left move is used so the machine $\mathcal{M}_L^+$ follows the instructions in the middle of Figure 13 and so $(w'\#^i, i+3, q) \vdash^{10}_{\mathcal{M}_L^+} (w'\#^{i-1}, i, r_q)$.

The computation of the machine $\mathcal{M}_L^-$ starts by the instructions on the top left of Figure 14 so that $(w'\#^{i-1}, i, r_q) \vdash^3_{\mathcal{M}_L^-} (w'\#^{i-1}, i+3, q)$. Then since after this, the computation stays inside the indices $[i+3, n+1]$, we have that $(w'\#^{i-1}, i+3, q) \vdash^+_{\mathcal{M}_L^-} (w'\#^{i-1}, i_\alpha+1, q_\alpha)$.

If $i = -1$, then $w'\#^i = w'\#^{n,1}$. Let $t = 2$ if $i_\alpha = 0$ or $t = t_0$ otherwise. Let $(w\#^{n,1}, i_\alpha+1, q_\alpha) \vdash^{n_{t-1}}_{\mathcal{M}_L^+} (w'\#^{n,1}, 2, q)$. As now $w'^{\#^{n,1}}_0 = w'^{\#^{n,1}}_{-1} = \#$, the computation will follow the upper path of Figure 13 and so $(w'\#^{n,1}, 2, q) \vdash^7_{\mathcal{M}_L^+} (w'\#^{n,1}, 0, r_q)$.

The computation then transitions to the machine $\mathcal{M}_F^-$ and follows the instructions of Figure 13 so that $(w'\#^{n,1}, 0, r_q) \vdash^2_{\mathcal{M}_F^-} (w'\#^{n,1}, 2, q)$. Then by definition of t

in the last paragraph, the computation never makes a left move at index 0 and so $(w'^{\#n,1}, 2, q) \vdash^{+}_{\mathcal{M}_F^-} (w^{\#n,1}, i_\alpha + 1, q_\alpha)$.

Finally, because q_ω is reachable from q_α, we have that $(w^{\#n,1}, i_\alpha + 1, q_\alpha) \vdash^{+}_{\mathcal{M}_F^+}$ $(w^{\#n,1}, i_\omega+1, q_\omega)$ and then instructions of Figures 16 and 12 are used and as the original computation stays within indices $[0, n]$, we have that $(w^{\#n,1}, i_\omega + 1, q_\omega) \vdash^{+}_{\mathcal{M}_R^-}$ $(w^{\#n,1}, i_\alpha + 1, q_{\alpha,\mathcal{M}_R^+})$.

Hence by combining all of the above we have shown that $(w^{\#n,0}, i_\alpha, q_\alpha) \vdash^{+}_{\mathcal{M}_{wp}}$ $(w^{\#n,1}, i_\alpha + 1, q_\alpha)$.

Suppose then that there exists a strictly weakly periodic configuration in $\mathcal{M}_{wp}$. We will show that then necessarily q_ω must be reachable from q_α in $\mathcal{M}$.

So let $(x, i, q_0) \vdash^{n} (\sigma(x)^{-k}, i + k, q_n)$ for $k \neq 0$ and let $\{q_j \mid j \in [0, n]\}$ be the set of all the states visited. Clearly if the set of states $\{q_j \mid 0 \leq j \leq n\}$ contains states from both $\mathcal{M}'^+$ and $\mathcal{M}'^-$ we would have a periodic configuration and so $k = 0$. Therefore we can assume that all the states are from $\mathcal{M}'^+$.

Let us first assume that q_0 is a state from the machine $\mathcal{M}_R^+$. Since the original machine $\mathcal{M}$ is aperiodic it follows from the construction that there needs to exist a time $t_0 \in (0, n)$ such that q_{t_0} is a state from either a machine $\mathcal{M}_R^-$ or $\mathcal{M}_L^-$. This can only happen using instructions from Figures 10 or 11 and in both cases a symbol $\#$ at some position $j \in \mathbb{Z}$ shifts one step left. Suppose we have the former case. Since our configuration is weakly periodic, there must exist a time $t_1 > t_0$ such that $q_{t_1} \in Q_R^+$. Since the machine $\mathcal{M}_R^-$ behaves as the inverse of the original machine, we also have $t_2 \geq t_1$ such that we have $(x, i, q_0) \vdash^{t_2} (x', i, q_0)$, where $x_j = \# = x'_{j-1}$, $x_{j-1} = x'_j$ and $x_{j'} = x'_{j'}$ for each $j' \in \mathbb{Z} \setminus [j-1, j]$. Hence we must have $n > t_2$, as all that has happened is the nearest symbol in the right side of cell i has moved one step left. This computation now necessarily must repeat until the symbol $\#$ is eventually at the cell $i + 1$. Hence there exists $t_3 > t_2$ and $x' \in \Gamma^{\mathbb{Z}}$ such that $(x, i, q_0) \vdash^{t_3} (x', i, q_\alpha)$ and $x'_{i+1} = \#$. Then necessarily instructions from Figure 11 are used and thus there exists such $t_4 > t_3$, $x' \in \Gamma^{\mathbb{Z}}$ and $q \in Q_L^-$, that $(x, i, q_0) \vdash^{t_4} (x', i + 1, q)$. The computation then pushes the symbol $\#$ left while it alternates between machines $\mathcal{M}_L^+$ and $\mathcal{M}_L^-$, until such moment that the instructions from the top of Figure 13 are used. This will happen eventually as by the assumption of weakly periodicity we will eventually return to a state in Q_R^+. Hence there exist such $t_5 > t_4$, $x' \in \Gamma^{\mathbb{Z}}$, $i' \in \mathbb{Z}$ and $q \in Q_F^-$, that $(x, i, q_0) \vdash^{t_5} (x', i', q)$. Now the computation can not remain in the states of the machine $\mathcal{M}_F^-$, hence by Figure 15 we have such $t_6 > t_5$, $x' \in \Gamma^{\mathbb{Z}}$ and $i' \in \mathbb{Z}$, that $(x, i, q_0) \vdash^{t_6} (x', i', q_{\alpha,\mathcal{M}_F^+})$. As the computation can not remain in $\mathcal{M}_F^+$ either, it must eventually use the instructions in Figure 16. Since the machine $\mathcal{M}_F^+$ is just an exact copy of the original one, we have that q_ω is reachable from q_α. We are done, but let us continue the computation a few steps further. Eventually the instructions of Figure 12 are used to return to the states of machine $\mathcal{M}_R^+$. Now we have cycled through all the machines. If q_0 belongs to any of the other machines $\mathcal{M}_x^y$, for $x \in \{F, L, R\}$ and $y \in \{+, -\}$, then we must repeat all the above steps in different order. Hence if there exists a strictly weakly periodic it must mean that q_ω is reachable from q_α.

□

Immediately we get the two following corollaries:

***Theorem* 3.6.3.** RCTM ZERO SPEED *is undecidable.*

Proof. From Theorem 3.5.6 we know that a Turing machine has non-zero speed if and only if there exists a weakly periodic configuration. From Theorem 3.6.2 we have that RCTM STRICTLY WEAKLY PERIODIC CONFIGURATION is undecidable and hence RCTM ZERO SPEED is also undecidable as otherwise we could use the same algorithm to solve both of the problems. □

***Theorem* 3.6.4.** RCTM ZERO ENTROPY *is undecidable.*

Proof. By combining Theorems 3.5.7 and 3.5.8 , for a given TM $\mathcal{M}$, we can build another TM $\mathcal{M}^A$ such that $\frac{h_{\mathcal{M}^A}}{\ln|A|} \geq s(\mathcal{M}) = s(\mathcal{M}^A) \geq \frac{h_{\mathcal{M}^A}}{\ln|\Gamma \times A|}$, where Γ is the set of symbols of $\mathcal{M}$ and A is a finite set of symbols such that $|A| > 1$. Hence the speed of TM $\mathcal{M}$ is equal to 0 if and only if the topological entropy of the TM $\mathcal{M}^A$ is equal to 0. Therefore if we had an algorithm, which tells if a given ARCTM has an entropy of value zero or not, we would also have an algorithm that tells whether a given TM has a speed zero or not. Therefore the claim follows by Theorem 3.6.3. □

4 One-Dimensional Cellular Automata

In this chapter we will study several decision problems related to topological entropy of cellular automata. We will show that the problems we consider are decidable if we restrict ourselves to the class of group cellular automata. On the other hand we will show that in the framework of reversible cellular automata the problems that we consider are undecidable. We will also construct a sensitive cellular automaton that has no configurations with positive pointwise Lyapunov exponents. With the exception of the group cellular automata, all our constructions embed and simulate computations of Turing machines in them. This is why we begin the chapter by introducing the details of such embeddings.

4.1 Simulating Turing Machines Inside Cellular Automata

We can simulate the computations of Turing machines inside cellular automata by using the construction method of Turing machine with moving head (TMH). The only issue that needs to be dealt with is the question of what should the CA do when a configuration has multiple states depicting Turing machine heads. This is typically dealt with the introduction of arrows, which subdivide each configuration into independent simulation areas. Then we just have to decide what should happen when the simulations run out of space. Furthermore we do not require the TMs to be complete to be able to use this kind of construction as we can add rules that deal with the cases when the TM transition is undefined. In the following subchapters we will describe a couple of ways of how a given Turing machine can be simulated in sets of simulation words.

4.1.1 Direct Embedding

Definition 4.1.1. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine and denote $A = \{\rightarrow, \leftarrow\}$. Let $\Sigma_{\mathcal{M}} = Q_1 \cup T_1$, where $Q_1 = \Gamma \times Q$ and $T_1 = \Gamma \times A$. We call elements in Q_1 the *head symbols* and elements in T_1 the *tape symbols*. The alphabet $\Sigma_{\mathcal{M}}$ is called the *TM alphabet*.

We define a relation R_1 in a following way: For $a \in \Sigma_{\mathcal{M}}$ and $b \in \Sigma_{\mathcal{M}}$

$$aR_1b \text{ iff } \begin{cases} & a \in \Gamma \times \{\rightarrow\} \wedge b \in (\Gamma \times \{\rightarrow\}) \cup Q_1 \\ \vee & a \in Q_1 \wedge b \in \Gamma \times \{\leftarrow\} \\ \vee & a \in \Gamma \times \{\leftarrow\} \wedge b \in \Gamma \times \{\leftarrow\}. \end{cases}$$

Define $S^{\alpha}_{\mathcal{M}} = \{w \in \Sigma^{\alpha}_{\mathcal{M}} \mid w_j R_1 w_{j+1} \forall j \text{ and } w_{Q_1} = 1\}$, where $\alpha \in \{\mathbb{Z}, \mathbb{Z}_-, \mathbb{N}, *\}$. Additionally we define $S^{\Omega}_{\mathcal{M}} = S^{\mathbb{Z}}_{\mathcal{M}} \cup S^{\mathbb{Z}_-}_{\mathcal{M}} \cup S^{\mathbb{N}}_{\mathcal{M}} \cup S^{*}_{\mathcal{M}}$. Elements in any of these sets will be called *simulation words*. Next we will define a partial function on these sets, which simulates the computations of a given Turing machine.

Let $w \in S^{\Omega}_{\mathcal{M}}$. If j is such an index that $w_j \in Q_1$, then we define $f_L(w) = j$. If furthermore $w_j = (a, q)$, then $f_Q(w) = q$.

Let $\# \notin \Gamma$. We define a padding function $p : S^{\Omega}_{\mathcal{M}} \to (\Gamma \cup \{\#\})^{\mathbb{Z}}$ such that $p(w)_i = a$, when $w \in S^{\alpha}_{\mathcal{M}}$, $w_i = (a, b) \in \Sigma_{\mathcal{M}}$, $i \in \alpha$, where $\alpha \in \{\mathbb{Z}, \mathbb{Z}_-, \mathbb{N}, *\}$ and $p(w)_i = \#$ otherwise.

Using these notations we can define an injective mapping from the simulation words to Turing machine configurations.

Definition 4.1.2. Let $\mathcal{M}$ be a TM. Define $\tau : S^{\Omega}_{\mathcal{M}} \to (\Gamma \cup \{\#\})^{\mathbb{Z}} \times \mathbb{Z} \times Q$ in such a way that $\tau(w) = (p(w), f_L(w), f_Q(w))$.

Finally we can define a partial function that simulates the computation of a Turing machine in the simulation words:

Definition 4.1.3. Let $\mathcal{M}$ be a TM. Define a partial function $f_{S_{\mathcal{M}}} : S^{\Omega}_{\mathcal{M}} \to S^{\Omega}_{\mathcal{M}}$ in such a way that $f_{S_{\mathcal{M}}}(w) = \tau^{-1} \circ \vdash \circ\, \tau(w)$ if $\vdash$ is defined for $\tau(w)$ and $\vdash \circ\, \tau(w) = (w', j, q)$ and $w'_j \neq \#$.

The function of the above definition behaves on the simulation words just as the Turing machine does on configurations as long as the Turing machine head stays inside the domains of the simulation words.

4.1.2 Conveyor Belt Model

We will then describe an alternative model of embedding configurations of Turing machines to configurations of cellular automata. The model we are going to describe is called the conveyor belt model. The name comes from the fact that the tape of the Turing machine is folded into two parts of equal length and tied together resembling a similar loop that a conveyor belt makes around two pulleys. The embedding method has the important property that a Turing machine head will visit each boundary of a finite simulation area infinitely many times if the underlying TM is aperiodic. We need this property in Chapter 4.3. This embedding method has been previously used at least in [28] by Pierre Guillon and Ville Salo.

Definition 4.1.4. Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine and denote $A = \{\rightarrow, \leftarrow\}$. Let $\Sigma_{\mathcal{M},2} = Q_2 \cup T_2$, where $Q_2 = \Gamma^2 \times Q \times \{0, 1\}$ and $T_2 = \Gamma^2 \times A$. We call elements in Q_2 the *head symbols* and elements in T_2 the *tape symbols*. The alphabet $\Sigma_{\mathcal{M},2}$ is called the *TM conveyor belt alphabet*.

In the above notation if $(a, b, q, i) \in Q_2$, then if $i = 0$ it is interpreted as the Turing machine head being at the top side of the conveyor belt and if $i = 1$ it is interpreted as the Turing machine head being at the bottom side of the conveyor belt.

Define a relation R_2 in a following way: Let $a \in \Sigma_{\mathcal{M},2}$ and $b \in \Sigma_{\mathcal{M},2}$, then

$$aR_2b \text{ if } \begin{cases} & a \in \Gamma^2 \times \{\rightarrow\} \wedge b \in (\Gamma^2 \times \{\rightarrow\}) \cup Q_2 \\ \vee & a \in Q_2 \wedge b \in \Gamma^2 \times \{\leftarrow\} \\ \vee & a \in \Gamma^2 \times \{\leftarrow\} \wedge b \in \Gamma^2 \times \{\leftarrow\}. \end{cases}$$

Define $S^{\alpha}_{\mathcal{M},2} = \{w \in \Sigma^{\alpha}_{\mathcal{M},2} \mid w_j R_2 w_{j+1} \forall\, j \text{ and } w_{Q_2} = 1\}$, where $\alpha \in \{\mathbb{Z}, \mathbb{Z}_-, \mathbb{N}, *\}$. Additionally we define $S^{\Omega}_{\mathcal{M},2} = S^{\mathbb{Z}}_{\mathcal{M},2} \cup S^{\mathbb{Z}_-}_{\mathcal{M},2} \cup S^{\mathbb{N}}_{\mathcal{M},2} \cup S^{*}_{\mathcal{M},2}$. Elements in any of these sets will be called *simulation words*.

Similarly to the case of the direct embedding, we need some auxiliary functions that allow us to map a given simulation word to a configuration of a Turing machine.

Let $w \in S^{\alpha}_{\mathcal{M},2}$, where $\alpha \in \{\mathbb{Z}, \mathbb{Z}_-, \mathbb{N}, *\}$. Let j be such an index that $w_j \in Q_2$. Suppose $w_j = (a, b, q, i)$. Then we define $f_Q(w) = q$ and $f_F(w) = i$. The location function $f_L : S^{\Omega}_{\mathcal{M},2} \rightarrow \mathbb{Z}$ is a bit more involved as we need to unwrap the conveyor belt: If $i = 0$, then $f_L = j$. If $i = 1$ and $\alpha = *$, then $f_L = 2|w| - 1 - j$. If $i = 1$ and $\alpha = \mathbb{N}$, then $f_L = -j - 1$. If $i = 1$ and $\alpha = \mathbb{Z}_-$, then $f_L = -j + 1$. If $i = 1$ and $\alpha = \mathbb{Z}$, then $f_L = -j$.

Next we define the unwrapping function $f_U : S^{\Omega}_{\mathcal{M},2} \rightarrow S^{\Omega}_{\mathcal{M},2}$. Let $w \in S^{\alpha}_{\mathcal{M},2}$, where $\alpha \in \{\mathbb{Z}, \mathbb{Z}_-, \mathbb{N}, *\}$. Let $u = \pi_0(w)$ and $v = \pi_1(w)$, be the tape unwrapped and cut into two parts of equal length at the ends. Then f_U is defined in the following way: If $\alpha \in \{*, \mathbb{Z}_-\}$, then $f_U(w) = uv^R$. If $\alpha = \mathbb{N}$, then $f_U(w) = v^R u$. Denote $f_F(w) = i$. If $\alpha = \mathbb{Z}$, then $f_U(w) = u$ if $i = 0$ and $f_U(w) = v^R$ if $i = 1$.

Definition 4.1.5. Let $\mathcal{M}$ be a TM. Define $\tau : S^{\alpha}_{\mathcal{M},2} \rightarrow \Gamma^G \times G \times Q$, where $\alpha \in \{\mathbb{Z}, \mathbb{Z}_-, \mathbb{N}, *\}$, in such a way that $\tau(w) = (f_U(w), f_L(w), f_Q(w))$, where $G = \mathbb{Z}_{2|w|}$ if $\alpha = *$ and $G = \mathbb{Z}$ otherwise.

The above function is an injection for each $\alpha \in \{\mathbb{Z}_-, \mathbb{N}, *\}$. For $\alpha = \mathbb{Z}$ the function is not an injection, but can be extended to one if we store the side of the conveyor belt where the Turing machine head is not located at.

We define the partial function that simulates the computation of a Turing machine for simulation words in the following way:

Definition 4.1.6. Let $\mathcal{M}$ be a TM. Define a partial function $f_{S_{\mathcal{M}},2} : S^{\alpha}_{\mathcal{M},2} \rightarrow S^{\alpha}_{\mathcal{M},2}$, for each $\alpha \in \{\mathbb{Z}_-, \mathbb{N}, *\}$, in such a way that $f_{S_{\mathcal{M}},2}(w) = \tau^{-1} \circ \vdash \circ\, \tau(w)$ if $\vdash$ is defined for $\tau(w)$. For $\alpha = \mathbb{Z}$ the function is defined in such a way that if $\vdash$ is defined for $\tau(w)$ then $f_{S_{\mathcal{M}},2}(w) = w'$, where $\pi_i(w') = \tau^{-1} \circ \vdash \circ \tau(w)$ and $\pi_j(w') = \pi_j(w)$, where $i = f_F(w)$ and $j \in \{0, 1\}$ such that $i \neq j$.

4.1.3 Simulation Areas

If the set of states of a given CA contains either a TM alphabet or a TM conveyor belt alphabet as a subset, we can recognize simulation areas in the configurations of such CA and use the functions $f_{S_{\mathcal{M}}}$ and $f_{S_{\mathcal{M}},2}$ to simulate Turing machine computations in those areas. We will describe this process next.

Let $\mathcal{M} = (Q, \Gamma, \delta)$ be a Turing machine. Let Σ_1 and Σ_2 be such sets of symbols that $\Sigma_{\mathcal{M}} \subseteq \Sigma_1$ and $\Sigma_{\mathcal{M},2} \subseteq \Sigma_2$.

Let $i \in \{1, 2\}$. For each configuration $c \in \Sigma_i^{\mathbb{Z}}$, we define a set of locations for the Turing machine heads as

$$H_c = \{j \in \mathbb{Z} \mid c_j \in Q_i\}.$$

Recall that aR^cb is used to denote that (a, b) is not in a relation R. Next we define the *simulation bounds* as functions $l_c : H_c \to \mathbb{Z} \cup \{-\infty\}$ and $r_c : H_c \to \mathbb{Z} \cup \{\infty\}$ in the following way:

$$l_c(j) = \sup\{k \in \mathbb{Z} \mid k \leq j \text{ and } c_{k-1} R_i^c c_k\}$$

and

$$r_c(j) = \inf\{k \in \mathbb{Z} \mid j \leq k \text{ and } c_k R_i^c c_{k+1}\}.$$

We will also denote $L_c = l_c(H_c)$ and $R_c = r_c(H_c)$. From these bounds we can define the set of cells that are not part of any simulation area as

$$U_c = \mathbb{Z} \setminus \Big(\bigcup_{j \in H_c} [l_c(j), r_c(j)] \Big).$$

Using the simulation bounds, we can define two partial functions, which simulate the computations of the given Turing machine in their designated simulation areas as $f'_{\mathcal{M}} : \Sigma_i^{\mathbb{Z}} \to \Sigma_i^{\mathbb{Z}}$, where

$$\begin{aligned} f'_{\mathcal{M}}(c)_{[l_c(j), r_c(j)]} &= f(c_{[l_c(j), r_c(j)]}) \text{ if } f \text{ is defined } \forall j \in H_c \text{ and} \\ f'_{\mathcal{M}}(c)_k &= c_k \ \forall k \in U_c, \end{aligned}$$

where $f = f_{S_{\mathcal{M}}}$ if $i = 1$ and $f = f_{S_{\mathcal{M}},2}$ if $i = 2$.

The partial functions $f'_{\mathcal{M}}$ can be completed to cellular automata. As each simulation area behaves as a computation of a Turing machine would, we can define partial radius-1 local rules, which behave exactly as our partial functions do in the simulation areas. Then we just need to decide what the CA do with simulation words

$c_{[l_c(j),r_c(j)]}$ such that $f_{S_\mathcal{M}}$ or $f_{S_\mathcal{M},2}$ is undefined, i.e. TM sees an error pair or an error state or the computation tries to move outside the simulation area in the direct embedding. For example if the TM is reversible we can reverse the computation when $f'_\mathcal{M}$ is undefined or runs out of space and we would get a reversible CA.

Hence let us finally define $f_\mathcal{M} : \Sigma_i^\mathbb{Z} \to \Sigma_i^\mathbb{Z}$ in the following way for complete Turing machines: If $i = 2$, then $f_\mathcal{M} = f'_\mathcal{M}$. If $i = 1$, then we demand that $\mathcal{M}$ is constructed from some reversible Turing machine by reversing the computation. For each $c \in \Sigma_1^\mathbb{Z}$ define $c' \in \Sigma_1^\mathbb{Z}$ in such a way that if $j \in H_c$ and $c_j = (a, q^x)$ where $x \in \{+,-\}$, then $c'_j = (a, q^y)$, where $y \in \{+,-\}$ and $x \neq y$. Then we can define

$$\begin{aligned} f_\mathcal{M}(c)_{[l_c(j),r_c(j)]} &= f_{S_\mathcal{M}}(c_{[l_c(j),r_c(j)]})\ \forall j \in H_c \text{ such that } f_{S_\mathcal{M}} \text{ is defined,} \\ f_\mathcal{M}(c)_{[l_c(j),r_c(j)]} &= c'_{[l_c(j),r_c(j)]}\ \forall j \in H_c \text{ such that } f_{S_\mathcal{M}} \text{ is undefined and} \\ f_\mathcal{M}(c)_k &= c_k\ \forall k \in U_c. \end{aligned}$$

4.2 Decision Problems: Topological Entropy, Lyapunov Exponents and Gliders

4.2.1 Decision Problems for Reversible Cellular Automata

RCA GLIDER: Given a reversible cellular automaton, decide if there exists a glider.

RCA ZERO GLOBAL LYAPUNOV EXPONENTS: Given a reversible cellular automaton, decide if $\lambda_+ = \lambda_- = 0$.

RCA ZERO ONE-SIDED GLOBAL LYAPUNOV EXPONENT: Given a reversible cellular automaton, decide if $\lambda_+ = 0$ (or analogously if $\lambda_- = 0$).

RCA ZERO ENTROPY: Given a reversible cellular automaton, decide if the topological entropy is zero.

In this chapter we will use the CA $f_\mathcal{M}$ multiple times as was constructed in the Chapter 4.1 from a given Turing machine $\mathcal{M}$. It is useful to keep in mind the notations from the subchapter when going through the following proofs.

***Theorem* 4.2.1.** RCA ZERO GLOBAL LYAPUNOV EXPONENTS *is undecidable.*

Proof. Let $\mathcal{M}$ be a reversible TM constructed from some Turing machine by reversing the computation and let us consider the CA $f = f_\mathcal{M}$. We will first show that the global Lyapunov exponents are bounded from above by the speed of $\mathcal{M}$. Let $c \in \Sigma_\mathcal{M}^\mathbb{Z}$ and let $B_l = \{l_c(j) \in \mathbb{Z}_- \mid j \in H_c\}$, $B_r = \{r_c(j) \in \mathbb{Z}_- \mid j \in H_c\}$ and $S = B_l \cup B_r$. If $S \neq \emptyset$, then let $m \in S$. Then for any $c' \in W_m^+(c)$, it holds that

$f^i(c') \in W^+_{m+1}(f^i(c)) \subseteq W^+_0(f^i(c))$ for each $i \in \mathbb{N}$. This is because the simulation areas work independently from each other.

Suppose then that there exists $j \in H_c$ such that $l_c(j) = -\infty$ and $r_c(j) \geq 0$. Let $n \in \mathbb{N}$ and $c' \in W^+_m(c)$, where $m = -f_M(n)$. Since there are $f_M(n) + 1$ cells in the interval $[-f_M(n), 0]$, there does not exist such a configuration that the computation would visit both cells 0 and $-f_M(n)$ in n steps. So if we take two configurations where we edit the tape symbols only in the cells $j' < -f_M(n)$, then either the computation visits those cells and the edits affect the computation, but the machine head does not visit cell 0 within n steps; or the machine visits cell 0 within n steps, but the edits do not affect the computation in n steps as those edited cells are not reached.

Hence $f^i(c') \in W^+_0(f^i(c))$ for each $i \leq n$. And so $f^i(W^+_m(c)) \subseteq W^+_0(f^i(c))$ for each $i \leq n$. We have shown that $I^+_n(c) \leq f_M(n)$ for each $c \in \Sigma_1^{\mathbb{Z}}$. Therefore we have that $\lambda^+ \leq f_S(\mathcal{M})$. Similarly, $\lambda^- \leq f_S(\mathcal{M})$.

Therefore if $f_S(\mathcal{M}) = 0$, then $\lambda^- = \lambda^+ = 0$.

On the other hand we can show that $\max\{I^+_n(c), I^-_n(c)\}$ is bounded from below by $\frac{f_M(n)}{2}$ for each $n \in \mathbb{N}$. Towards this end let (x, i, q) be such a configuration of the Turing machine $\mathcal{M}$ that $|f_V((x, i, q), n)| = f_M(n)$. Then let $[l, r] = \{f_T((x, i, q), j) \mid j \leq n\}$ be the set of indices that are visited by the configuration. Then necessarily either $|l - i| \geq \frac{f_M(n)}{2}$ or $|r - i| \geq \frac{f_M(n)}{2}$. Suppose its the former, then we take a shifted configuration $(\sigma^l(x), -l + i, q)$. The shifted configuration visits the cells $[0, -l + r]$ so especially there exists $j \leq n$ such that $(\sigma^l(x), -l + i, q) \vdash^j (x', 0, q')$.

Now take any configuration c of the CA f such that $f_{S_{\mathcal{M}}}(c_{[l_c(-l+i), r_c(-l+i)]}) = (x'', -l + i, q)$, where $x''_{[0,-l+r]} = \sigma^l(x)_{[0,-l+r]}$. Consider a configuration c' such that $c = c'$ for each $j \neq -l + i$, but $c'_{-l+i} \notin Q_1$, in other words we remove the Turing machine head from index $-l + i$ in the new configuration c'. Then as there exists $j \leq n$ such that $(\sigma^l(x), -l + i, q) \vdash^j (x', 0, q')$, we have that $f^j(c)_0 \in Q_1$, but $f^j(c')_0 \notin Q_1$. This means that $f^j(c') \notin W^-_0(f^j(c))$, but $c' \in W^-_{-l+i-1}(c)$ and so $I^-_n(c) \geq -l + i = |l - i| \geq \frac{f_M(n)}{2}$. Now if instead $|r - i| \geq \frac{f_M(n)}{2}$ we could by a symmetric argument show that then $I^+_n(c) > \frac{f_M(n)}{2}$.

Thus if $f_S(\mathcal{M}) > 0$, we have that $\lambda^+ > 0$ or $\lambda^- > 0$.

We have shown that $\lambda^+ = \lambda^- = 0$ if and only if $f_S(\mathcal{M}) = 0$. Therefore by Theorem 3.6.3 we have that RCA ZERO GLOBAL LYAPUNOV EXPONENTS is undecidable. □

The next corollary follows easily:

Corollary 4.2.2. RCA ZERO ONE-SIDED GLOBAL LYAPUNOV EXPONENT is undecidable.

Proof. Suppose that we could decide if $\lambda_+ = 0$ for a given CA f. Let f' be the CA derived from f by mirroring the local rule and the neighbourhood. Then λ_+ of f' is the same as λ_- of f. Thus if we could decide if the positive Lyapunov exponent is zero or not, we could test if the positive Lyapunov exponents of a given CA and its mirrored version are both zero or not to decide if $\lambda_+ = \lambda_- = 0$. □

***Theorem* 4.2.3.** RCA GLIDER *is undecidable.*

Proof. Let $\mathcal{M}$ be an aperiodic reversible Turing machine and let $\mathcal{M}_{wp} = (Q, \Gamma, \delta)$ be the Turing machine as constructed in Theorem 3.6.2. Let $\mathcal{A} = (\Sigma, N, h)$ be a cellular automaton, where $\Sigma = \Sigma_{\mathcal{M}_{wp}}$ and $(\#, \rightarrow)$ is the quiescent state. Let $S = \{(a, q) \in Q_1 \mid a \neq \#, (q, +, r) \in \delta_{\mathcal{M}_R^+}\}$. We define a global rule $g : \Sigma^{\mathbb{Z}} \rightarrow \Sigma^{\mathbb{Z}}$ in such a way that

$$g(c)_i = \begin{cases} (\#, \leftarrow) & \text{if } c_{[i-1,i+1]} \in S(\#, \rightarrow)(\#, \rightarrow), \\ (\#, \rightarrow) & \text{if } c_{[i-1,i+1]} \in S(\#, \leftarrow)(\#, \rightarrow) \text{ and} \\ c_i & \text{otherwise.} \end{cases}$$

The second line in the definition of g is not used in any of the orbits that are of interest to us, but rather exists only to make g reversible. Let $f = f_{\mathcal{M}_{wp}} \circ g$.

We will show that there exists a glider in f if and only q_ω is reachable from q_α in $\mathcal{M}$.

Assume first that q_ω is reachable from q_α. Let (x, i_α, q_α) be such a configuration that $(x, i_\alpha, q_\alpha) \vdash (x', i_\omega, q_\omega)$ and assume again that the computation only visits the cells $[0, n]$ as in the proof of Theorem 3.6.2 and let $w = x_{[0,n]}$. Let $c \in \Sigma^{\mathbb{Z}}$ be such that $(c_i)_0 = w_i^{\#n}$ and $(c_i)_1 = \rightarrow$ if $i < i_\alpha$ or $i > n$, $(c_i)_1 = q_\alpha$ if $i = i_\alpha$ and $(c_i)_1 = \leftarrow$ otherwise.

Now let $t \in \mathbb{N}$, be the smallest integer such that $(w^{\#n}, i_\alpha, q_\alpha) \vdash^{t-1} (y, n, q)$, where $(q, +, r) \in \delta_{\mathcal{M}_R^+}$. If we look back into the proof of Theorem 3.6.2, we can see that this happens either right after the first time we move right from index $n-1$ to n, using the instructions of Figure 10 or if $n = i_\alpha$ then $t = 1$ and we use the instruction of Figure 11. In both cases we move right from index n when the tape symbol at index n is some $a \neq \#$ once and then when the computation comes back to index n it replaces the symbol a by $\#$, so the second time that the computation moves right from n, i.e. at move $t + 4$, there will be $\#$ at index n. After this there will be a tape symbol $a \neq \#$ at index $n + 1$, and that index is not visited again at a state from the machine $\mathcal{M}_R$ until the machine goes through all the stages of $\mathcal{M}_R$, $\mathcal{M}_L$ and $\mathcal{M}_F$.

This means that at step $t - 1$ the simulation word gets extended by one from the right side by applying the CA g once. And because the conditions of g happen only once during each iteration of all the stages of $\mathcal{M}_{wp}$, the finite configuration c is a glider.

Let us then assume that q_ω is not reachable from q_α. By definition a glider must be a strictly weakly periodic configuration of the form $c = {}^\infty awa^\infty$, where $a = (\#, \rightarrow)$ and $w \in \Sigma^*$, so suppose there exists one. We can also assume that $w_0 \neq (\#, \rightarrow)$, $w_n \neq (\#, \rightarrow)$ and $c_{[0,n]} = w$. Then there must exist a time $t > 0$ such that $f^t(c)_{n+1} \neq (\#, \rightarrow)$ or $f^t(c)_{-1} \neq (\#, \rightarrow)$. Suppose the former. Then by definition of g there is a simulation word $u \in \Sigma^+$ such that $w = vu$. Since the Turing machine head of the simulation area is at a state belonging to machine $\mathcal{M}_R^+$, then by the construction of $\mathcal{M}_{wp}$, the machine in the simulation area of u, will move the symbol $\#$ left one step at a time. Just as we saw in Theorem 3.6.2, the machine then must eventually pass through all the stages of the machines $\mathcal{M}_R$, $\mathcal{M}_L$ and $\mathcal{M}_F$ and this would mean that q_ω is reachable from q_α, which would be a contradiction. If it did not, then eventually the computation would run out of space and reverse and we would have a time $t_2 > t$ such that $f^{t_2}(c)_{n+1} = (\#, \rightarrow)$. Thus c would not be strictly weakly periodic, which is again a contradiction.

If on the other hand $f^t(c)_{-1} \neq (\#, \rightarrow)$, then that would mean that $w = vu$, where v is a simulation area. But the only way that $\#$ is moved right is if we are at states of the machine $\mathcal{M}'^-$. Recall that $\mathcal{M}'^-$ is made out of the inverse machines of $\mathcal{M}_x^y$, where $x \in \{L, R, F\}$ and $y \in \{+, -\}$. The machine keeps moving the symbol $\#$ right unless it goes to a state from the machine $\mathcal{M}'^+$ or it runs out of space, in which case the simulation will eventually reverse and go to the states of the machine $\mathcal{M}'^+$ or it reaches q_ω from q_α in the inverse machine of $\mathcal{M}_R^-$, which is a contradiction. If the simulation goes to the states of the machine $\mathcal{M}'^+$, then eventually for some $t_2 > t$, we have that $f^{t_2}(c)_{-1} = (\#, \rightarrow)$ and hence c is not strictly weakly periodic, which is once again a contradiction.

Hence if there exists a glider, it must follow that q_ω is reachable from q_α. Therefore by Theorem 3.6.1 we have that RCA GLIDER is undecidable. □

***Theorem* 4.2.4.** RCA ZERO ENTROPY *is undecidable.*

Proof. Let $\mathcal{M} = (Q, \Gamma, \delta)$. We construct $\mathcal{M}_{wp}$ almost the same way as before, except that we add two new symbols $\#_0$ and $\#_1$ instead of $\#$ such that both behave as they would if they were just $\#$. Furthermore we add four more copies of $\mathcal{M}$ and four more copies of its inverse machine. We denote the new machines as $\mathcal{M}_y^x$, where $x \in \{+, -\}$ describing whether the copy is of the original or the inverse machine and $y \in \{F_2, F_3, S, S_2\}$. We remove the transitions of Figure 16. Then we add transitions $(q_\omega, a, a, q_\omega)$ from $\mathcal{M}_F^+$ to $\mathcal{M}_S^-$, $\mathcal{M}_S^+$ to $\mathcal{M}_{F_2}^-$, $\mathcal{M}_{F_2}^+$ to $\mathcal{M}_{S_2}^-$, $\mathcal{M}_{S_2}^+$ to $\mathcal{M}_{F_3}^-$ and $\mathcal{M}_{F_3}^+$ to $\mathcal{M}_R^-$ for each $a \in \Gamma$. We also add transitions $(q_\alpha, a, a, q_\alpha)$ from $\mathcal{M}_{F_2}^-$ to $\mathcal{M}_{F_2}^+$ and $\mathcal{M}_{F_3}^-$ to $\mathcal{M}_{F_3}^+$. We have intentionally left out transitions from $\mathcal{M}_S^-$ to $\mathcal{M}_S^+$ and $\mathcal{M}_{S_2}^-$ to $\mathcal{M}_{S_2}^+$ as these will be dealt by a cellular automaton. Finally we finish the construction by reversing the computation and denote the machine as $\mathcal{M}_E$.

This newly constructed machine behaves exactly as $\mathcal{M}_{wp}$ until q_ω is reached in $\mathcal{M}_F^+$. After that instead of transitioning to $\mathcal{M}_R^-$ for a new iteration, the computation goes through the stages (with the exception that there are some missing transitions) of the new machines $\mathcal{M}_y^x$ in order of F to S to F_2 to S_2 to F_3 until it transitions from $\mathcal{M}_{F_3}^+$ to $\mathcal{M}_R^-$ as $\mathcal{M}_{wp}$ would. But each one behaves just as $\mathcal{M}_F^x$, for both $x \in \{+, -\}$. Hence this does not affect the reachability in any way, rather just some computation stages are repeated.

Let Σ_1 be the TM alphabet of $\mathcal{M}_E$, i.e. $\Sigma_1 = \Sigma_{\mathcal{M}_E}$. And let $\Sigma = \Sigma_1 \cup \{(\#_0, \cdot), (\#_1, \cdot)\}$. Let $\#, \#', \#'' \in \{\#_0, \#_1\}$ and define g as:

$$g(c)_{[i,i+2]} = \begin{cases} (\#', \cdot)(\#, \rightarrow)(q_{S^+}, a) & \text{if } c_{[i-1,i+2]} = (\#'', \cdot)(\#', \rightarrow)(\#, \rightarrow)(q_{S^-}, a), \\ (\#', \rightarrow)(\#, \rightarrow)(q_{S^-}, a) & \text{if } c_{[i-1,i+2]} = (\#'', \cdot)(\#', \cdot)(\#, \rightarrow)(q_{S^+}, a), \\ (q_{S_2^+}, a)(\#, \leftarrow)(\#', \cdot) & \text{if } c_{[i,i+2]} = (q_{S_2^-}, a)(\#, \cdot)(\#', \cdot), \\ (q_{S_2^-}, a)(\#, \cdot)(\#', \cdot) & \text{if } c_{[i,i+2]} = (q_{S_2^+}, a)(\#, \leftarrow)(\#', \cdot) \text{ and} \\ c_{[i,i+2]} & \text{otherwise,} \end{cases}$$

where q_{S^+} is any state of the machine $\mathcal{M}_S^+$ and q_{S^-} is the copy of the state q_{S^+} in the inverse machine $\mathcal{M}_S^-$. Analogously $q_{S_2^+}$ is any state of the machine $\mathcal{M}_{S_2}^+$ and $q_{S_2^-}$ is the copy of the state $q_{S_2^+}$ in the inverse machine $\mathcal{M}_{S_2}^-$. It is easy to see that g is reversible. Let $f = f_{\mathcal{M}} \circ g$.

We will now prove the claim by a reduction to Theorem 3.6.3.

Suppose first that q_ω is reachable from q_α.

Consider a configuration of the original Turing machine such that $(x, i_\alpha, q_\alpha) \vdash (x', i_\omega, q_\omega)$, and suppose the machine visits indices $[0, n]$ and let $w = x_{[0,n]}$. Then consider a configuration in Σ such that the simulation word in the cells $[0, n]$ correspond to w and the state of the Turing machine head is q_α at location i_α. In this paragraph we will use the symbol $\#$ to denote either $\#_0$ or $\#_1$. Suppose $c_k = (\#, \cdot)$ for $k \in \{-3, -2, n+2, n+3\}$ and $c_{-1} = (\#, \rightarrow)$ and $c_{n+1} = (\#, \leftarrow)$. Then the computation behaves as in the proof of Theorem 3.6.2, until the machine transitions from $\mathcal{M}_F^+$ to $\mathcal{M}_S^-$. At this point we will have $c_0 = c_{-1} = (\#, \rightarrow)$ and the word w is written in the cells $[1, n+1]$. Now there will be a first step when the original machine visits cell 0 so the cellular automaton will visit cell 1 and as we are at states of the machine $\mathcal{M}_S^-$, the rule of g is applied and the content at cell -1 will change to $(\#, \cdot)$ from $(\#, \rightarrow)$, the machines state changes to $\mathcal{M}_S^+$ and the computation rewinds that of $\mathcal{M}_S^-$. From there the computation goes through $\mathcal{M}_{F_2}^-$ followed by $\mathcal{M}_{F_2}^+$ to $\mathcal{M}_{S_2}^-$ by changing every time after q_α or q_ω is reached depending if the original computation is simulated backwards or forwards. Now there will be a first step the cell $n+1$ is reached and g is applied once more changing the content of cell $n+2$ from $(\#, \cdot)$ to $(\#, \leftarrow)$ and the state is changed to one corresponding to that of the machine $\mathcal{M}_{S_2}^+$. After this the computation is reversed and again goes through the states of the machines $\mathcal{M}_{S_2}^+$ to $\mathcal{M}_{F_3}^-$ to $\mathcal{M}_{F_3}^+$ to $\mathcal{M}_R^-$ to $\mathcal{M}_R^+$ returning from where the computation started, but the content of cells $[-1, n+1]$ have been shifted to $[0, n+2]$.

Suppose one complete iteration when starting from from q_α in $\mathcal{M}_R^+$ and ending to it back again takes t steps in the CA. Now we want to take the word $w' = c_{[-3,n+3]}$ from the configuration c of the previous paragraph. And notice that the symbols $\#$ in indices $[-3,-1]$ and $[n+1, n+3]$ can be freely chosen to be either $\#_0$ or $\#_1$ as the subscript $i \in \{0,1\}$ does not affect the computation in any way. Hence for any $u \in \{\#_0, \#_1\}^6$ we can define words $w^u = ac_{[0,n]}b$, where $a = (u_0, \cdot)(u_1, \cdot)(u_2, \rightarrow)$ and $b = (u_3, \leftarrow)(u_4, \cdot)(u_5, \cdot)$. This means that each w^u is equivalent to $c_{[-3,n+3]}$, except with specific subscripts for the symbols $\#$. Let $v_i \in \{\#_0, \#_1\}^6$. Then consider a configuration such that the words of the form w^{v_i} are placed next to each other, i.e. we have configurations of the form $\ldots w^{v_{-2}} w^{v_{-1}} w^{v_0} w^{v_1} w^{v_2} \ldots$.

Now let $c' = \ldots w^{v_{-2}} w^{v_{-1}} w^{v_0} w^{v_1} w^{v_2} \ldots$ and suppose that $c'_{[0,n+6]} = w^{v_0}$ and denote $n+1 = n'$. Then we have that $f^k(c')_{[0,n+6]} = w^{v_{-k'}}$ where $k = 6tn'k'$ for each $k' \in \mathbb{N}$.

Now since we can freely fix the content of the subscripts of $\#$, we have that the number of subwords of length $n''k'$, where $n'' = 6tn'$, in the m-trace shift of the CA is at least $(2^6)^{k'} > 2^{k'}$, where $m = n+6$. In other words $P_m(n''k') \geq 2^{k'}$.

Hence $h_f \geq \lim_{k' \to \infty} \frac{\ln(P_m(n''k'))}{n''k'} \geq \lim_{k' \to \infty} \frac{\ln(2^{k'})}{n''k'} = \frac{\ln(2)}{n''} > 0$.

Suppose then that q_ω is not reachable from q_α. We will show that then $\lambda^+ = \lambda^- = 0$. It then follows from Theorem 2.7.13 that $h_f = 0$.

The important thing to note here is that now the simulation areas can change over time, and simulation areas could in theory interact in such a way that there would exist such configurations whose Lyapunov exponents would not be bounded from above by the speed of the Turing machine as in Theorem 4.2.1, however we will next show that is not the case.

Let c be a configuration such that $j \in H_c$. We want to show that a simulation area can only change its size from one side, hence we can assume that $l_c(j) \in \mathbb{Z}$ and $r_c(j) \in \mathbb{Z}$ as otherwise the statement is clearly true. Let us assume that there exists smallest such $t \geq 0$, that $g(f^t(c))_{l_c(j)} \neq f^t(c)_{l_c(j)}$, i.e. after step t is the first step when g affects the computation at all. It does this by either increasing or decreasing the simulation area and the Turing machine at the simulation area is at a state belonging to either $\mathcal{M}_S^-$ or $\mathcal{M}_S^+$. These are identical copies of the original Turing machine that was fed into the construction, so it behaves in the same way. Now if there also exists time $t' \geq 0$ such that $g(f^{t'}(c))_{r_c(j)} \neq f^{t'}(c)_{r_c(j)}$, then that would happen when the Turing machine in the simulation area is at the states of $\mathcal{M}_{S_2}^-$ or $\mathcal{M}_{S_2}^+$. The construction was built in such a way that we can go from $\mathcal{M}_S^+$ to $\mathcal{M}_{F_2}^-$ to $\mathcal{M}_{F_2}^+$ to $\mathcal{M}_{S_2}^-$ to $\mathcal{M}_{S_2}^+$ or from $\mathcal{M}_{S_2}^+$ to $\mathcal{M}_{F_3}^-$ to $\mathcal{M}_{F_3}^+$ to $\mathcal{M}_R^-$ through all the computations of $\mathcal{M}_{wp}$ to $\mathcal{M}_F^+$ to $\mathcal{M}_S^-$ to $\mathcal{M}_S^+$. Furthermore the computation can only go through the computations of the machines $\mathcal{M}_F^+$, $\mathcal{M}_{F_2}^+$ and $\mathcal{M}_{F_3}^+$ if and only if q_ω is reachable from q_α. Hence no such t' can exist.

We have shown in the previous paragraph that if q_ω is not reachable from q_α, then the size of the simulation areas can only change from one side. One can also show that the size can only change by one, as otherwise the computation would again have to go through the states of $\mathcal{M}_F^+$.

Let then c be a configuration and define $L = \{j \in H_c \mid r_c(j) < 0\}$, i.e. the set of Turing machine head locations such that the simulation areas are entirely on the left side of origin. If $L \neq \emptyset$, then let $j = \sup(L)$. Let us consider the simulation area $[l_c(j), r_c(j)]$ assuming that $l_c(j) \in \mathbb{Z}_-$. If the size of the simulation area changes from the right at some time-step, then it never changes from the left. In this case no changes in the left side of index $l_c(j)$ can ever affect the simulation inside $[l_c(j), r_c(j)]$. If on the other hand the size of the simulation area changes from the left at some time-step, then it never changes from the right. Thus no edits to the configuration in the left side of index $l_c(j)$ can ever affect cells in the indices $k > r_c(j)$. If no changes happen to the size of the simulation area, this could possibly change in a new configuration if some edits are made to cells left of $l_c(j)$, but if this change alters the size of the simulation area from the left, then it can't alter the simulation area from the right and again the cells in the indices $k > r_c(j)$ are unaffected. All in all, in each case we have that if $c' \in W^+_{l_c(j)}(c)$, then $f^t(c') \in W^+_{r_c(j)+1}(f^t(c)) \subseteq W^+_0(f^t(c))$ for each $t \in \mathbb{N}$. So for such configurations we have that $I^+_n(c) \leq l_c(j)$ for each $n \in \mathbb{N}$.

On the other hand if c is such that $L \neq \emptyset$ and $j \in L$ such that $l_c(j) = -\infty$, then if $c' \in W^+_{r_c(j)-f_M(n)}(c)$, then $f^t(c') \in W^+_{r_c(j)+1}(f^t(c)) \subseteq W^+_0(f^t(c))$ for each $t \leq n$. And if $r_c(j) < -f_M(n) - 1$ then if $c' \in W^+_{r_c(j)}(c)$, then $f^t(c') \in W^+_0(f^t(c))$ for each $t \leq n$.

Let then $L \neq \emptyset$ and $j = \sup(L)$. If $r_c(j) < -f_M(n) - 1$, we can choose $m = -f_M(n) - 1$, if $r_c(j) \geq -f_M(n) - 1$ then either $l_c(j) < r_c(j) - f_M(n)$ and we can choose $m = -f_M(n) * 2 - 1$ or $l_c(j) \geq r_c(j) - f_M(n)$ and we can again choose $m = -f_M(n) * 2 - 1$ to have that if $c' \in W^+_m(c)$, then $f^t(c') \in W^+_0(c)$ for each c and each $t \leq n$.

If $L = \emptyset$, then the case is almost analogous to the proof of Theorem 4.2.1 and again have that for all such c if $m = -f_M(n)$ then if $c' \in W^+_m(c)$, then $f^t(c') \in W^+_0(c)$ for each $t \leq n$.

Thus $\lambda^+ \leq f_S(\mathcal{M}) = 0$ where the equality follows by Theorem 3.6.3 as we assumed that q_ω is not reachable from q_α.

By a symmetric argument we can also show that $\lambda^- = 0$ if q_ω is not reachable from q_α.

Therefore we have shown that q_ω is reachable from q_α if and only if the entropy h_f is non-zero.

□

As a consequence we have that one can not decide if the topological entropies of two given reversible cellular automata are equal. This is because we can fix one of the reversible cellular automata so that it has zero topological entropy, for example the cellular automaton with the identity map rule. Then if we could decide if any given reversible cellular automaton had the same topological entropy, we would also be able to decide if a given reversible cellular automaton had zero topological entropy. Furthermore clearly one can not have an algorithm that computes the topological entropy of a given reversible cellular automaton as again we could then decide if the topological entropy was zero or not. Of course the more interesting question still remains open, which is whether we can estimate the topological entropy of a given reversible cellular automaton up to a given precision.

4.2.2 Decision Problems for Group Cellular Automata

SGCA ZERO GLOBAL LYAPUNOV EXPONENTS: Given a surjective group cellular automaton, decide if $\lambda_+ = \lambda_- = 0$.

GCA ZERO ENTROPY: Given a group cellular automaton, decide if the entropy is zero.

GCA SENSITIVITY: Given a group cellular automaton, decide if it is sensitive.

Let Σ be a group, then $\Sigma^{\mathbb{Z}}$ is a group as an infinite direct product, and therefore for each $x \in \Sigma^{\mathbb{Z}}$ and $y \in \Sigma^{\mathbb{Z}}$ we have that $(xy)_i = x_i y_i$ for each $i \in \mathbb{Z}$.

A subshift $X \subseteq \Sigma^{\mathbb{Z}}$ is a *group shift* if it is a subgroup. A cellular automaton is a *group cellular automaton* if it is a group homomorphism. So for a group CA we have that $f(xy) = f(x)f(y)$.

The following theorem was proved in [50] by Bruce Kitchens and Klaus Schmidt.

***Theorem* 4.2.5.** *[50] A group shift is a SFT.*

The above Theorem combined with the following Lemma that was proved in [6] by Pierre Béaur and Jarkko Kari shows that the column subshifts are SFTs.

Lemma 4.2.6. [6] Any column subshift of a group CA is a group shift.

We also need the following two facts from [6].

Lemma 4.2.7. [6] A group CA is either equicontinuous or sensitive.

***Theorem* 4.2.8.** *[6]* GCA SENSITIVITY *is decidable.*

Lemma 4.2.9. Let (X, f) be a sensitive group CA and let ϵ be a sensitivity constant. Then

$$\forall \delta > 0 \, \exists n \geq 0 \, \forall x \in X \, \exists y \in B_\delta(x) \, : \, f^n(y) \notin B_\epsilon(f^n(x)).$$

Proof. By the definition of sensitivity we have that

$$\forall \delta > 0 \, \forall x \in X \, \exists y \in B_\delta(x) \, \exists n \geq 0 : f^n(y) \notin B_\epsilon(f^n(x))$$

and therefore it especially holds that

$$\forall \delta > 0 \, \exists x \in X \, \exists y \in B_\delta(x) \, \exists n \geq 0 : f^n(y) \notin B_\epsilon(f^n(x)).$$

Then by reordering the existential quantifiers we have that

$$\forall \delta > 0 \, \exists n \geq 0 \; \exists x \in X \exists y \in B_\delta(x) \; : f^n(y) \notin B_\epsilon(f^n(x)).$$

Let $c \in X$, then because $(cyx^{-1})_i = c_i$ for all such $i \in \mathbb{Z}$, that $y_i = x_i$ and thus because $y \in B_\delta(x)$, we have that $cyx^{-1} \in B_\delta(c)$. On the other hand $f^n(c)_i = f^n(cc')_i = f^n(c)_i f^n(c')_i$ if and only if $f^n(c')_i$ is the neutral element and so we have that $f^n(cyx^{-1}) \notin B_\epsilon(f^n(c))$. The claim follows from this. □

Lemma 4.2.10. Every group CA has the shadowing property.

Proof. By Lemma 4.2.6 every column shift of a group CA is a group shift and by Theorem 4.2.5 every group shift is a SFT. Thus it follows from Theorem 2.7.8 that every group CA has the shadowing property. □

The following theorem follows from the results in [53] by Petr Kůrka, but is stated explicitly in [54] by the same author.

***Theorem* 4.2.11.** *[54] Let $(\Sigma^{\mathbb{Z}}, f)$ be an equicontinuous CA. Then $h_f = 0$.*

Lemma 4.2.12. Let $(\Sigma^{\mathbb{Z}}, f)$ be a sensitive group CA. Then $h_f > 0$.

Proof. Let ϵ be a sensitivity constant of the CA. By Lemma 4.2.10, any group CA has the shadowing property. Therefore let δ be such that every δ-chain is ϵ-shadowed. By Lemma 4.2.9 there exists $n \geq 0$ such that for each $x \in X$ there exists $y \in B_\delta(x)$ such that $f^n(y) \notin B_\epsilon(f^n(x))$.

For each $u \in \{0,1\}^*$, we define configurations $x^u \in \Sigma^{\mathbb{Z}}$ and $y^u \in \Sigma^{\mathbb{Z}}$ as follows: Let x^0 and x^1 be any two configurations such that $x^1 \notin B_\epsilon(x^0)$. Then let $|u| \geq 1$ and assume that x^u has been defined. Let $y^u \in B_\delta(x^u)$ be such that $f^n(y^u) \notin B_\epsilon(f^n(x^u))$. We can then inductively define $x^{u0} = f^n(x^u)$ and $x^{u1} = f^n(y^u)$.

For each $u \in \{0,1\}^*$ such that $|u| > 1$ we define a δ-chain (x_i) such that for each $k \in \{0, \ldots, |u|-2\}$ let $x_{kn} = x^{u_{[0,k]}}$ if $u_{k+1} = 0$ and $x_{kn} = y^{u_{[0,k]}}$ if $u_{k+1} = 1$. For $j \in \{1, 2, \ldots, n-1\}$ and $k \in \{0, 1, \ldots, |u|-2\}$ let $x_{j+kn} = f(x_{j-1+kn}) = f^j(x_{kn})$. And finally let $x_{(|u|-1)n} = f(x_{(|u|-1)n-1})$.

Let us confirm that our chain is indeed a δ-chain. Let $k \in \{1, \ldots, |u|-2\}$. We will first show that $f^n(x_{(k-1)n}) = x^{u_{[0,k]}}$.

Suppose first that $x_{(k-1)n} = x^{u_{[0,k-1]}}$. Then we know that $u_k = 0$ and so $f^n(x_{(k-1)n}) = f^n(x^{u_{[0,k-1]}}) = x^{u_{[0,k]}}$. Suppose on the other hand that $x_{(k-1)n} = y^{u_{[0,k-1]}}$. Then we know that $u_k = 1$ and so $f^n(x_{(k-1)n}) = f^n(y^{u_{[0,k-1]}}) = x^{u_{[0,k]}}$.

Now if $u_{k+1} = 0$, then $x_{kn} = x^{u_{[0,k]}}$. If on the other hand $u_{k+1} = 1$, then $x_{kn} = y^{u_{[0,k]}} \in B_\delta(x^{u_{[0,k]}})$. And since $x_{j+(k-1)n} = f(x_{j-1+(k-1)n})$ for each $j \in \{1, 2, \ldots, n-1\}$ and $k \in \{1, 2, \ldots, |u|-1\}$, we especially have that $x_{n-1+(k-1)n} = f^{n-1}(x_{(k-1)n})$ and therefore $f(x_{n-1+(k-1)n}) = f^n(x_{(k-1)n}) = x^{u_{[0,k]}}$ and thus $d(x_{kn}, f(x_{n-1+(k-1)n})) < \delta$ regardless of whether $u_{k+1} = 0$ or $u_{k+1} = 1$.

Finally for $k = |u| - 1$, straight from definition we have that $x_{j+(k-1)n} = f(x_{j-1+(k-1)n})$ or each $j \in \{1, 2, \ldots, n\}$.

By the shadowing property there exists a configuration $\bar{x}^u$ for each δ-chain, that has been constructed from a word $u \in \{0,1\}^*$ in the manner described previously, that ϵ-shadows the chain.

Now let u and v be two distinct words of the same length. Suppose k is the smallest index such that $u_k \neq v_k$. We can assume that $u_k = 0$ and $v_k = 1$. If $k = 0$, then x^0 and x^1 are the first elements in the chains u and v and $x^1 \notin B_\epsilon(x^0)$. If $k > 0$, then let $w = u_{[0,k-1]}$. By definition of the chains, the $nk-$th elements in the chains u and v are x^{w0} and x^{w1} respectively and by definition $x^{w0} \notin B_\epsilon(x^{w1})$.

By the shadowing property, we have that $f^{nk}(\bar{x}^u) \in B_\epsilon(x^{w0})$ and $f^{nk}(\bar{x}^v) \in B_\epsilon(x^{w1})$ and therefore $f^{nk}(\bar{x}^u) \notin B_\epsilon(f^{nk}(\bar{x}^v))$.

Let $j > 0$ be such that $2^{-j} < \epsilon$. Then by the above we have that $\tau_j(\bar{x}^u)_{[0,nk]} \neq \tau_j(\bar{x}^v)_{[0,nk]}$ for each distinct u and v of length k. Therefore $P_{nk}(j) \geq 2^k$ and so we have the following:

$$h(\Sigma_j(f), \sigma) = \lim_{k\to\infty} \frac{\ln P_k(j)}{k} = \lim_{k\to\infty} \frac{\ln P_{nk}(j)}{nk} \geq \lim_{k\to\infty} \frac{\ln(2^k)}{nk} = \frac{\ln 2}{n}.$$

This holds for each $j' \geq j$ so then by Theorem 2.7.7 we have that $h(\Sigma^{\mathbb{Z}}, f) \geq \frac{\ln 2}{n}$.

□

***Theorem* 4.2.13.** GCA ZERO ENTROPY *is decidable.*

Proof. According to Lemma 4.2.7, a group CA is either sensitive or equicontinuous. By Theorem 4.2.11 equicontinuous CA have zero entropy and by Lemma 4.2.12 sensitive group CA have positive entropy. Since according to Theorem 4.2.8 it is decidable if a given group CA is equicontinuous or not, we can use the same algorithm to determine whether a given group CA has zero entropy or not.

□

Lemma 4.2.14. Let (X, f) be a group CA. Then $\lambda^+(c) = \lambda^+$ and $\lambda^-(c) = \lambda^-$ for each $c \in X$.

Proof. It is enough to show that for any $n \in \mathbb{N}$ and any two configurations $c \in X$ and $c' \in X$ we have that $I_n^+(c) \leq I_n^+(c')$. The claim follows for λ^+ from this and the case for λ^- is symmetric.

Suppose $I_n^+(c) = m$. Then by definition there exists such $c'' \in W^+_{-(m-1)}(c)$ and $i \leq n$, that $f^i(c'') \notin W_0^+ f^i(c)$.

It then follows that $c'c^{-1}c'' \in W^+_{-(m-1)}(c')$ and $f^i(c'c^{-1}c'') \notin W_0^+ f^i(c')$ and so $I_n^+(c') \geq m = I_n^+(c)$.

As c and c' were arbitrarily chosen their roles can be exchanged and so $I_n^+(c') = I_n^+(c)$ for each $n \in \mathbb{N}$ and the claim follows. □

Lemma 4.2.15. Let (X, f) be an equicontinuous CA. Then $\lambda^+(c) = \lambda^+ = \lambda^- = \lambda^-(c) = 0$ for each $c \in X$.

Proof. We will show that the set $I_c = \{I_n^+(c) \in \mathbb{N} \mid n \in \mathbb{N}\}$ is bounded if c is equicontinuous.

Suppose r is the radius of the neighborhood of the CA and let $\epsilon < 2^{-r}$. Let $\delta > 0$ be such that for each $y \in B_\delta(c)$ and $n \geq 0$ we have that $f^n(y) \in B_\epsilon(f^n(c))$.

Then if $j \in \mathbb{N}$ is such that $2^{-j} < \delta$ we have that for each $y \in W_j^-(c) \subset B_\delta(x)$ and $n \geq 0$ we have that $f^n(y) \in B_\epsilon(f^n(c))$ and so $f^n(y)_i = f^n(c)_i$ for each $i \in [-r, r]$. And since $y_i = c_i$ for each $i \leq 0$, we have that $f(y)_i = f(c)_i$ for each $i \leq 0$. As $f^n(y)_i = f^n(c)_i$ for each $i \in [-r, r]$ holds for each $n \in \mathbb{N}$ it then inductively follows that for each $n \in \mathbb{N}$ and $i \leq 0$, we have that $f^n(y)_i = f^n(c)_i$.

Hence for each $c \in X$ it holds that $I_c \leq j$, for any $j > 0$ such that $2^{-j} < \delta$. Therefore $\lambda^- = \lambda^-(c) = 0$ for each $c \in X$. Symmetrically we can show that $\lambda^+ = \lambda^+(c) = 0$. □

***Theorem* 4.2.16.** SGCA ZERO GLOBAL LYAPUNOV EXPONENTS *is decidable.*

Proof. According to Lemma 4.2.7, a group CA is either sensitive or equicontinuous. By Lemma 4.2.15 $\lambda^+ = \lambda^- = 0$ for equicontinuous CA. By Lemma 4.2.12 $h_f > 0$ for sensitive CA and by Theorem 2.7.13 $h_f \leq (\lambda^+ + \lambda^-)\ln(\Sigma)$ for surjective CA, hence $0 < \lambda^+ + \lambda^-$. Since according to 4.2.8 it is decidable if a given group CA is equicontinuous or not, we can use the same algorithm to determine whether $\lambda^+ = \lambda^- = 0$ or not.

□

4.3 Lyapunov Exponents for Sensitive Cellular Automata

In this chapter we study the relation between sensitivity and the value of the pointwise Lyapunov exponents. It was conjectured in [9] by Xavier Bressaud and Pierre Tisseur and again in [55] by Petr Kůrka that if a cellular automaton is sensitive then necessarily either the left or the right pointwise Lyapunov exponent of at least one configuration would be positive. We prove in Theorem 4.3.2 that this conjecture, i.e. Conjecture 4.3.1, is false by constructing a sensitive cellular automaton from an aperiodic and complete Turing machine.

Conjecture 4.3.1. [9][55] Let $(\Sigma^{\mathbb{Z}}, f)$ be a sensitive cellular automaton. Then there exists a configuration $c \in \Sigma^{\mathbb{Z}}$ such that $\lambda^+(c) > 0$ or $\lambda^-(c) > 0$.

***Theorem* 4.3.2.** *There exists a sensitive one-dimensional cellular automaton* $(\Sigma^{\mathbb{Z}}, f)$ *such that* $\lambda^+(c) = \lambda^-(c) = 0$ *for every configuration* $c \in \Sigma^{\mathbb{Z}}$.

Proof. Let $\mathcal{M}_0 = (Q^0, \Gamma, \delta_0)$ be a complete aperiodic Turing machine. Such machines exist and were constructed first in [8]. Let $\mathcal{M}_1 = (Q^1, \Gamma, \delta_1)$ be a copy of $\mathcal{M}_0$. Let $\mathcal{M}$ be the union of $\mathcal{M}_0$ and $\mathcal{M}_1$. Recall the notations from Definition 4.1.4. Denote $Q_2^i = \Gamma^2 \times Q^i \times \{0, 1\}$ and then $Q_2 = Q_2^0 \cup Q_2^1$. If $q_i = (a, b, r_i, d) \in Q_2^i$, where $r_i \in Q^i$ then we denote the equivalent state in Q_2^j as $q_j = (a, b, r_j, d)$, where $r_j \in Q^j$ and $i \in \{0, 1\}$ and $j \in \{0, 1\}$ and $i \neq j$.

We are ready to define the set of states for our cellular automata as $\Sigma = \Sigma_{\mathcal{M},2} \cup \{>\}$. We will refer the state $>$ as the *eraser*. The point of this state is to increase the size of the simulation areas from the right side by one cell, erasing the previous content of that cell. This happens when a head symbol of the simulation area visits the right boundary. We also set conditions that this can not happen repeatedly unless the head symbol visits the left boundary at each interim. We will next define the global rules that accomplish this.

Let $a \in \Gamma^2 \times \{\leftarrow\}$ be fixed. Define $e : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ in the following way:

$$e(c)_i = \begin{cases} a & \text{if } c_{[i-1,i]} \in Q_2^1{>}, \\ > & \text{if } c_{[i-2,i-1]} \in Q_2^1{>}, \\ q_0 & \text{if } c_{[i-1,i+1]} \in Sq_1{>}, \text{ where } S = \Sigma \setminus \{>\}, \\ c_i & \text{otherwise.} \end{cases}$$

And we also define $e' : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ in the following way:

$$e'(c)_i = \begin{cases} q_1 & \text{if } c_{[i-1,i]} \in Sq_0, \text{ where } S = Q_2 \cup (T^2 \times \{\leftarrow\}), \\ a & \text{if } c_{[i-1,i]} \in {>}Q_2, \\ c_i & \text{otherwise.} \end{cases}$$

The function e is used to move the erasers one cell towards the right. The state of the cell that contained the eraser changes to an arbitrarily chosen tape symbol with a left arrow. This increases the size of the simulation area by one cell from the right side. These changes take place only when a head symbol from the machine $\mathcal{M}_1$ is seen at the cell left to a cell containing an eraser. And if so then the head symbol is changed to an equivalent one from the machine $\mathcal{M}_0$.

The function e' decides what should be done when a head symbol is at the left boundary of the simulation area. If there is an eraser at the cell on the left side of the left boundary, then the head symbol is changed to state a, which essentially removes the simulation area. If there is no eraser next to left boundary then a head symbol from the machine $\mathcal{M}_0$ is changed to the equivalent symbol from the machine $\mathcal{M}_1$.

We are ready to define our cellular automaton of interest as $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, where $f = f_{\mathcal{M}} \circ e' \circ e$. The behaviour of the CA is depicted in Figure 18.

In Lemmas 4.3.11 and 4.3.12 we prove, that the CA we constructed has the desired properties and therefore the claim follows.

□

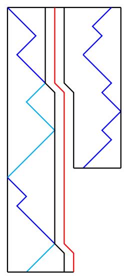

Figure 18. In this figure, we have depicted the behaviour of the CA constructed in Theorem 4.3.2. The black lines represent the left and the right simulation bounds, the blue and the cyan lines represent head symbols from the sets Q_2^1 and Q_2^0, respectively, and the red lines represent the erasers. We also note, that in the figure, time increases from top to bottom. One can witness several types of behaviour in the two simulation areas. When a head symbol from the set Q_2^1 of the left simulation area visits the right boundary, then in the next time-step the eraser moves one cell to the right, which also moves the left boundary of the second simulation area. In the same time-step, the head symbol switches to a state from the set Q_2^0. Such head symbols do not move the eraser states as witnessed when the cyan coloured line visits the right boundary. For the Turing machine to be allowed to move the eraser state again, it needs to switch back to a state from Q_2^1, which happens if and only if it visits the left boundary. The right simulation area does not have an eraser on the right side and hence its simulation area can never increase in size. In the middle of the image we can see that the head symbol on the right simulation area visits a cell within distance of two from a cell containing an eraser state and hence gets removed. This happens eventually in all simulation areas in which at some time-step an eraser state has moved next to the left boundary of the simulation area.

Before diving into the main lemmas of this subchapter, let us prove some useful facts about the functions $f : \mathbb{R} \to \mathbb{R}$ of the form $C\frac{n}{\ln n}$, where $C \in \mathbb{R}$. These will be useful to know in the proof of Lemma 4.3.11, because as we recall if a Turing machine $\mathcal{M}$ is aperiodic then for its movement bound f_M the fact that $f_M = \mathcal{O}(\frac{n}{\ln n})$ holds.

Definition 4.3.3. Let $(a, b)_{\mathbb{R}} \subseteq \mathbb{R}$. A function $f : (a, b)_{\mathbb{R}} \to \mathbb{R}$ is *concave* if

$$(1-t)f(x) + tf(y) \leq f((1-t)x + ty)$$

for each $x \in (a,b)_{\mathbb{R}}$, $y \in (a,b)_{\mathbb{R}}$ and $t \in [0,1]_{\mathbb{R}}$.

***Theorem* 4.3.4.** *Let $(a,b)_{\mathbb{R}} \subseteq \mathbb{R}$. A differentiable function $f : (a,b)_{\mathbb{R}} \to \mathbb{R}$ is concave if and only if the derivative f' is non-increasing. A twice differentiable function $f : (a,b)_{\mathbb{R}} \to \mathbb{R}$ is concave if and only if the second derivative f'' is non-positive everywhere in its domain.*

Lemma 4.3.5. Let $(a,b)_{\mathbb{R}} \subseteq \mathbb{R}_+$ and $C \in \mathbb{R}_+$. The function $f : (a,b)_{\mathbb{R}} \to \mathbb{R}$ defined as $f(x) = C\frac{x}{\ln x}$ is concave if $a \geq \mathrm{e}^2$ and it is increasing if $a > \mathrm{e}$.

Proof. Recall that the derivative of $\ln x$ is $\frac{1}{x}$, then using the quotient rule we get that $f'(x) = C\frac{-1+\ln x}{\ln^2 x}$. Then $f'(x) > 0$ if $x > \mathrm{e}$ so the second claim follows from this. Denote $g_1(x) = -1 + \ln x$ and $g_2(x) = \ln^2 x$. Then $g_1'(x) = \frac{1}{x}$ and using the chain rule we have that $g_2'(x) = \frac{2\ln x}{x}$. And further we have that $g_1(x)g_2'(x) = \frac{(-1+\ln x)(2\ln x)}{x}$ and $g_1'(x)g_2(x) = \frac{\ln^2 x}{x}$. Then using the quotient rule we have that:

$$\begin{aligned} f''(x) &= C\frac{g_1'(x)g_2(x) - g_1(x)g_2'(x)}{g_2^2(x)} \\ &= C\left(\frac{\ln^2 x}{x} - \frac{(-1+\ln x)2\ln x}{x}\right)\frac{1}{\ln^4 x} \\ &= C\frac{-\ln^2 x + 2\ln x}{x\ln^4 x} \\ &= C\frac{-\ln x + 2}{x\ln^3 x}. \end{aligned}$$

The function $x\ln^3 x$ is positive when $x > \mathrm{e}$, so $f''(x)$ is negative when $-\ln x + 2 \leq 0$, which is when $x \geq \mathrm{e}^2$. Hence by Theorem 4.3.4 $f : (a,b)_{\mathbb{R}} \to \mathbb{R}$ is concave if $a \geq \mathrm{e}^2$. □

The following theorem is due to [40] by Johan L. W. V. Jensen although it was already known earlier in the framework of twice differentiable functions as proven in [36] by Otto Hölder.

***Theorem* 4.3.6** (Jensen's Inequality)**.** *Let $f : (a,b)_{\mathbb{R}} \to \mathbb{R}$ be a concave function. Let $x_i \in (a,b)_{\mathbb{R}}$ for each $i \in I$, where I is a finite set of indices. Then*

$$\sum_{i\in I} t_i f(x_i) \leq f\Big(\sum_{i\in I} t_i x_i\Big)$$

for any set of $t_i \in [0,1]_{\mathbb{R}}$ such that $\sum_{i\in I} t_i = 1$.

We need a special case of Jensen's Inequality, which was already proven for twice differentiable functions in [27] by Jules Grolous.

***Theorem* 4.3.7** (Grolous' Inequality)**.** *Let $f : (a,b)_{\mathbb{R}} \to \mathbb{R}$ be a concave function. Let $x_i \in (a,b)_{\mathbb{R}}$ for each $i \in I$, where I is a finite set of indices. Then*

$$\sum_{i\in I} f(x_i) \leq |I| f\Big(\sum_{i\in I} \frac{x_i}{|I|}\Big).$$

Proof. Let $t_i = \frac{1}{|I|}$ for each $i \in I$. From Jensen's Inequality 4.3.6 we get immediately that

$$\sum_{i \in I} \frac{f(x_i)}{|I|} \leq f(\sum_{i \in I} \frac{x_i}{|I|})$$

and the claim follows my multiplying both sides with $|I|$. □

Lemma 4.3.8. Let $q \in \mathbb{N}$ and $n \in \mathbb{N}$. The inequality

$$\sum_{i=1}^{n} 2q^i \leq q^{n+1}$$

holds if $q > 2$.

Proof. We prove this by induction. The base case is clear. By induction hypothesis the claim holds for each $k < n$. Then for the induction step we have that

$$\sum_{i=1}^{n} 2q^i \leq q^n + 2q^n \leq 3q^n \leq q^{n+1}.$$

□

Lemma 4.3.9. Let $q \in \mathbb{N}$ and $n \in \mathbb{N}$. The inequality

$$\sum_{i=2}^{n} \frac{q^i}{\ln(i)} \leq \frac{q^{n+1}}{\ln(n+1)}$$

holds if $q > 3$ and $n > 1$.

Proof. We prove this by induction. First of all we have that $\frac{q^2}{\ln(2)} \leq \frac{q^3}{\ln(3)}$ as $\frac{\ln(3)}{\ln(2)} < 2$ and so the base case holds. By induction hypothesis the claim holds for each $k < n$. Then for the induction step we have that

$$\sum_{i=2}^{n} \frac{q^i}{\ln(i)} \leq \frac{2q^n}{\ln(n)} \leq \frac{q^{n+1}}{\ln(n+1)},$$

where the last inequality holds because

$$\frac{\ln(n+1)}{\ln(n)} \leq 2 \leq \frac{q}{2},$$

for each $n \geq 2$. □

Lemma 4.3.10. Let $b \in \mathbb{R}$ and $f : [1, b)_{\mathbb{R}} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{\ln(\frac{b}{x})}$. Then f is increasing.

Proof. We have that $f'(x) = \frac{1}{x\ln^2(\frac{b}{x})} \geq 0$ for each $x \in [1, b)_{\mathbb{R}}$ and so f is increasing. □

Lemma 4.3.11. The cellular automaton (Σ, N, f) constructed in Theorem 4.3.2 has the property that $\lambda^+(c) = \lambda^-(c) = 0$ for every configuration $c \in \Sigma^{\mathbb{Z}}$.

Proof. We will begin the proof by introducing tracking functions for the eraser states, the simulation bounds and the Turing machine heads. The point of them is that, given an initial configuration and a cell containing a Turing machine head, an eraser or a simulation bound, we can tell to which cell said symbol has travelled to in time. In the output of each function, we will use the symbol $-$ to denote, that the symbol no longer exists, i.e. it has been destroyed by the symbol $>$. These functions will be useful when we study the propagation speed of differences between configurations.

For each $c \in \Sigma^{\mathbb{Z}}$ define a set $E_c = \{i \in \mathbb{Z} \mid c_i = >\}$.

Then for each $c \in \Sigma^{\mathbb{Z}}$ we define a function $e_{c,1} : E_c \to \mathbb{Z} \cup \{-\}$ in the following way:

$$e_{c,1}(i) = \begin{cases} i+1 & \text{if } c_{[i-1,i]} \in Q_2^1 >, \\ - & \text{if } c_{[i-2,i-1]} \in Q_2^1 >, \\ i & \text{otherwise.} \end{cases}$$

Then as $f_{\mathcal{M}}$ and e' have no effect on the location of the erasers it follows from the definition of e that for each $k \in E_{f(c)}$ there exists unique i such that $e_{c,1}(i) = k$. Then for each $c \in \Sigma^{\mathbb{Z}}$ we define $e_c : E_c \times \mathbb{N} \to \mathbb{Z} \cup \{-\}$ in a way such that $e_c(i, j) = e^j_{c,1}(i)$. Then we have that for each $k \in E_{f^j(c)}$ there exists unique $i \in E_c$ such that $e_c(i, j) = j$. This function allows us to track the locations of erasers in the orbit of a given configuration.

Next we will define analogous tracking functions for the left and right boundaries of the simulation areas and the head symbols. Define $l_{c,1} : L_c \to \mathbb{Z} \cup \{-\infty, -\}$ in the following way:

$$l_{c,1}(i) = \begin{cases} i+1 & \text{if } c_{[i-2,i-1]} \in Q_2^1 > \text{ and } c_i \notin Q_2, \\ - & \text{if } c_{[i-2,i-1]} \in Q_2^1 > \text{ and } c_i \in Q_2, \\ i & \text{otherwise.} \end{cases}$$

Define $l_{c,2} : L_c \cup \{-\} \to \mathbb{Z} \cup \{-\infty, -\}$ in the following way:

$$l_{c,2}(i) = \begin{cases} - & \text{if } c_{[i-1,i]} \in >Q_2, \\ i & \text{otherwise.} \end{cases}$$

Now for each $k \in L_{f(c)}$ there exists unique $i \in L_c$ such that $l_{c,3}(i) = k$, where $l_{c,3}(i) = l_{e(c),2}(l_{c,1}(i))$. This follows for one because the function $f_{\mathcal{M}}$ does not move the boundaries of the simulation area. And for two because either $l_{c,1}$ removes a left boundary of a simulation area if the function e moves the state $>$ over a head symbol or it moves the simulation area when the function e moves an eraser next to the left boundary of the simulation area. The function $l_{c,2}$ destroys a left boundary only when e' does so by changing a head symbol to state a and destroys essentially the whole simulation area. This happens when a head symbol visits the left boundary and there is an eraser at the cell next to the left boundary. Then we define $l_c : L_c \times \mathbb{N} \to \mathbb{Z} \cup \{-\infty, -\}$ in a way such that $l_c(i,j) = l_{c,3}^j(i)$. Then we have that for each $k \in L_{f^j(c)}$ there exists unique $i \in L_c$ such that $l_c(i,j) = j$.

Analogously we define for the right boundary a function $r_{c,1} : R_c \to \mathbb{Z} \cup \{\infty, -\}$ in the following way:

$$r_{c,1}(i) = \begin{cases} i+1 & \text{if } c_{[i-1,i]} \in Q_2^1 >, \\ i & \text{otherwise.} \end{cases}$$

Define $r_{c,2} : R_c \to \mathbb{Z} \cup \{\infty, -\}$ in the following way:

$$r_{c,2}(i) = \begin{cases} - & \text{if } l_{c,3}(i) = -, \\ i & \text{otherwise.} \end{cases}$$

Now for each $k \in R_{f(c)}$ there exists unique $i \in R_c$ such that $r_{c,3}(i) = k$, where $r_{c,3}(i) = r_{e(c),2}(r_{c,1}(i))$. This follows because $r_{c,1}$ moves the simulation area only when the function e moves an eraser next to the right boundary of the simulation area and the function $r_{c,2}$ destroys a right boundary at the same time the left boundary is removed. Then we define $r_c : R_c \times \mathbb{N} \to \mathbb{Z} \cup \{\infty, -\}$ in a way such that $r_c(i,j) = r_{c,3}^j(i)$. Then we have that for each $k \in R_{f^j(c)}$ there exists unique $i \in R_c$ such that $r_c(i,j) = k$.

Finally we define the tracking function for the head symbols as $h_c : H_c \times \mathbb{N} \to \mathbb{Z} \cup \{-\}$.

$$h_c(i,j) = \begin{cases} i & \text{if } j = 0, \\ k & \text{if } l_c(i,j) \leq k \leq r_c(i,j) \\ & \text{and } \ f^j(c)_k \in Q_2, \\ - & \text{if } l_c(i,j) = -. \end{cases}$$

Again we can show that for each $k \in H_{f^j(c)}$ there exists unique $i \in H_c$ such that $h_c(i,j) = k$. This follows directly from the fact that we can uniquely track the simulation bounds and each simulation word contains exactly one head symbol.

Next we show that all the tracking functions are bounded from above by the movement bound f_M of the Turing machine $\mathcal{M}$. That is $|\alpha_c(i,j) - i| \leq f_M(j) + 1$ for each $j \in \mathbb{N}$, $\alpha \in \{h, l, r, e\}$ and for each i such that $\alpha_c(i,j)$ is defined and in $\mathbb{Z}$. Intuitively this is clear from the definition of the cellular automaton as the movements of erasers and simulation bounds are tied directly to the movement of the head symbols. Let us be more precise and show this formally. The claim is clear for h_c as the function tracks a head symbol inside a single simulation word.

For the movement bound of the erasers we can argue as follows. Let $i \in E_c$. Suppose that $k = e_c(i,j) - i > 0$. Then there exists unique $i' \in H_c$ such that $i' = \max\{i'' \in H_c \mid i'' < i\}$. The head symbol i' has then necessarily visited at least all the cells in $[i-1, e_c(i,j)-2]$ in $j-1$ steps. Hence $k - 1 \leq f_M(e_c(i,j) - i)$.

For the left boundary of the simulation area we can show that it is similarly bounded by an analogous argument. Let $i \in L_c$ and suppose that $i < l_c(i,j) = k$ and let $k - i = k'$. Then by the definition of l_c there exists unique $i' \in E_c$ such that $i' < i$ and $e_c(i',j) = k-1$. Hence $e_c(i',j) - i' = k - 1 - i' \geq k - i = l_c(i,j) - i$. Analogously we can prove that r_c is bounded from above by e_c for suitable pairs of indices.

We are ready to prove that $\lambda^+(c) = \lambda^+ = 0$ for each $c \in \Sigma^+$. To do this we will analyse how a difference can propagate in configurations of our cellular automaton.

Let $c \in \Sigma^{\mathbb{Z}}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ and r be the radius of the neighbourhood of f. Suppose that $c' \in W^+_{-m}(c) \setminus \{c\}$ and that $m \geq 2r + Cf_M(n)$, where $C = 2$ for the Cases 1 and 2 and $C = 3$ for the Case 3. Let $i_n \in \mathbb{Z}$ be the smallest such that $f^j(c') \in W^+_{i_n}(f^j(c))$ for each $j \in [0,n]$. We want to show that $i_n \leq 0$. We will analyse three cases.

Case 1: $> \not\sqsubset f^j(c)_{[-m,0]}$ for each $j \in [0,n]$.
Case 2: $> \not\sqsubset f^n(c)_{[-m,0]}$ and $> \sqsubset f^j(c)_{[-m,0]}$ for some $j \in [0,n-1]$.
Case 3: $> \sqsubset f^n(c)_{[-m,0]}$.

Let us start from Case 1. Therefore let us assume that $> \not\sqsubset f^j(c)_{[-m,0]}$ for each $j \in [0,n]$. Suppose that $i_n > 0$. Then let $t \in [0,n]$ be the smallest such that $f^t(c)_0 \neq f^t(c')_0$. Such t exists because there exists a non-decreasing sequence of time-steps j_i such that $f^{j_i}(c)_i \neq f^{j_i}(c')_i$ for $i \in [-m, i_n]$ due to the definition of f. Also by the definition of f and since at the time-step $t-1$ we have that $f^{t-1}(c)_{[0,r]} = f^{t-1}(c')_{[0,r]}$, the difference is caused by the presence of an eraser state or a head symbol in the cells $[-r,0]$ of either the configuration $f^{t-1}(c)$ or $f^{t-1}(c')$. Let us assume first that $f^{t-1}(c')_{i_0} = >$ for some $i_0 \in [-r,0]$. Then there exists a unique such $i \in E_{c'}$ that $e_{c'}(i,t-1) = i_0$. From the movement bound we have that $i_0 - i \leq f_M(t-1) \leq f_M(n)$, hence $i \in [-r - f_M(n), 0] \in [-m, 0]$. But then $i \in E_c \cap [-m,0]$ and hence $> \sqsubset c_{[-m,0]}$, which is a contradiction. Therefore the difference is caused without the presence of an eraser state and thus must come from the application of $f_{\mathcal{M}}$. Let $c'' \in \{c, c'\}$. Assume that $f^{t-1}(c'')_{i_0} \in Q_2$, where $i_0 \in [-r, 0]$. There exists a unique such $i \in H_{c''}$ such that $h_{c''}(i, t-1) = i_0$. From the movement bound we have that $i \in [-r - f_M(n), f_M(n)]$. This means that the same computation of the Turing machine is simulated in the cells $[-r - f_M(n), f_M(n)]$

of the orbits of both of the configurations c and c' for $t-1$ steps unless one of the simulation areas gets destroyed by an eraser state. This destruction cannot take place in the orbit of c because of the assumption. If it took place in the orbit of c', then again from the movement bound we would have that $> \sqsubset c'_{[-2r-2f_M(n),0]}$, which is again a contradiction. Hence $i_n \leq 0$.

Let us then move to Case 2. We assume that $> \not\sqsubset f^n(c)_{[-m,0]}$ and also that $> \sqsubset f^j(c)_{[-m,0]}$ for some $j \in [0, n-1]$. By the movement bound all of the erasers that the cells $[-m, 0]$ will see in the orbit of c must already exist in the cells $[-m, 0]$ at the first time-step. Let $i \in [-m, 0]$ be the smallest such that $c_i = >$. Let $t \in [0, n]$ be the smallest such time-step that $e_c(i, t) = -1$. By the definition of f, or more precisely e, this means that $f^{t-1}(c)_{-3} \in Q_2$. Let $i'_0 \in H_c$ be such that $h_c(i'_0, t-1) = -3$. Then from the movement bound we have that $h_c(i'_0, j) \in [-3 - f_M(n), e_c(i, j) - 1]$ for each $j \in [0, n]$, hence $i'_0 \in H_{c'}$. If there existed such $j_0 \in [0, n]$ that $h_c(i'_0, j_0) \neq h_{c'}(i'_0, j_0)$, then it would mean that $> \sqsubset f^{j_0-1}(c')_k$ where $k \in [i'_0 - r, i'_0]$. Then there would exist a unique such $i' \in E_{c'}$ such that $e_{c'}(i', j_0 - 1) = k$. But from the movement bound we would have that $i' \in [-2f_M(n) - 2r, 0] \subseteq [-m, 0]$ which is a contradiction. Therefore $i_n \leq 0$.

Finally let us consider Case 3. So let us assume that $> \sqsubset f^n(c)_{[-m,0]}$. One may refer to Figure 19 for visual depiction of this case, but many of the notations will be introduced only later. Suppose that $m \geq 2r + 3f_M(n)$ and that $i_n > 0$. We will first analyse all the possible ways of how a difference can propagate between the orbits of two configurations. Let $i_0 \in [-m, 0]$ be the largest such that $f^n(c)_{i_0} = >$. Let $k \in \mathbb{N}$ and $\epsilon_k \in E_c$ be such that $e_c(\epsilon_k, n) = i_0$. Suppose that $\epsilon_k < -m$, which means that $i_0 < -m + f_M(n) \leq -2r - 2f_M(n) = m'$. This means that $> \not\sqsubset f^n(c)_{[-m',0]}$, hence from Cases 1 and 2, we have that $i_n \leq 0$. So we can assume that $\epsilon_k \geq -m$ and hence $\epsilon_k \in E_{c'}$. If $e_c(\epsilon_k, j) = e_{c'}(\epsilon_k, j)$ for each $j \in [0, n]$, then $f^j(c') \in W^+_{e_c(\epsilon_k, j)}(f^j(c))$ for each $j \in [0, n]$ and so $i_n \leq i_0$. Hence we can assume that there exists smallest such t'_k, that $e_c(\epsilon_k, t'_k) \neq e_{c'}(\epsilon_k, t'_k)$. Suppose that $e_c(\epsilon_k, t'_k) < e_{c'}(\epsilon_k, t'_k)$ otherwise we can relabel the configurations. Let $\eta_k = \max\{i \in H_{c'} \mid i < \epsilon_k\}$. Suppose that $l_{c'}(\eta_k) \in [-m, 0]$. Then $\eta_k \in H_c$. There must then exist smallest such $t_k < t'_k$, that $h_c(\eta_k, t_k) \neq h_{c'}(\eta_k, t_k)$. But since the simulation areas are the same at the beginning and the left boundary is within the cells $[-m, 0]$, then one of them must be destroyed at time-step t_k and the other one does not. And since $e_c(\epsilon_k, t'_k) < e_{c'}(\epsilon_k, t'_k)$, we have that $h_c(\eta_k, t_k) = -$. Then there must exist $\epsilon_{k-1} \in E_c$ such that $e_c(\epsilon_{k-1}, t_k) \in [h_c(\eta_k, t_k) - 2, h_c(\eta_k, t_k) - 1]$. Now either $\epsilon_{k-1} < -m$ or $\epsilon_{k-1} \in E_{c'}$. In the case that $\epsilon_{k-1} \in E_{c'}$, there must exist such $t'_{k-1} < t_k$, that $e_c(\epsilon_{k-1}, t'_{k-1}) > e_{c'}(\epsilon_{k-1}, t'_{k-1})$. We can keep repeating this argument finitely many times while we remain in the cells $[-m, 0]$. We will need to calculate how fast a difference can propagate between chains of such pairs of alternately destructible simulation areas. But first let us look at the end of such chain. Let $t \in [0, n]$ be the smallest such that $f^t(c)_0 \neq f^t(c')_0$. Then there exists such $i'_0 \in [-r, 0]$ that

$f^{t-1}(c)_{i_0'} \in Q_2$ or $f^{t-1}(c')_{i_0'} \in Q_2$. Suppose the former, otherwise we just relabel the configurations. Then there exists unique $\eta \in H_c$ such that $h_c(\eta, t-1) = i_0'$. We must have that $\eta > \epsilon_k$ or $\eta = \max\{i \in H_c \mid i < \epsilon_k\}$, as no head symbol starting left of ϵ_k can reach origin, other than possibly the one whose right boundary is at the cell $\epsilon_k - 1$. If $\eta < \epsilon_k$ then we must have that $\eta \in [f_M(n) - r, 0]$. If $\eta > \epsilon_k$, then $\eta \in H_{c'}$. Then there must exist such $t_0 \in [0, t]$ that $h_{c'}(\eta, t_0) = -$ as otherwise the same computation would be simulated in both of the orbits of the configurations of c and c' in the simulation areas where the cell η belongs to. From the movement bound we have that $\eta \in [f_M(n) - r, 0]$. Hence if $m \geq \max\{2r + 3f_M(n), 2r + 2f_M(n) + \alpha(n)\}$, where $\alpha(n)$ is an upper bound for the propagation of difference between alternately destructing simulation areas, then $i_n \leq 0$. Next we will want to find an upper bound for such α. If the bound is sub-linear for infinitely many indices then the we have that $\lambda^+(c) = \lambda^+ = 0$ for each $c \in \Sigma^+$.

Let $k + 1 \in \mathbb{N}$ be the amount of the alternately destructing simulation areas. Let $\eta_j \in H_c \cap H_{c'}$ and $\epsilon_j \in E_c \cap E_{c'}$ be such that $\epsilon_j = r_c(\eta_j) + 1$ for each $j \in [0, k]$ and (η_j) is an increasing sequence. Let $t_j \in [0, n]$ for each $j \in [1, k]$ be the smallest such that $h_c(\eta_j, t_j) = - \neq h_{c'}(\eta_j, t_j)$ when j is odd and $h_c(\eta_j, t_j) \neq - = h_{c'}(\eta_j, t_j)$ when j is even. Suppose that j is odd and so $h_c(\eta_j, t_j) = -$. Denote $i = h_c(\eta_j, t_j - 1)$. Then we have that $e_c(\epsilon_{j-1}, t_j - 1) \in [i - 2, i - 1]$. This further means that there exists such $t \in [t_{j-1} - 1, t_j - 1]$ that $h_c(\eta_{j-1}, t) = e_c(i_{e,j-1}, t_j - 1) - 2$. Denote $i' = h_c(\eta_{j-1}, t_{j-1} - 1) = h_{c'}(\eta_{j-1}, t_{j-1} - 1)$. Then since $h_{c'}(\eta_{j-1}, t_{j-1}) = -$ we have that $e_{c'}(\epsilon_{j-2}, t_{j-1} - 1) \in [i' - 2, i' - 1]$. This means that the head symbol at cell η_{j-1}, must visit all the cells in $[e_{c'}(\epsilon_{j-2}, t_{j-1} - 1) + a_{j-1}, e_c(\epsilon_{j-1}, t_j - 1) - 2]$, where $a_{j-1} \in [1, 2]$ in $t_j - 1 - (t_{j-1} - 1) = t_j - t_{j-1}$ steps. Denote $b_j = e_c(\epsilon_{j-1}, t_j - 1) - e_{c'}(\epsilon_{j-2}, t_{j-1} - 1)$ for each odd $j \in [2, k]$. All of the above holds if we flip the roles of c and c'. Therefore we analogously define $b_j = e_{c'}(\epsilon_{j-1}, t_j - 1) - e_c(\epsilon_{j-2}, t_{j-1} - 1)$ when $j \in [2, k]$ is even. As the head symbol visits at least $b_j - 4$ cells, we have that $b_j - 1 \leq f_M(t_j - t_{j-1})$.

By Theorem 3.5.5 there exists such $g : \mathbb{N} \to \mathbb{N}$, that $f_M(n) \leq g(n) = C\frac{n}{\ln(n)}$ for each $n \in \mathbb{N}$ and where $C \in \mathbb{R}_+$. By Lemma 4.3.5 we have that g is concave and increasing in the domain $[\mathrm{e}^2, \infty)_{\mathbb{R}}$. Let $I \subseteq [2, k]$. We will partition the set I into two sets I_A and I_B, in such a way that $j \in I_A$ if $t_{j+1} - t_j \geq \mathrm{e}^2$ and $j \in I_B$ otherwise for each $j \in I$. Denote $C' = \max\{f_M(n) \in \mathbb{N} \mid n \in [1, \mathrm{e}^2)_{\mathbb{R}} \cap \mathbb{N}\}$. Then for each $j \in I_B$ we have that $f_M(n)(t_{j+1} - t_j) \leq C'$. We have that

$$\begin{aligned}\sum_{j\in I} b_j &\leq 4|I| + \sum_{t_j\in I} f_M(t_{j+1}-t_j)\\ &= 4|I| + \sum_{t_j\in I_B} f_M(t_{j+1}-t_j) + \sum_{t_j\in I_A} f_M(t_{j+1}-t_j)\\ &\leq 4|I| + |I_B|C' + \sum_{t_j\in I_A} g(t_{j+1}-t_j)\\ &\leq |I|(4+C') + |I_A| g(\sum_{t_j\in I_A}\frac{t_{j+1}-t_j}{|I_A|})\\ &\leq |I|(4+C') + |I_A| g(\frac{n}{|I_A|})\\ &= |I|(4+C') + C\frac{n}{\ln(\frac{n}{|I_A|})},\end{aligned}$$

where the the third inequality from the top follows from Jensen's Inequality 4.3.6 or Grolous's Inequality 4.3.7 for concave functions and the last inequality follows because g is increasing. If $x \in [1, n)$ then the function $\frac{1}{\ln(\frac{n}{x})}$ is increasing by Lemma 4.3.10. Hence if $|I| \in [1, n)$ we have that $C\frac{n}{\ln(\frac{n}{|I_A|})} \leq C\frac{n}{\ln(\frac{n}{|I|})}$ and then in such case $\sum_{j\in I} b_j \leq |I|(4 + C') + C\frac{n}{\ln(\frac{n}{|I|})}$. Then the upper bound is maximized when $|I|$ is maximized.

Denote $p = |\Sigma|$ and $s : \mathbb{N} \to \mathbb{N}$ $s(n) = \min\{t \in \mathbb{N} \mid f_M(t) = n\}$. The function s is well-defined for aperiodic Turing machines. Since the head of a Turing machine moves at most one cell at a time, it takes at least n steps to reach n new cells, so we have that $s(n) \geq n + 1$ for each $n \in \mathbb{N}$.

Let $t \in \mathbb{N}$. Denote $d_j = e_{c'}(\epsilon_{j-1}, t_j - 1)$ and $a_{j,t} = e_{c'}(\epsilon_{j-1}, p^t)$ when $j \in [1, k]$ is even and $d_j = e_c(\epsilon_{j-1}, t_j - 1)$ and $a_{j,t} = e_c(\epsilon_{j-1}, p^t)$ when $j \in [1, k]$ is odd. With these notations $b_j = d_j - d_{j-1}$ for each $j \in [2, k]$. Also denote $b_{j,t} = a_{j,t} - a_{j-1,t}$ for each $j \in [2, k]$. By definition none of the simulation areas where head symbols $i_{h,j}$ belong to get destroyed before at least t_j time-steps. The size of each simulation word such that there is an eraser state at the cell next to the right boundary, is increased every time when the head symbol visits first the left boundary and then the right boundary (or just the right boundary if it is the first time this happens). The sizes of simulation words are decreased only when an eraser state hits the cell next to the left boundary. Then as there are only p^t words of length t it follows that $2p^t$ is always enough time for a simulation word of length t to increase to a simulation word of length $t + 1$. By Lemma 4.3.8 we have that $\sum_{i=1}^{t-1} 2p^i \leq p^t$ and so in p^t time-steps any simulation word that has not seen an eraser on the cell next to the left boundary is at least of length t. On the other hand if there is an eraser state next to the cell on the left boundary of a simulation word of length less than t, then the simulation word gets destroyed somewhere along the way during p^t time-steps. Suppose that $t_j \geq p^t$. Then the head symbol starting initially from the cell $i_{h,j}$ does not get destroyed until p^t steps. Thus we have that $b_{j,t} \geq t$.

Define then a partition of I made out of sets I_t such that $j \in I_t$ if $t_j \in [p^t, p^{t+1})$.

Recall that the head symbol at the cell $i_{h,j}$ visits at least $b_j - 1$ cells in either the orbit of c or c' between the time-steps from t_j to t_{j+1}. This means that $s(b_j - 4) \leq t_{j+1} - t_j$. Suppose $I_t \neq \emptyset$ and denote $I_t = (j_t, j_{t+1}]$.

We have that

$$\begin{aligned} |I_t|(t-4) &\leq -4|I_t| + \sum_{j\in I_t} b_{j,t} \\ &= -4|I_t| + a_{j_{t+1},t} - a_{j_t,t} \\ &\leq -4|I_t| + d_{j_{t+1}} - a_{j_t,t} \\ &= -4|I_t| + d_{j_{t+1}} - d_{j_t} + d_{j_t} - a_{j_t,t} \\ &= d_{j_t} - a_{j_t,t} + \sum_{j\in I_t} b_j - 4 \\ &\leq s(d_{j_t} - a_{j_t,t}) + \sum_{j\in I_t} s(b_j - 4) \\ &\leq p^{t+1} \end{aligned}$$

and so if $t \geq 5$ then $|I_t| \leq \frac{p^{t+1}}{(t-4)}$. So from the previous upper bound we get for the index sets I_t that

$$\begin{aligned} \sum_{j\in I_t} b_j &\leq |I_t|(4 + C') + C\frac{p^{t+1}}{\ln(\frac{p^{t+1}}{|I_t|})} \\ &\leq \frac{p^{t+1}}{(t-4)}(4 + C') + C\frac{p^{t+1}}{\ln(\frac{p^{t+1}}{p^{t+1}/(t-4)})} \\ &= \frac{p^{t+1}}{(t-4)}(4 + C') + C\frac{p^{t+1}}{\ln(t-4)} \\ &\leq C''\frac{p^t}{\ln(t-4)}, \end{aligned}$$

for each $t \in \mathbb{N}$ where $t > 5$ and $C'' = p(4 + C') + pC$. Let $C''' \in \mathbb{R}$ be such that $\sum_{t=1}^{5} \sum_{j\in I_t} b_j \leq C'''$. We can choose for example $C''' = rp^6$. And so by Lemma 4.3.9 we have that

$$\begin{aligned} \alpha(p^n) &\leq \sum_{t=1}^{n} \sum_{j\in I_t} b_j \\ &\leq C''' + \sum_{t=6}^{n} C''\frac{p^t}{\ln(t-4)}, \\ &= C''' + p^4 \sum_{t=2}^{n-4} C''\frac{p^t}{\ln(t)}, \\ &\leq C''' + C''\frac{p^{n+1}}{\ln(n-3)} \end{aligned}$$

for each $n \in [6, \infty)$.

This implies that $\lambda^+(c) = \lambda^+ = 0$ for each $c \in \Sigma^+$.

The fact that $\lambda^-(c) = 0$ holds for each $c \in \Sigma^{\mathbb{Z}}$ is much easier to see. First of all we have seen that the eraser states travel only to the right direction. Hence if there exists such $i \in \mathbb{N}$ that $c_i = >$, for some $i \geq 0$, it means that if $c' \in W_i^-(c)$, then $f^n(c') \in W_i^-(f^n(c))$, for each $n \in \mathbb{N}$. Hence any difference coming from right must propagate within a single simulation area. But then it follows from the movement bound that $I_n^-(c) \leq f_M(n)$ and hence $\lambda^-(c) = 0$ for each $c \in \Sigma^{\mathbb{Z}}$.

$\square$

Lemma 4.3.12. The cellular automaton (Σ, N, f) constructed in Theorem 4.3.2 is sensitive.

Proof. Let $c \in \Sigma^{\mathbb{Z}}$ and r be the neighbourhood of f and denote $I_c = \{i \in \mathbb{N} \mid f^i(c)_0 = >\}$. Let $m \in \mathbb{N}$ and $k < -mr$. Suppose that $c' \in W_k^+(c)$ is such that $c'_{[k-2,k-1]} \in Q_2^1>$ and $c'_i = a$ for each $i < k-2$. Then the set $I_{c'}$ is bounded from below by m and bounded from above by some constant $C_m \in \mathbb{N}$. Therefore if I_c is bounded from above by m', then for each $m > m'$ we can find a configuration $c' \in W_{-mr-1}^+(c)$, such that the orbits will differ at the origin. On the other hand if I_c is not bounded from above, we can find for each $m \in \mathbb{N}$ a configuration $c' \in W_{-mr-1}^+(c)$, such that $I_{c'}$ is bounded from above and so the difference will again propagate to the origin. $\square$

A couple of related conjectures from literature remain open. First one is also due to Xavier Bressaud and Pierre Tisseur as stated in [9].

Conjecture 4.3.13. [9] Let X be an irreducible SFT. Let (X, f) be a sensitive and surjective cellular automaton. If μ is the Parry measure of X, then $I_\mu^+ + I_\mu^- > 0$.

The following conjecture is from T.K. Subrahmonian Moothathu in [66] and can be also found restated in [55]. As all transitive CA are surjective by Theorem 2.7.12 it would suffice to find such transitive cellular automaton whose both average Lyapunov exponents (with respect to the uniform measure) are zero to disprove it. Or analogous to what we have done here, to show that the maximal Lyapunov exponents are zero.

Conjecture 4.3.14. [66] Let $(\Sigma^{\mathbb{Z}}, f)$ be a transitive cellular automaton. Then $h_f > 0$.

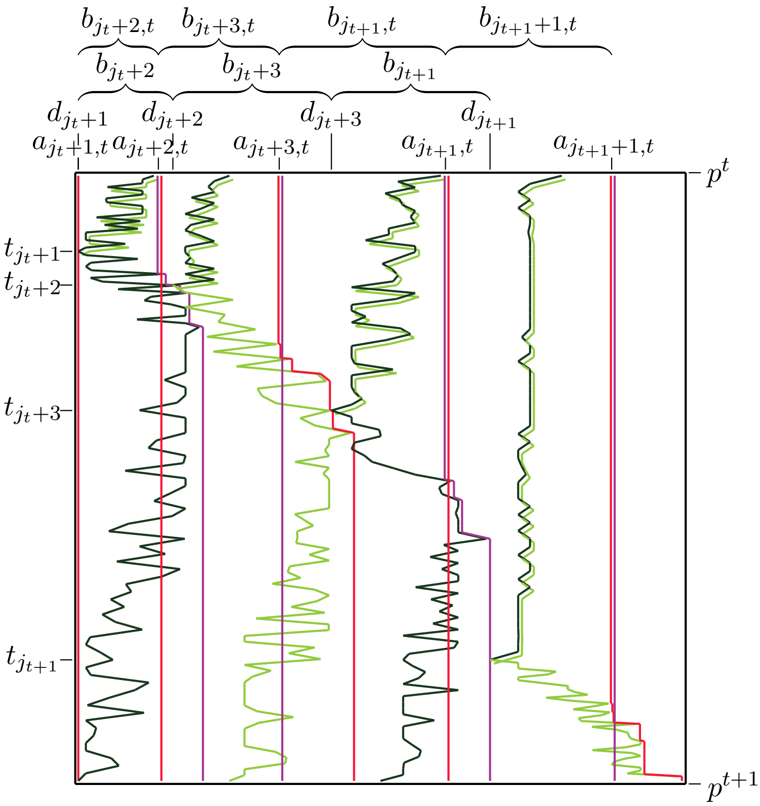

Figure 19. A depiction of the dynamics of the alternately destructing simulation areas from the time p^t to the time p^{t+1}. Two simplified space-time diagrams have been drawn on top of each other. The dark green lines and the purple lines depict the locations of the head symbols and erasers, respectively, of the configuration c. The light green lines and the red lines depict the locations of the head symbols and erasers, respectively, of the configuration c'. Recall that t_j are the times when a head symbol η_j gets destroyed in exactly one of the configurations, while d_j tells the location of the eraser state ϵ_{j-1} that destroys that head symbol at one step prior. The symbols $a_{j,t}$ tell where the erasers are located at the time p^t. The symbols b_j measure distance between the consecutive d_js. Finally the symbols $b_{j,t}$ measure distances between the consecutive $a_{j,t}$s. Note that the need to move to the left border between being able to move the right border more than once is not taken to account in this depiction as it is not important here.

5 Equivariant Dynamical Systems

5.1 Amenable Groups

In this chapter we will assume all groups to be countable. The original definition for amenable groups is due to John von Neumann in [70] and was motivated by the study of the Banach–Tarski paradox. By his definition a group is amenable if it admits a finitely-additive, left invariant probability measure on its power set. Since then many equivalent definitions have been formulated. For instance amenable groups are exactly the non-paradoxical groups. For several equivalent definitions see for example [41] by Kate Juschenko. An interesting equivalence, proven in parts in [13] by Tullio G. Ceccherini-Silberstein, Antonio Machì and Fabio Scarabotti and in [5] and [3] by Laurent Bartholdi, is that the amenable groups are precisely the groups such that any cellular automata over them satisfies the Garden-of-Eden Theorem.

We will use the definition given by Erling Følner in [20], as we will need the sequences defined by him in the definition of our entropy.

Definition 5.1.1. Let G be a group. A sequence (F_n) of finite subsets of G is called a *Følner sequence* if for each $g \in G$ and $\epsilon > 0$, there exists n_ϵ such that $|gF_n \triangle F_n| < \epsilon|F_n|$, whenever $n > n_\epsilon$. If a group admits a Følner sequence it is called an *amenable* group.

Examples of amenable groups include for instance all finite groups, all groups of subexponential growth and all solvable groups. The last class of these on the other hand includes all polycyclic groups, all supersolvable groups, all nilpotent groups, all abelian groups and all cyclic groups. We do not define these classes of groups, but one can find definitions and proofs of these results in many sources treating amenable groups. See for example [41].

The following two lemmas can be derived from well known properties of Følner sequences presented for example in [41]. Lemma 5.1.3 implies that a group extension of an amenable group by amenable group is itself an amenable group, we need this result in a bit more technical form.

Lemma 5.1.2. Let G be a group and (F_n) be a Følner sequence. For any sequence (H_n) of finite non-empty subsets of G there exists a subsequence (F_{n_i}) such that $(H_iF_{n_i})$ is a Følner sequence and $\lim\limits_{n\to\infty} \frac{|H_iF_{n_i}|}{|F_{n_i}|} = 1$.

Lemma 5.1.3. Let G be a group and $N \trianglelefteq G$ such that G/N and N are amenable groups. Let $T = \{t_1, t_2, ..., t_k\}$ be such that $\langle t_1N, t_2N, ..., t_kN\rangle = G/N$. Then for any Følner sequence $(F_n^{G/N})$ of G/N, there exists a Følner sequence (F_n^N) of N such that $(T_nF_{k_n}^N)$ is a Følner sequence of G, where $F_{k_n}^N$ is a subsequence of (F_n^N), the set T_n is a finite subset of $\langle T\rangle$ and $\varphi(T_n) = (F_n^{G/N})$, where $\varphi : G \to G/N$ is the canonical epimorphism.

Denote by $\mathrm{Fin}(G)$ the set of finite subsets of a group G. Let $f : \mathrm{Fin}(G) \to \mathbb{R}$.

f is called *non-decreasing* if	$f(F) \leq f(E)$ for each $F \subseteq E$.
f is called *non-negative* if	$0 \leq f(F)$ for each $F \neq \emptyset$.
f is called *shift invariant* if	$f(F) = f(Fg)$ for each $g \in G$.
f is called *sub-additive* if	$f(F \cup E) \leq f(F) + f(E)$ for each $E, F \in \mathrm{Fin}(G)$.

The following lemma, which we will need in our definition, originates from [72] by Donald Ornstein and Benjamin Weiss. In the following form it can be found stated in [58] by Elon Lindenstrauss and Benjamin Weiss.

Lemma 5.1.4 (Ornstein - Weiss)**.** Let G be an amenable group. Let $f : \mathrm{Fin}(G) \to \mathbb{R}$ be non-negative, non-decreasing, shift invariant and sub-additive and let (F_n) be a Følner sequence. Then the limit

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|}$$

exists and is independent of the choice of the Følner sequence.

The Lemma 5.1.4 by Ornstein and Weiss can be used to show that the limit exists in the following generalization of measure-theoretic entropy for actions of amenable groups. The definition in its following form can be found for example in [57] by Elon Lindenstrauss, but is also credited to [72].

Definition 5.1.5. Let $(X, \mathcal{B}, \mu, G)$ be a measure-preserving dynamical system, where G is amenable. The *measure-theoretic entropy of the group action with respect to a partition* α is defined as

$$\bar{h}^G_{\mu,\alpha} = \lim_{n\to\infty} \frac{H_\mu(\alpha_{F_n})}{|F_n|},$$

where (F_n) is an arbitrary Følner sequence. Furthermore the *measure-theoretic entropy of the group action* is defined as

$$\bar{h}^G_\mu = \sup\{\bar{h}^G_{\mu,\alpha} \mid \alpha \text{ is a partition of } X\}.$$

5.2 Generalized Entropy

In this chapter we define a new type of measure-theoretic entropy for measure-preserving equivariant dynamical systems. Our definition generalizes the usual definition of measure-theoretic entropy and is an invariant under a stronger type of isomorphism.

We start by defining an auxiliary function from the collection of finite subsets of a group to the set of real numbers. We show that this function is non-negative, non-decreasing, shift invariant and sub-additive and therefore by Lemma 5.1.4 we can define a converging sequence using Følner sequences. These limits are then used to define the entropy.

Definition 5.2.1. Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving dynamical system. Let G be measure-preserving. If f is G-equivariant, we call the system *G-equivariant*, or just *equivariant* if G is not specified.

Definition 5.2.2. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system. Define $h_{\mu,\alpha} : \text{Fin}(G) \to \mathbb{R}$ such that

$$h_{\mu,\alpha}(F) = \lim_{n\to\infty} \frac{H_\mu((\alpha_F)^n)}{n} = h_\mu(\alpha_F).$$

The above definition is well-defined as the limit exists due to Theorem 2.5.21.

Lemma 5.2.3. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system, then the function $h_{\mu,\alpha}$ is non-decreasing, non-negative, shift invariant and sub-additive.

Proof. Non-decreasing: For any F and E such that $F \subseteq E$, we have, that α_E is a refinement of α_F and hence $h_{\mu,\alpha}(F) \leq h_{\mu,\alpha}(E)$.

Non-negative: Clear.

Shift invariant: Let n be arbitrary. For any $F \in \mathrm{Fin}(G)$ and $g \in G$, we have that

$$\begin{aligned}
(\alpha_{Fg})^n &= \bigvee_{i=0}^{n-1} f^{-i}(\alpha_{Fg}) \\
&= \bigvee_{i=0}^{n-1} f^{-i}(\{\bigcap_{h \in Fg} h^{-1}(A_h) \mid A_h \in \alpha\}) \\
&= \{\bigcap_{i=0}^{n-1} f^{-i}(\bigcap_{h \in Fg} h^{-1}(A_{i,h})) \mid A_{i,h} \in \alpha\} \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{h \in Fg} f^{-i}(h^{-1}(A_{i,h})) \mid A_{i,h} \in \alpha\} \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{h \in Fg} h^{-1}(f^{-i}(A_{i,h})) \mid A_{i,h} \in \alpha\} \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{h' \in F} (h'g)^{-1}(f^{-i}(A_{i,h'})) \mid A_{i,h'} \in \alpha\} \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{h' \in F} g^{-1}(h'^{-1}(f^{-i}(A_{i,h'}))) \mid A_{i,h'} \in \alpha\} \\
&= \{g^{-1}(\bigcap_{i=0}^{n-1} \bigcap_{h' \in F} h'^{-1}(f^{-i}(A_{i,h'})) \mid A_{i,h'} \in \alpha\} \\
&= \{g^{-1}(\bigcap_{i=0}^{n-1} f^{-i}(\bigcap_{h' \in F} h'^{-1}(A_{i,h'})) \mid A_{i,h'} \in \alpha\} \\
&= g^{-1}(\bigvee_{i=0}^{n-1} \{\bigcap_{h \in F} h^{-1}(A_h) \mid A_h \in \alpha\}) \\
&= g^{-1}(((\alpha_F)^n) \\
&= ((\alpha_F)^n)_g.
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
H_\mu((\alpha_{Fg})^n) &= -\sum_{A \in (\alpha_{Fg})^n} \mu(A)\ln(\mu(A)) \\
&= -\sum_{A \in ((\alpha_F)^n)_g} \mu(A)\ln(\mu(A)) \\
&= -\sum_{A \in (\alpha_F)^n} \mu(g^{-1}(A))\ln(\mu(g^{-1}(A))) \\
&= -\sum_{A \in (\alpha_F)^n} \mu(A)\ln(\mu(A)) \\
&= H_\mu((\alpha_F)^n).
\end{aligned}$$

Hence $h_{\mu,\alpha}(F) = h_{\mu,\alpha}(Fg)$.

Sub-additive: For any partitions α and β, we have that

$$\begin{aligned}(\alpha \vee \beta)^n &= \bigvee_{i=0}^{n-1} f^{-i}(\{A \cap B | A \in \alpha, B \in \beta\}) \\ &= \bigvee_{i=0}^{n-1} (\{f^{-i}(A) \cap f^{-i}(B) | A \in \alpha, B \in \beta\}) \\ &= \bigvee_{i=0}^{n-1} (f^{-i}(\alpha) \vee f^{-i}(\beta)) \\ &= \alpha^n \vee \beta^n.\end{aligned}$$

Then, because $\alpha_{F \cup F'} = \bigvee_{g \in F \cup F'} g^{-1}\alpha = \bigvee_{g \in F} g^{-1}\alpha \vee \bigvee_{g' \in F'} g'^{-1}\alpha = \alpha_F \vee \alpha_{F'}$, we get that

$$\begin{aligned}H_\mu((\alpha_{F \cup F'})^n) &= H_\mu((\alpha_F \vee \alpha_{F'})^n) \\ &= H_\mu((\alpha_F)^n \vee (\alpha_{F'})^n) \\ &\leq H_\mu((\alpha_F)^n) + H_\mu((\alpha_{F'})^n),\end{aligned}$$

for each $n \in \mathbb{N}$ and $F, F' \in \mathrm{Fin}(G)$. Dividing by n and taking a limit, it follows that

$$h_{\mu,\alpha}(F \cup F') \leq h_{\mu,\alpha}(F) + h_{\mu,\alpha}(F').$$

□

Combining the above Lemma together with the Lemma of Ornstein and Weiss we are able to define our entropy.

Definition 5.2.4. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system, where G is amenable. Let $H \leq G$ and (F_n) be a Følner sequence of H. We define the (H, f)*-entropy of the system with respect to a partition* α to be

$$h^H_{\mu,\alpha} = \lim_{n \to \infty} \frac{h_{\mu,\alpha}(F_n)}{|F_n|}.$$

By Lemmas 5.2.3 and 5.1.4 the limit exists and is independent of the choice of the Følner sequence. Furthermore we define the (H, f)*-entropy of the system* to be

$$h^H_\mu = \sup\{h^H_{\mu,\alpha} \mid \alpha \text{ is a finite partition of } X\}.$$

In the above notation if we are considering multiple functions, we might add a function as a superscript and denote $h^H_\mu = h^{H,f}_\mu$ for example.

Notice that letting $H = \{1\}$ gives the usual definition of the measure-theoretic entropy. In this sense our entropy is a generalization of the classical measure-theoretic entropy.

In the next Lemma we show that just like with the usual measure-theoretic entropy, one can calculate the entropy as the limit of a generating sequence of partitions.

Lemma 5.2.5. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system, where G is amenable. Let $H \leq G$. If (α_n) is a generating sequence, then

$$\lim_{n\to\infty} h^H_{\mu,\alpha_n} = h^H_\mu.$$

Proof. For any two partitions α and β, we have that $h_\mu(\beta) \leq h_\mu(\alpha) + H_\mu(\beta|\alpha)$, which implies that

$$\begin{aligned} h_\mu(\beta_F) &\leq h_\mu(\alpha_F) + H_\mu(\beta_F|\alpha_F) \\ &\leq h_\mu(\alpha_F) + \sum_{g\in F} H_\mu(\beta_g|\alpha_F) \\ &\leq h_\mu(\alpha_F) + \sum_{g\in F} H_\mu(\beta_g|\alpha_g) \\ &\leq h_\mu(\alpha_F) + |F| H_\mu(\beta|\alpha). \end{aligned}$$

For a generating sequence, we have that for each $\epsilon > 0$ there exists n_ϵ such that $H_\mu(\beta|\alpha_n) < \frac{\epsilon}{2}$, whenever $n_\epsilon < n$. Then for any $\epsilon > 0$ and $F \in \mathrm{Fin}(G)$, we have that

$$\begin{aligned} \frac{h_{\mu,\beta}(F)}{|F|} &= \frac{h_\mu(\beta_F)}{|F|} \\ &\leq \frac{h_\mu((\alpha_n)_F)}{|F|} + \frac{|F| H_\mu(\beta|\alpha_n)}{|F|} \\ &\leq \frac{h_\mu((\alpha_n)_F)}{|F|} + \frac{\epsilon}{2} \\ &= \frac{h_{\mu,\alpha_n}(F)}{|F|} + \frac{\epsilon}{2}, \end{aligned}$$

whenever $n_\epsilon < n$. By taking any Følner sequence of H we see, that for any $\epsilon > 0$, there exists such n_ϵ, that $h^H_{\mu,\beta} \leq h^H_{\mu,\alpha_n} + \frac{\epsilon}{2}$, whenever $n > n_\epsilon$.

Let $\epsilon > 0$. By the properties of supremum, we can choose a partition β such, that $h^H_{\mu,\beta} \geq h^H_\mu - \frac{\epsilon}{2}$. Therefore it follows that there exists such n_ϵ, that

$$h^H_\mu \leq h^H_{\mu,\beta} + \frac{\epsilon}{2} \leq h^H_{\mu,\alpha_n} + \epsilon,$$

whenever $n_\epsilon < n$. This shows that

$$\lim_{n\to\infty} h^H_{\mu,\alpha_n} = h^H_\mu.$$

□

We can generalize the property that $(\alpha_g)^n = ((\alpha)^n)_g$ seen in the proof of Lemma 5.2.3 to show that $(\alpha_F)^n = (\alpha^n)_F$ for G-equivariant functions. This allows use the notation α^n_F without brackets as the order of the operation does not matter.

Lemma 5.2.6. Let X be a set such that a group G acts on it. Let $f : X \to X$ be a function such that $f \circ g = g \circ f$ for each $g \in G$. Then for any partition α, $n \in \mathbb{N}$ and $F \subseteq G$, we have that $(\alpha_F)^n = (\alpha^n)_F = \alpha^n_F$.

Proof. We already saw in the proof of Lemma 5.2.3, that $(\alpha_g)^n = (\alpha^n)_g$ for any $g \in G$. Hence we have that:

$$\begin{aligned}
(\alpha^n)_F &= \bigvee_{g\in F} (\alpha^n)_g \\
&= \bigvee_{g\in F} (\alpha_g)^n \\
&= \bigvee_{g\in F} \bigvee_{i=0}^{n} f^{-i}(\alpha_g) \\
&= \bigvee_{g\in F} \bigvee_{i=0}^{n} f^{-i}(g^{-1}(\alpha)) \\
&= \bigvee_{i=0}^{n} \bigvee_{g\in F} f^{-i}(g^{-1}(\alpha)) \\
&= \bigvee_{i=0}^{n} \{ \bigcap_{g\in F} f^{-i}(g^{-1}(A_g)) \mid A_g \in \alpha\} \\
&= \bigvee_{i=0}^{n} \{f^{-i}(\bigcap_{g\in F} g^{-1}(A_g)) \mid A_g \in \alpha\} \\
&= \bigvee_{i=0}^{n} f^{-i}(\alpha_F) \\
&= (\alpha_F)^n.
\end{aligned}$$

□

Lemma 5.2.7. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system, where G is amenable. Let $H \leq G$. Then for any finite $F \subseteq H$, $m > 0$ and partition α, we have that $h^H_{\mu,\alpha} = h^H_{\mu,\alpha^m_F}$.

Proof. Let (F_n) be a Følner sequence of H.

$$\begin{aligned}
h^H_{\mu,\alpha^m_F} &= \lim_{n\to\infty} \frac{h_{\mu,\alpha^m_F}(F_n)}{|F_n|} \\
&= \lim_{n\to\infty} \frac{h_{\mu,\alpha}(FF_n)}{|F_n|} \\
&= \lim_{n\to\infty} \frac{h_{\mu,\alpha}(FF_n)}{|F_n|} \frac{|FF_n|}{|FF_n|} \\
&= \lim_{n\to\infty} \frac{h_{\mu,\alpha}(FF_n)}{|FF_n|} \frac{|FF_n \setminus F_n| + |FF_n \cap F_n|}{|F_n|} \\
&\leq \lim_{n\to\infty} \frac{h_{\mu,\alpha}(FF_n)}{|FF_n|} \frac{|FF_n \triangle F_n| + |F_n|}{|F_n|} \\
&= \lim_{n\to\infty} \frac{h_{\mu,\alpha}(FF_n)}{|FF_n|} \\
&= h^H_{\mu,\alpha},
\end{aligned}$$

where the last equality follows because (FF_n) is a Følner sequence of H. The converse $h^H_{\mu,\alpha} \leq h^H_{\mu,\alpha^m_F}$ holds because α^m_F is a refinement of α. □

From the above lemma we get the following corollary, which is analogous to the property of having a dynamically generating sequence in the setting of the usual measure-theoretic entropy.

Corollary 5.2.8. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system, where G is amenable. Let α be a partition. If (α_i) is such a generating sequence, that for every $i \in \mathbb{N}$, there exists such $m_i \in \mathbb{N}$ and a finite subset H_i of H, that $\alpha_i = \alpha_{H_i}^{m_i}$, then $h_{\mu,\alpha}^H = h_\mu^H$.

Proof. Follows immediately from Lemmas 5.2.5 and 5.2.7. □

Definition 5.2.9. Let $i \in \{0,1\}$ and $\mathcal{D}_i = (X_i, \mathcal{B}_i, \mu_i, f_i)$ be two G-equivariant measure-preserving dynamical systems, where G is amenable. If there exists a measure-preserving G-equivariant surjective mapping $T : X_0 \to X_1$ such that $T \circ f_0 = f_1 \circ T$, then $\mathcal{D}_1$ is a *strong factor* of $\mathcal{D}_0$ and T is a strong factor map. If T is bijective and the inverse function is measure-preserving then the systems are called *strongly isomorphic*.

The following lemma shows that a strong factor of a given measure-preserving dynamical system, cannot have a larger entropy, than the original system. As a corollary, we get that strongly isomorphic systems have the same entropy.

***Theorem* 5.2.10.** *Let $i \in \{0,1\}$ and $\mathcal{D}_i = (X_i, \mathcal{B}_i, \mu_i, f_i)$ be two G-equivariant measure-preserving dynamical systems, where G is amenable. Let $H \leq G$. If $\mathcal{D}_1$ is a strong factor of $\mathcal{D}_0$, then $h_{\mu_0}^{H,f_0} \geq h_{\mu_1}^{H,f_1}$.*

Proof. Let $T : X_0 \to X_1$ be a strong factor map. Then for any partition β of X_1, we have that

$$\begin{aligned}
T^{-1}((\beta_F)^n) &= T^{-1}(\{\bigcap_{i=0}^{n-1} \bigcap_{g\in F} f_1^{-i}(g^{-1}(B_{i,g}))|B_{i,g} \in \beta\}) \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{g\in F} T^{-1}(f_1^{-i}(g^{-1}(B_{i,g})))|B_{i,g} \in \beta\} \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{g\in F} f_0^{-i}(T^{-1}(g^{-1}(B_{i,g}))|B_{i,g} \in \beta\} \\
&= \{\bigcap_{i=0}^{n-1} \bigcap_{g\in F} f_0^{-i}(g^{-1}(T^{-1}(B_{i,g}))|B_{i,g} \in \beta\} \\
&= ((T^{-1}(\beta))_F)^n,
\end{aligned}$$

holds for any $F \in \mathrm{Fin}(G)$ and $n \in \mathbb{N}$. Therefore we have that

$$\begin{aligned}
h_{\mu_0}^{H,f_0} &= \sup\{h_{\mu,\alpha}^{H,f_0} \mid \alpha \text{ is a finite partition of } X_0\} \\
&\geq \sup\{h_{\mu,T^{-1}(\beta)}^{H,f_0} \mid \beta \text{ is a finite partition of } X_1\} \\
&= \sup\{h_{\mu,\beta}^{H,f_1} \mid \beta \text{ is a finite partition of } X_1\} \\
&= h_{\mu_1}^{H,f_1}.
\end{aligned}$$

□

Corollary 5.2.11. Let $i \in \{0,1\}$ and $\mathcal{D}_i = (X_i, \mathcal{B}_i, \mu_i, f_i)$ be two G-equivariant measure-preserving dynamical systems, where G is amenable. Let $H \leq G$. If $\mathcal{D}_0$ and $\mathcal{D}_1$ are strongly isomorphic, then $h_{\mu_0}^{H,f_0} = h_{\mu_1}^{H,f_1}$.

Proof. Follows immediately from Theorem 5.2.10. □

Lemma 5.2.12. Let $\mathcal{A} = (\Sigma^G, V, w, G)$ be a cellular automaton and let μ be the uniform measure. Suppose that $G/N \cong \mathbb{Z}$. Let $g \in G$ be such that $\langle gN \rangle = G/N$. If $\mathcal{A}$ is left or right permutive with respect to $h \in G$, where $k \in \mathbb{Z}$ is such that $h \in g^k N$, then $h_\mu^N \geq |k| \log(|\Sigma|)$

Proof. We only prove the theorem in the case that $\mathcal{A}$ is left permutive as the proofs are analogous. If $k = 0$ the lower bound clearly holds, because $h_\mu^N \geq 0$, therefore we can assume that $k \neq 0$.

Denote the set of patterns $D_F = \{x|_F \mid x \in \Sigma^G\}$ for each finite $F \subseteq G$. In this notation we will omit the brackets around singleton sets. We will first show that for each $x \in D_F$ and $y \in D_{FV \setminus Fh}$ there exists unique $z \in D_{Fh}$ such that $f(c)|_F = x$ for each $c \in \Sigma^G$ such that $c|_{FV\setminus Fh} = y$ and $c|_{Fh} = z$. From this it follows that the number of patterns in D_{FV} that map to any given pattern in D_F is $|\Sigma|^{|FV|-|Fh|}$.

Let $a \in G$. From the definition of h-permutivity it follows that for each $x \in D_a$ and $y \in D_{aV\setminus ah}$ there exists unique $z \in D_{ah}$, such that $f(c)|_a = x$ for each $c \in \Sigma^G$ such that $c|_{aV\setminus ah} = y$ and $c|_{ah} = z$. Suppose then that for a finite $A \subseteq G$ we have that $A \subseteq g^{k'}N$ for some $k' \in \mathbb{Z}$. From left permutivity we have that $aV \cap g^{k'+k}N = \{ah\}$ for each $a \in A$. Therefore for each distinct $a \in A$ and $b \in A$, we have that $aV \cap bV \cap g^{k'+k}N = \emptyset$. Hence it follows that for each $x \in D_A$ and $y \in D_{AV\setminus Ah}$ there exists a unique $z \in D_{Ah}$ such that $f(c)|_A = x$ for each $c \in \Sigma^G$ such that $c|_{AV\setminus Ah} = y$ and $c|_{Ah} = z$. Let then F be a finite set and let us partition it into sets A_i such that $A_i \subseteq g^i N$. Let then A_i and A_j be two members of the partition and suppose that $i < j$. Then $A_i h \cap A_j V = \emptyset$. Thus it follows that for each $x \in D_F$ and $y \in D_{FV\setminus Fh}$ there exists unique $z \in D_{Fh}$ such that $f(c)|_F = x$ for each $c \in \Sigma^G$ such that $c|_{FV\setminus Fh} = y$ and $c|_{Fh} = z$.

Let $A = F\{1_G, g, g^2, \dots g^{|k|-1}\}$, where $F \subseteq N$ is finite. Let $A_0 = A$ and $A_l = A \cup A_{l-1}V$. Because $A \cap A_l h = \emptyset$, we have that the number of patterns in D_{A_l}, that map to a specific pattern in $D_{A_{l-1}\cup A}$ is $m_l = |\Sigma|^{|A_l|-|A_{l-1}|-|F||k|}$. Then each element in the intersection $P_A(x_0) \cap f^{-1}(P_A(x_1)) \cap \dots f^{-(n-1)}(P_A(x_{n-1})) \in \alpha_n$ contains $\prod_{i=0}^{n-1} m_i = |\Sigma|^{|A_{n-1}|-|A_0|-(n-1)|F||k|} = |\Sigma|^{|A_{n-1}|-n|F||k|}$ patterns. Thus for $B \in \alpha_n$ we have that $\mu(B) = \frac{|\Sigma|^{|A_{n-1}|-n|F||k|}}{|\Sigma|^{|A_{n-1}|}} = |\Sigma|^{-n|F||k|}$ as the measure is the uniform measure. Thus we have that $H_\mu(\alpha^n) = -\ln \mu(B) = n|F||k| \ln |\Sigma|$ and so $h_\mu(\alpha) = |F||k| \ln |\Sigma|$. Since this holds for each finite $F \subseteq N$ it follows that $h_\mu^N \geq |k| \ln |\Sigma|$. □

5.3 Generalized Lyapunov Exponents

In this chapter we generalize the notion of Lyapunov exponents of one-dimensional cellular automata by Shereshevsky. They were first introduced in [75] by Mark A. Shereshevsky. Notice that in this chapter we do not require groups to be amenable, but we require them to be finitely generated. In the context of cellular automata our definition of λ_α^H can be thought of as the maximum speed of which a difference in the cells far from H propagates to the cells of H.

Example 5.3.1. Let (Σ^G, G) be a shift. Let α be the partition $\{P_{1_G}(x) \mid x \in \Sigma^G\}$. We saw in the Subchapter 2.7 that $\alpha_F = \{P_{F^{-1}}(x) \mid x \in X\}$. Then $\mathrm{diam}(\alpha_{B_n}) = 2^{-(n+1)}$ and so $\lim_{n\to\infty} \mathrm{diam}(\alpha_{B_n}) = 0$ and thus α is a topologically *G-generating* partition.

Definition 5.3.2. Let X be a set, let G be a group acting on it and let f be a self-map with domain X. Let α be a partition of X and $H \subseteq G$. For each $x \in X$ we define

$$L_{\alpha,n}^H(x) = \inf\{k \in \mathbb{N} \mid f^i(\alpha_{B_k H^{-1}}(x)) \subseteq \alpha_{H^{-1}}(f^i(x)) \text{ for each } i \leq n\}.$$

Furthermore for $Y \subseteq X$, we define

$$L_{\alpha,n}^H(Y) = \sup\{L_{\alpha,n}^H(x) \mid x \in Y\}.$$

For $L_{\alpha,n}^H(X)$, we will use the shorthand notation $L_{\alpha,n}^H$. Finally we define

$$\lambda_\alpha^H(Y) = \limsup_{n\to\infty} \frac{L_{\alpha,n}^H(Y)}{n},$$

and give it the name *Lyapunov exponent* of the set Y, with respect to α and H.

Fixing Y in a specific way will give us generalizations of the different types of Lyapunov exponents found in literature. If Y is a singleton $\{x\}$, we call it the Lyapunov exponent of the point x or a *pointwise Lyapunov exponent* and denote it as $\lambda_\alpha^H(x)$. If $Y = Gx$ for some point $x \in X$, then we call $\lambda_\alpha^H(Gx)$ the *shift invariant Lyapunov exponent* of the point x. If $Y = X$, we call it the *global Lyapunov exponent* and denote it by λ_α^H.

Let us look at a motivational example for the above definition.

Example 5.3.3. Let (Σ^G, f) be a cellular automaton. Let α be the partition $\{P_{1_G}(x) \mid x \in \Sigma^G\}$. Let $k = L_{\alpha,n}^H(x)$. Recall from Example 5.3.1 that $\alpha_{H^{-1}} = \{P_H(x) \mid x \in X\}$ and thus $\alpha_{B_k H^{-1}} = \{P_{HB_k}(x) \mid x \in X\}$. Then if $y \in P_{HB_k}(x)$ we have that $f^i(y) \in P_H(f^i x)$ for each $i \leq n$. In other words the content of the cells in H coincide at least for the first n iterations for the orbits of x and y if the content of their cells in HB_k coincide. Hence in the context of cellular automata, we can think of

$\lambda^H_\alpha(x)$ to mean how fast a difference can propagate towards cells in H. A depiction of Lyapunov exponents for cellular automata can be found in Figure 20.

If $G = \mathbb{Z}$, then setting $\alpha = \{P_{1_G}(x) \mid x \in X\}$ and either $H = \mathbb{N}$ or $H = -\mathbb{N}$ gives us the Lyapunov exponents of one-dimensional cellular automata. Lemma 5.3.4 shows that for cellular automata $L^H_{\alpha,n}(X) \in \mathbb{N}$.

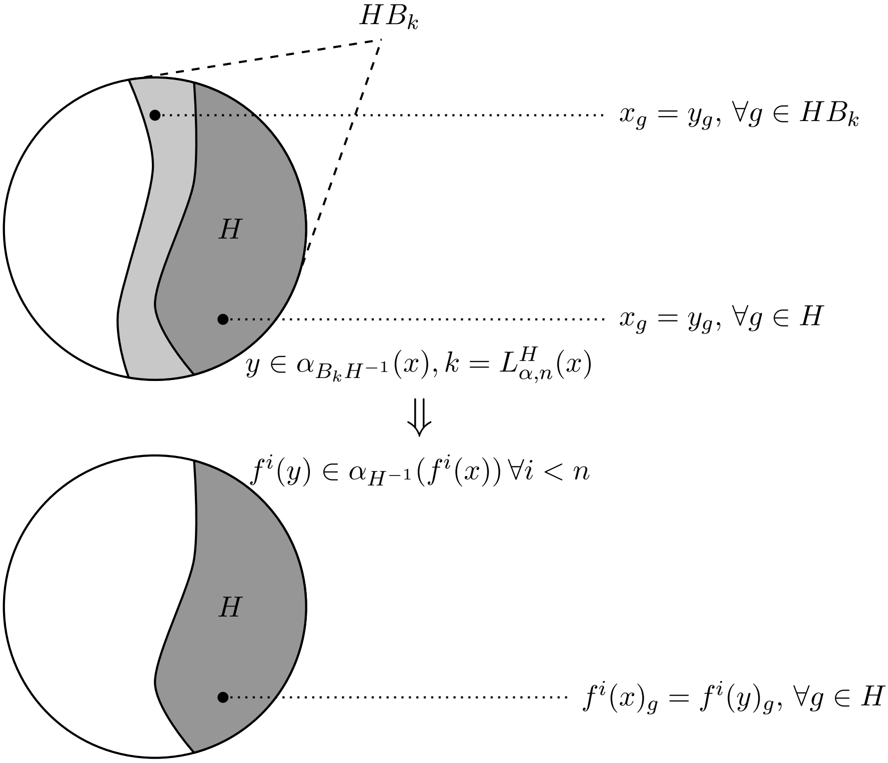

Figure 20. A depiction of the Lyapunov exponent $L^H_{\alpha,n}$ for a cellular automaton over some group G. Here $H \subseteq G$ and $\alpha = \{P_{1_G}(x) \mid x \in X\}$ as in Example 5.3.3.

Lemma 5.3.4. Let (X, f) be a G-equivariant dynamical system. Let α be a topologically G-generating open partition of X. Let $H \subseteq G$. Then

$$L^H_{\alpha,n}(Y) = \min\{k \in \mathbb{N} \mid f^i(\alpha_{B_k H^{-1}}(x)) \subseteq \alpha_{H^{-1}}(f^i(x)) \, \forall\, i \leq n \, \forall\, x \in Y\}$$

exists for any $Y \subseteq X$ and $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$ be a Lebesgue number of the partition α. Then for any $x \in X$, we have that $B_\epsilon(x) \subseteq \alpha(x)$. Because f is a continuous function on a compact metric space it is uniformly continuous, therefore there exists such $\delta > 0$, that for every $x \in X$ and $i \leq n$, we have that if $y \in B_\delta(x)$, then $f^i(y) \in B_\epsilon(f^i(x))$. Because α is a topologically generating partition, there exists such $n_\delta \in \mathbb{N}$, that

$\alpha_{B_{n_\delta}}(x) \subseteq B_\delta(x)$ holds for each $x \in X$. Putting everything together, we have that whenever $y \in \alpha_{B_{n_\delta}}(x)$, then $f^i(y) \in \alpha(f^i(x))$ for any $x \in X$ and $i \leq n$. As $\alpha_g(x) = \alpha(gx)$ and because f is G-equivariant, we have that for any $g \in H^{-1}$, $x \in X$ and $i < n$, it holds that $y \in \alpha_{B_{n_\delta}g}(x) = \alpha_{B_{n_\delta}}(gx)$ implies $f^i(y) \in \alpha(f^i(gx)) = \alpha_g(f^i(x))$. Finally $f^i(\alpha_{B_{n_\delta}H^{-1}}(x)) \subseteq \bigcap_{g\in H^{-1}} f^i(\alpha_{B_{n_\delta}g}(x)) \subseteq \bigcap_{g\in H^{-1}} \alpha_g(f^i(x)) = \alpha_{H^{-1}}(f^i(x))$. Therefore $L^H_{\alpha,n}(Y) \leq L^H_{\alpha,n} \leq n_\delta$, which proves the claim. □

The following lemma is by Michael Fekete proven in [18].

Lemma 5.3.5 (Fekete's Subadditive Lemma)**.** Let (x_n) be a sequence of real numbers. If the inequality $x_{n+m} \leq x_n + x_m$ holds for every n and m in $\mathbb{N}$, then

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf\{\frac{x_n}{n} \mid n \in \mathbb{N}\}.$$

Lemma 5.3.6. Let (X, f) be a G-equivariant dynamical system. Let α be a topologically G-generating open partition of X. Let $H \subseteq G$ be such that $H^{-1}B_k = B_kH^{-1}$ for each $k \in \mathbb{N}$. If $Y \subseteq X$ is such that $Gx \subseteq Y$ for each $x \in Y$, then

$$L^H_{\alpha,i+j}(Y) \leq L^H_{\alpha,i}(Y) + L^H_{\alpha,j}(f^i(Y)).$$

Proof. Denote $I = L^H_{\alpha,i}(Y)$ and $J = L^H_{\alpha,j}(f^i(Y))$. Let $i' \leq i$ and $j' \leq j$. For each $x \in Y$, we have that

$$\begin{aligned}
f^{i'+j'}(\alpha_{B_{I+J}H^{-1}}(x)) &= f^{i'+j'}(\alpha_{B_IH^{-1}B_J}(x)) \\
&= f^{i'+j'}(\bigcap_{g\in B_J} \alpha_{B_IH^{-1}g}(x)) \\
&\subseteq f^{j'}(\bigcap_{g\in B_J} f^{i'}(\alpha_{B_IH^{-1}g}(x))) \\
&= f^{j'}(\bigcap_{g\in B_J} f^{i'}(\alpha_{B_IH^{-1}}(gx))) \\
&\subseteq f^{j'}(\bigcap_{g\in B_J} \alpha_{H^{-1}}(f^{i'}(gx))) \\
&= f^{j'}(\bigcap_{g\in B_J} \alpha_{H^{-1}g}(f^{i'}(x))) \\
&= f^{j'}(\alpha_{H^{-1}B_J}(f^{i'}(x))) \\
&= f^{j'}(\alpha_{B_JH^{-1}}(f^{i'}(x))) \\
&\subseteq \alpha_{H^{-1}}(f^{i'+j'}(x)).
\end{aligned}$$

Thus we have shown that $L^H_{\alpha,i+j}(Y) \leq L^H_{\alpha,i}(Y) + L^H_{\alpha,j}(f^i(Y))$. □

Corollary 5.3.7. Let (X, f) be a G-equivariant dynamical system. Let α_G be a topologically G-generating open partition of X. Let $H \subseteq G$ be such that $H^{-1}B_i = B_iH^{-1}$ for each $i \in \mathbb{N}$. If $Y \subseteq X$ is such that $Gx \subseteq Y$ for each $x \in Y$ and $f^i(x) \in Y$, for each $i \in \mathbb{N}$, then

$$\lambda^H_\alpha(Y) = \lim_{n\to\infty} \frac{L^H_{\alpha,n}(Y)}{n}.$$

Proof. By Lemma 5.3.6, we have that $L^H_{\alpha,i+j}(Y) \leq L^H_{\alpha,i}(Y) + L^H_{\alpha,j}(f^i(Y))$ and since $f^i(Y) \subseteq Y$ the claim follows from Fekete's Subadditive Lemma 5.3.5. $\square$

In the above Lemma and Corollary the condition $H^{-1}B_i = B_iH^{-1}$ is satisfied for example when H^{-1} is a subset of the center of G, i.e. $H^{-1} \subseteq Z(G) = \{z \in G \mid \forall\, g \in G : zg = gz\}$. So especially if $H^{-1} = \{1_g\}$ or G is an abelian group.

We can analogously prove a measure-theoretic version of the above by using Kingman's Theorem from [49] by John F. C. Kingman.

***Theorem* 5.3.8** (Kingman's Subadditive Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space. If f is a measure-preserving function and (g_n) is a sequence of L^1 functions such that*

$$g_{i+j}(x) \leq g_i(x) + g_j(f^i(x)),$$

for every $i \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$ and μ-almost every x, then there exists a function $g : X \to \mathbb{R} \cup \{-\infty, \infty\}$ such that almost everywhere we have that

$$\lim_{n\to\infty} \frac{g_n(x)}{n} = g(x) \geq -\infty$$

and g is μ-invariant. If furthermore f is μ-ergodic, then g is constant almost everywhere.

Corollary 5.3.9. Let $(X, \mathcal{B}, \mu, f)$ be a G-equivariant measure-preserving dynamical system, where X is a compact metric space. Let α_G be a topologically G-generating open partition of X. Let $H \subseteq G$ be such that $H^{-1}B_i = B_iH^{-1}$ for each $i \in \mathbb{N}$. Then for μ-almost every x we have that

$$\lambda^H_\alpha(Gx) = \lim_{n\to\infty} \frac{L^H_{\alpha,n}(Gx)}{n}.$$

If additionally μ is f-ergodic, then $\lambda^H_\alpha(Gx)$ is constant almost everywhere.

Proof. Define a sequence of functions $l_i : X \to \mathbb{R}$ such that $l_i(x) = L^H_{\alpha,i}(Gx)$ for each $i \in \mathbb{N}$ and $x \in X$. By Lemma 5.3.4, we have that $L^H_{\alpha,i}(Gx) \leq L^H_{\alpha,i} < \infty$ for each $i \in \mathbb{N}$, hence (l_i) is a sequence of integrable functions. By Lemma 5.3.6, we have that

$$l_{i+j}(x) \leq l_i(x) + l_j(f^i(x)),$$

therefore the claim follows from Kingman's Subadditive Ergodic Theorem 5.3.8. $\square$

Definition 5.3.10. Let (X, f) be a G-equivariant dynamical system. A partition α is called *strongly irreducible* if whenever $x \in X$, $y \in X$, $H \subseteq G$ and $H' \subseteq G$ such that $H \cap H' = \emptyset$ implies that $\alpha_H(x) \cap \alpha_{H'}(y) \neq \emptyset$.

The following splitting property for strongly irreducible partitions can be used to derive an upper bound relation between the entropy and a specific pair of Lyapunov exponents as is done in Chapter 5.5.

Lemma 5.3.11. Let (X, f) be a G-equivariant dynamical system. Let α a partition of X. Let $\mathcal{H}$ be such a finite collection of subsets of G, that for each $H \in \mathcal{H}$, there exists a finite $B_H \subseteq G$ such that $y \in \alpha_{B_H H}(x)$ implies that $f(y) \in \alpha_H(f(x))$, for every $x \in X$. If α is strongly irreducible then $y \in \alpha_A(x)$ implies that $f(y) \in \alpha_K(f(x))$, for every $x \in X$, where

$$A = \bigcap_{H \in \mathcal{H}} B_H H$$

and

$$K = \bigcap_{H \in \mathcal{H}} H.$$

Proof. By induction. Assume, that $\mathcal{H} = \{H, H'\}$ and let $x \in X$ and $y \in \alpha_A(x)$. As α is strongly irreducible, there exists such a point $z \in X$, that $z \in \alpha_{B_H H}(x)$ and $z \in \alpha_C(y)$, where $C = B_{H'}H' \setminus B_H H$. Then as $z \in \alpha_A(x) = \alpha_A(y)$ we have that $z \in \alpha_{B_{H'}H'}(y)$. By assumption, this means that $f(z) \in \alpha_H(f(x)) \subseteq \alpha_K(f(x))$ and $f(z) \in \alpha_{H'}(f(y)) \subseteq \alpha_K(f(y))$. Therefore $\alpha_K(f(x)) = \alpha_K(f(z)) = \alpha_K(f(y))$, implying $f(y) \in \alpha_K(f(x))$. As an induction hypothesis, we assume that the claim holds whenever the collection $\mathcal{H}$ contains less than k elements. Finally let $\mathcal{H}$ contain k elements. For $H \in \mathcal{H}$ we denote $\mathcal{H}' = \mathcal{H} \setminus \{H\}$. Denote by

$$A' = \bigcap_{H \in \mathcal{H}'} B_H H$$

and

$$K' = \bigcap_{H \in \mathcal{H}'} H.$$

Let $x \in X$ and $y \in \alpha_A(x)$. By the strong irreducibility, there exists $z \in \alpha_{A'}(x) \cap \alpha_{B_H H}(y)$. By the induction hypothesis we have that $f(z) \in \alpha_{K'}(f(x)) \subseteq \alpha_K(f(x))$ and by the assumption we have that $f(z) \in \alpha_H(f(y)) \subseteq \alpha_K(f(y))$. Therefore we again have that $\alpha_K(f(x)) = \alpha_K(f(z)) = \alpha_K(f(y))$, implying $f(y) \in \alpha_K(f(x))$. □

Lemma 5.3.12. Let (X, f) be a G-equivariant dynamical system. Let α be a strongly irreducible partition of X. Let $\mathcal{H}$ be a finite collection of subsets of G. Denote by $K = \bigcap_{H \in \mathcal{H}} H^{-1}$. Then

$$L^K_{\alpha,n}(X) \leq \max\{L^H_{\alpha,n}(X) \mid H \in \mathcal{H}\}$$

holds for each $n \in \mathbb{N}$.

Proof. Let $x \in X$ and let us denote $k_H = L^H_{\alpha,n}(x)$ and $A = \bigcap_{H \in \mathcal{H}} B_{k_H} H^{-1}$. As we have that $f^i(\alpha_{B_{k_H} H^{-1}}(x)) \subseteq \alpha_{H^{-1}}(f^i(x))$ for each $i \leq n$, then by Lemma 5.3.11 we have that $f^i(\alpha_A(x)) \subseteq \alpha_K(f^i(x))$ for each $i \leq n$. Let $k = \max\{k_H \mid H \in \mathcal{H}\}$. Then $A \subseteq B_k K$. Hence we have that $f^i(\alpha_{B_k K}(x)) \subseteq f^i(\alpha_A(x)) \subseteq \alpha_K(f^i(x))$ for each $i \leq n$. Therefore the claim follows. □

Of course as a corollary from the previous lemma, we also get that $\lambda^K_\alpha(X) \leq \max\{\lambda^H_\alpha(X) \mid H \in \mathcal{H}\}$.

5.3.1 Invariance of the Generalized Lyapunov Exponents

If we take a supremum over all finite topologically G-generating partitions we get an invariant preserved under strong topological conjugacies, which we will see in this subchapter.

Definition 5.3.13. Let (X, f) be a G-equivariant dynamical system. Let $H \subseteq G$. Denote by $\mathcal{A}$ the set of all finite topologically G-generating open partitions of X.

Then we define $L^H_n(x) = \sup\{L^H_{\alpha,n}(x) \mid \alpha \in \mathcal{A}\}$, $L^H_n(Y) = \sup\{L^H_{\alpha,n}(Y) \mid \alpha \in \mathcal{A}\}$ and $\lambda^H(Y) = \sup\{\lambda^H_\alpha(Y) \mid \alpha \in \mathcal{A}\}$.

Lemma 5.3.14. Let (X, f) be a G-equivariant dynamical system. Let α be a finite open partition of X. Let $H \subseteq G$. Then $\lambda^H_\alpha(Y) \leq \lambda^H_{\alpha_{B_k}}(Y)$ for each $k \in \mathbb{N}$.

Proof. We have that $f^i(\alpha_{B_k B_{n'} H^{-1}}(x)) \subseteq \alpha_{B_k H^{-1}}(f^i(x)) \subseteq \alpha_{H^{-1}}(f^i(x))$, where $n' = L^H_{\alpha_{B_k},n}(x)$, holds for each $n \in \mathbb{N}$ and $x \in X$ when $i \leq n$. From this we see that $L^H_{\alpha,n}(x) \leq k + n'$ and so $L^H_{\alpha,n}(Y) \leq k + L^H_{\alpha_{B_k},n}(Y)$ for each $Y \subseteq X$ and $n \in \mathbb{N}$. Then dividing by n and taking the limit supremum, we get that $\lambda^H_\alpha(Y) \leq \lambda^H_{\alpha_{B_k}}(Y)$ as desired. □

As each α_{B_n} is a topologically G-generating partition whenever α is, then we can see that $\lambda^H(Y)$ is the supremum of limits $\limsup_{n\to\infty} \lambda^H_{\alpha_{B_n}}(Y) \geq \lambda^H_\alpha(Y)$. We show that these limits do not depend on the choice of the partition.

Lemma 5.3.15. Let (X, f) be a G-equivariant dynamical system. Let α and β be two topologically G-generating open partitions of X. Let $H \subseteq G$ and $Y \subseteq X$. Then $\limsup_{n\to\infty} \lambda^H_{\alpha_{B_n}}(Y) = \limsup_{n\to\infty} \lambda^H_{\beta_{B_n}}(Y) = \lambda^H(Y)$.

Proof. Since β is G-generating, there exists such $n_\beta \in \mathbb{N}$ that $\beta_{B_{n_\beta}}$ is a refinement of α. Likewise, since α is G-generating there exists such $n_\alpha \in \mathbb{N}$ that $\alpha_{B_{n_\alpha}}$ is a refinement of $\beta_{B_{n_\beta}}$.

Thus we have that the following chain of subset inclusions $f^i(\alpha_{B_{n_\alpha} B_{n'} H^{-1}}(x)) \subseteq f^i(\beta_{B_{n_\beta} B_{n'} H^{-1}}(x)) \subseteq \beta_{B_{n_\beta} H^{-1}}(f^i(x)) \subseteq \alpha_{H^{-1}}(f^i(x))$ holds for each $n \in \mathbb{N}$

and $x \in X$ when $i \leq n$ and where $n' = L^H_{\beta_{B_{n_\beta}},n}(x)$. Therefore we have that $L^H_{\alpha,n}(Y) \leq n_\alpha + L^H_{\beta_{B_{n_\beta}},n}(Y)$ for each $Y \subseteq X$ and $n \in \mathbb{N}$. Then dividing by n and taking the limit supremum it follows that $\lambda^H_\alpha(Y) \leq \lambda^H_{\beta_{B_{n_\beta}}}(Y)$.

As we can repeat the argument for any partition α_{B_k}, it immediately follows that $\limsup_{n\to\infty} \lambda^H_{\alpha_{B_n}}(Y) \leq \limsup_{n\to\infty} \lambda^H_{\beta_{B_n}}(Y)$. The claim then follows by exchanging the roles of α and β. □

Lemma 5.3.16. Let X and X' be two compact metric spaces and G be a group acting on them. Let T be a continuous mapping $T : X \to X'$ such that $g \circ T = T \circ g$ for each $g \in G$. Let β be a finite partition of X'. Then if $T^{-1}(\beta) = \alpha$ is a topologically G-generating partition of X, then β is a topologically G-generating partition of X'.

Proof. We saw in the proof of Theorem 5.2.10 that $T^{-1}(\beta_F) = T^{-1}(\beta)_F = \alpha_F$ holds for each finite $F \subseteq G$. While the theorem considered measure-preserving dynamical systems, this part of the proof used only set theory so it holds here too.

Because T is a continuous mapping from a compact metric space it is uniformly continuous. Let $\epsilon > 0$. Then there exists such $\delta > 0$, that for each $x \in X$ it holds that if $y \in B_\delta(x)$ then $T(y) \in B_\epsilon(T(x))$. Because α is topologically G-generating, then there exists such n_δ that $\alpha_{B_n}(x) \subseteq B_\delta(x)$ for each $n \geq n_\delta$. Then $y \in \alpha_{B_n}(x)$ implies that $T(y) \in B_\epsilon(T(x))$. On the other hand since $T(\alpha_{B_n}(x)) = \beta_{B_n}(T(x))$ we have that $\mathrm{diam}(\beta_{B_n}) < \epsilon$, whenever $n \geq n_\delta$. Therefore β is a topologically G-generating partition of X'. □

Corollary 5.3.17. Let X and X' be two compact metric spaces and G be a group acting on them. Let T be a homeomorphism $T : X \to X'$ such that $g \circ T = T \circ g$ for each $g \in G$. Let β be a finite partition of X' and $\alpha = T^{-1}(\beta)$ be a finite partition of X. Then α is a topologically G-generating partition of X if and only if β is a topologically G-generating partition of X'.

Proof. Follows from Lemma 5.3.16 as T^{-1} is a factor map from X' to X. □

By the next Theorem we see that the global Lyapunov exponents are invariants under a stronger form of conjugacy.

***Theorem* 5.3.18.** *Let $T : X \to X'$ be a conjugacy from (X, G) to (X', G) and from (X, f_0) to (X', f_1), where f_i are G-equivariant for $i \in \{0, 1\}$. Let $H \subseteq G$. Then $\lambda^H_{f_0} = \lambda^H_{f_1}$.*

Proof. Let β be a topologically G-generating finite partition of X'. Then $\alpha = T^{-1}(\beta)$ is a topologically G-generating finite partition of X by Corollary 5.3.17.

Let $k \in \mathbb{N}$ be such that $f_1^i(\beta_{B_k H^{-1}}(x)) \subseteq \beta_{H^{-1}}(f_1^i(x))$ holds for each $i \leq n$. Then $T^{-1}(f_1^i(\beta_{B_k H^{-1}}(x))) = f_0^i(T^{-1}(\beta_{B_k H^{-1}}(x))) = f_0^i(\alpha_{B_k H^{-1}}(T^{-1}(x)))$ and $T^{-1}(\beta_{H^{-1}}(f_1^i(x))) = \alpha_{H^{-1}}(T^{-1}(f_1^i(x))) = \alpha_{H^{-1}}(f_0^i(T^{-1}(x)))$. Here we used the fact that from the proof of Theorem 5.2.10 we have that $T^{-1}(\beta_F) = T^{-1}(\beta)_F = \alpha_F$ holds for each $F \subseteq G$. Therefore we have that the inclusion $f_0^i(\alpha_{B_k H^{-1}}(T^{-1}(x))) \subseteq \alpha_{H^{-1}}(f_0^i(T^{-1}(x)))$ holds for each $i \leq n$.

From the above we have that $L^H_{\alpha,n}(T^{-1}(x)) \leq L^H_{\beta,n}(x)$ for each $x \in X'$, $n \in \mathbb{N}$ and $H \subseteq G$. Because T is a conjugacy we can repeat the argument for the inverse mapping T^{-1} to attain that $L^H_{\beta,n}(T(x)) \leq L^H_{\alpha,n}(x)$. Therefore $L^H_{\alpha,n}(X) = L^H_{\beta,n}(X')$ for each $n \in \mathbb{N}$ and $H \subseteq G$. Then by dividing with n and taking the limit supremum we get that $\lambda^H_\alpha(X) = \lambda^H_\beta(X')$ for each $H \subseteq G$. Therefore it follows that $\lambda^H_{\alpha_{B_n}}(X) = \lambda^H_{\beta_{B_n}}(X')$ for each $H \subseteq G$. And furthermore $\limsup\limits_{n\to\infty} \lambda^H_{\alpha_{B_n}}(Y) = \limsup\limits_{n\to\infty} \lambda^H_{\beta_{B_n}}(Y)$ and so $\lambda^H_{f_0} = \lambda^H_{f_1}$. □

The next Theorem tells us that the supremum of the Lyapunov exponents $\lambda^H(Y)$ equals $\lambda^H_\alpha(Y)$ for any topologically generating partition α for some dynamical systems.

***Theorem* 5.3.19.** *Let (X, f) be a G-equivariant dynamical system. Let α be a G-generating open partition of X. Let $Y \subseteq X$ be such that $Gx \subseteq Y$ for each $x \in Y$. Let $H \subseteq G$ be such that $B_n H^{-1} = H^{-1} B_n$ for each $n \in \mathbb{N}$. Then $\lambda^H_\alpha(Y) = \lambda^H(Y)$.*

Proof. Recall first that $\alpha_g(x) = \alpha(gx)$ for each partition α, $g \in G$ and $x \in X$. Denote $k = L^H_{\alpha,t}(Y)$. Then since $\alpha_{B_n B_k H^{-1}}(x) = \alpha_{B_k H^{-1} B_n}(x)$ we get that

$$\begin{aligned}
f^i(\alpha_{B_n B_k H^{-1}}(x)) &= f^i(\alpha_{B_k H^{-1} B_n}(x)) \\
&\subseteq \bigcap_{g \in B_n} f^i(\alpha_{B_k H^{-1} g}(x)) \\
&= \bigcap_{g \in B_n} f^i(\alpha_{B_k H^{-1}}(gx)) \\
&\subseteq \bigcap_{g \in B_n} \alpha_{H^{-1}}(f^i(gx)) \\
&= \bigcap_{g \in B_n} \alpha_{H^{-1} g}(f^i(x)) \\
&= \alpha_{H^{-1} B_n}(f^i(x)) \\
&= \alpha_{B_n H^{-1}}(f^i(x)),
\end{aligned}$$

for each $i \leq t$ and $x \in Y$.

Therefore we have that $L^H_{\alpha_{B_n},t}(Y) \leq L^H_{\alpha,t}(Y)$ for each $n \in \mathbb{N}$ and $t \in \mathbb{N}$. Therefore we have that $\lambda^H_{\alpha_{B_n}}(Y) \leq \lambda^H_\alpha(Y)$ for each $n \in \mathbb{N}$. On the other hand by Lemma 5.3.14 we have that $\lambda^H_\alpha(Y) \leq \lambda^H_{\alpha_{B_n}}(Y)$ for each $n \in \mathbb{N}$. Therefore $\lambda^H_\alpha(Y) = \lambda^H_{\alpha_{B_n}}(Y)$ for each $n \in \mathbb{N}$. As this holds for each n and we have that $\limsup_{n\to\infty} \lambda^H_{\alpha_{B_n}}(Y) = \lambda^H(Y)$ holds for any topologically G-generating partition by Lemma 5.3.15, it follows that $\lambda^H_\alpha(Y) = \lambda^H(Y)$ as was claimed. □

As a corollary from Theorems 5.3.18 and 5.3.19 we get that the global Lyapunov exponents for one-dimensional cellular automata are invariants under strong conjugacy.

5.4 Generalized Notions of Stability

It is interesting to relate different dynamical properties with different values of Lyapunov exponents and entropy. For instance linear cellular automata are either equicontinuous or sensitive. If they are equicontinuous the classical entropies and Lyapunov exponents are zero. If they are sensitive, then in the case of one-dimensional cellular automata, the entropy is positive and finite, and in the multidimensional case entropy is infinite. On the other hand expansive cellular automata have positive and finite entropy in the one-dimensional case. It is well known that multidimensional expansive cellular automata do not exist however. It is then natural to generalize all these properties for equivariant dynamical systems and study their relation to our Lyapunov exponents and entropy.

Definition 5.4.1. Let (X, f) be a G-equivariant dynamical system. Let $H \subseteq G$ and α be a partition of X. Then a point $x \in X$ is (α, H)-*equicontinuous* if for every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that if $y \in \alpha_{B_k H^{-1}}(x)$, then it follows that $f^i(y) \in \alpha_{B_n H^{-1}}(f^i(x))$ for every $i \in \mathbb{N}$. The system (X, f) is (α, H)-equicontinuous if every point is (α, H)-equicontinuous and it is *almost* (α, H)-*equicontinuous* if the set of (α, H)-equicontinuous points is residual. If for every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that for every point x and $y \in \alpha_{B_k H^{-1}}(x)$, it holds that $f^i(y) \in \alpha_{B_n H^{-1}}(f^i(x))$ for every $i \in \mathbb{N}$, then (X, f) is *uniformly* (α, H)-*equicontinuous*.

Definition 5.4.2. Let (X, f) be a G-equivariant dynamical system. Let $H \subseteq G$ and α be a partition of X. The system (X, f) is (α, H)-*sensitive*, if there exists such $n \in \mathbb{N}$ that for any $x \in X$ and $k \in \mathbb{N}$, there exists such $i \in \mathbb{N}$ and $y \in \alpha_{B_k H^{-1}}(x)$, that $f^i(y) \notin \alpha_{B_n H^{-1}}(f^i(x))$.

Definition 5.4.3. Let (X, f) be a G-equivariant dynamical system. Let $H \subseteq G$ and α be a partition of X. The system (X, f) is (α, H)-*expansive*, if there exists such $n \in \mathbb{N}$ that for any $x \in X$ and $y \in X$ such that $x \neq y$, there exists such $i \in \mathbb{N}$, that $f^i(y) \notin \alpha_{B_n H^{-1}}(f^i(x))$.

Lemma 5.4.4. Let (X, f) be a G-equivariant dynamical system. Let α be topologically G-generating, strongly irreducible, open partition of X. Let $\mathcal{H}$ be such a collection of subsets of G, that for every $H \in \mathcal{H}$, we have that $1_G \in H^{-1}$ and $|\bigcap_{H \in \mathcal{H}} B_{k_H} H^{-1}| < \infty$, for any combination of $k_H \in \mathbb{N}$. If x is (α, H)-equicontinuous for each $H \in \mathcal{H}$ then x is equicontinuous.

Proof. By the assumption x is (α, H)-equicontinuous, for each $H \in \mathcal{H}$. Then for each $n \in \mathbb{N}$ and $H \in \mathcal{H}$, there exists $k_{H,n}$ such that $y \in \alpha_{B_{k_{H,n}} H^{-1}}(x)$ implies that $f^i(y) \in \alpha_{B_n H^{-1}}(f^i(x))$ for each $i \in \mathbb{N}$. Denote $A_n = \bigcap_{H \in \mathcal{H}} B_{k_{H,n}} H^{-1}$ and $K_n = \bigcap_{H \in \mathcal{H}} B_n H^{-1}$. By Lemma 5.3.11, we have that for each $n \in \mathbb{N}$, if $y \in \alpha_{A_n}(x)$ then $f^i(y) \in \alpha_{K_n}(f^i(x))$ for each $i \in \mathbb{N}$. Since α is G-generating, for each $\epsilon > 0$, there exists such n, that

$$\alpha_{K_n}(x) \subseteq \alpha_{B_n}(x) \subseteq B_\epsilon(x),$$

holds for each $x \in X$. Let $\delta > 0$ be a Lebesgue number of the partition α_{A_n}. Therefore by combining the above it follows, that for each $\epsilon > 0$, there exists such $\delta > 0$, that $y \in B_\delta(x)$ implies $f^i(y) \in B_\epsilon(f^i(x))$ for each $i \in \mathbb{N}$ and hence x is equicontinuous. □

Lemma 5.4.5. Let (X, f) be a G-equivariant dynamical system. Let α be topologically G-generating open partition of X. If x is equicontinuous, then x is (α, H)-equicontinuous for any $H \subseteq G$.

Proof. Let us assume that x is not (α, H)-equicontinuous, for some $H \subseteq G$. Then there exists such $n \in \mathbb{N}$, that for each $k \in \mathbb{N}$, we have that there exists such a point $y \in \alpha_{B_k H^{-1}}(x)$, that $f^i(y) \notin \alpha_{B_n H^{-1}}(f^i(x))$ for some $i \in \mathbb{N}$. Therefore there exists such $g \in H^{-1}$, that $y \in \alpha_{B_k H^{-1}}(x) \subseteq \alpha_{B_k g}(x) = \alpha_{B_k}(gx)$ and $f^i(y) \notin \alpha_{B_n g}(f^i(x)) = \alpha_{B_n}(f^i(gx))$ for some $i \in \mathbb{N}$. Let $\epsilon > 0$ be a Lebesgue number of the partition α_{B_n}. Since α is G-generating, for each $\delta > 0$, there exists such $k \in \mathbb{N}$, that for each $x \in X$ we have that $\alpha_{B_k}(x) \subseteq B_\delta(x)$. Hence there exists $\epsilon > 0$ such that for each $\delta > 0$, we have a point $y \in \alpha_{B_k}(gx) \subseteq B_\delta(gx)$ such that $f^i(y) \notin B_\epsilon(f^i(gx))$ for some $g \in H^{-1}$ and $i \in \mathbb{N}$. It is well known that if x is equicontinuous then each point in Gx are also equicontinuous and so the claim follows. □

Corollary 5.4.6. Let (X, f) be a G-equivariant dynamical system. Let α be topologically G-generating, strongly irreducible, open partition of X. Let $\mathcal{H}$ be such a collection of subsets of G, that for every $H \in \mathcal{H}$, we have that $1_G \in H^{-1}$ and $|\bigcap_{H \in \mathcal{H}} B_{k_H} H^{-1}| < \infty$, for any combination of $k_H \in \mathbb{N}$. Then (X, f) is (uniformly) (α, H)-equicontinuous, for each $H \in \mathcal{H}$ if and only if (X, f) is (uniformly) equicontinuous.

Proof. The non-uniform case follows directly from Lemmas 5.4.4 and 5.4.5. The uniform versions of the lemmas can be proven analogously, so the claim follows from those. □

Lemma 5.4.7. Let (X, f) be a G-equivariant dynamical system, $H \subseteq G$ and $x \in X$. Let α and β be two open partitions of X such that β is a refinement of α. Let α be topologically G-generating. Then x is (α, H)-equicontinuous if and only if x is (β, H)-equicontinuous.

Proof. Let x be (α, H)-equicontinuous. Let $\epsilon > 0$ be a Lebesgue number of β, then since α is G-generating, there exists such $n \in \mathbb{N}$, that $\alpha_{B_n}(x) \subseteq B_\epsilon(x) \subseteq \beta(x)$. Then from the assumption for any $m \in \mathbb{N}$, there exists such $k \in \mathbb{N}$, that

$y \in \alpha_{B_k H^{-1}}(x)$ implies that $f^i(y) \in \alpha_{B_{n+m} H^{-1}}(f^i(x)) \subseteq \beta_{B_m H^{-1}}(f^i(x))$ for each $i \in \mathbb{N}$. Since $\beta_{B_k H^{-1}}(x) \subseteq \alpha_{B_k H^{-1}}(x)$, x is (β, H)-equicontinuous.

Let x be (β, H)-equicontinuous. Then for any $n \in \mathbb{N}$, there exists such $k \in \mathbb{N}$, that $y \in \beta_{B_k H^{-1}}(x)$ implies that $f^i(y) \in \beta_{B_n H^{-1}}(f^i(x)) \subseteq \alpha_{B_n H^{-1}}(f^i(x))$ for each $i \in \mathbb{N}$. Since α is G-generating, there exists such $k' \in \mathbb{N}$, that $\alpha_{B_{k'} H^{-1}}(x) \subseteq \beta_{B_k H^{-1}}(x)$, and hence x is (α, H)-equicontinuous. □

Lemma 5.4.8. Let (X, f) be a G-equivariant dynamical system. Let $H \subseteq G$. Let α be topologically G-generating, strongly irreducible, open partition of X. Then $(L^H_{\alpha_{B_m},n}(X))$ is bounded over $n \in \mathbb{N}$ for each $m \in \mathbb{N}$ if and only if (X, f) is uniformly (α, H)-equicontinuous.

Proof. By definition

$$L^H_{\alpha_{B_m},n}(x) = \min\{k \in \mathbb{N} \mid f^i(\alpha_{B_k B_m H^{-1}}(x)) \subseteq \alpha_{B_m H^{-1}}(f^i(x)) \text{ for each } i < n\}.$$

If (X, f) is uniformly (α, H)-equicontinuous, then for each $m \in \mathbb{N}$, there exists such $k \in \mathbb{N}$, that $y \in \alpha_{B_k B_m H^{-1}}(x)$ implies $f^i(y) \in \alpha_{B_m H^{-1}}(f^i(x))$ for each $i \in \mathbb{N}$ and $x \in X$. Thus $L^H_{\alpha_{B_m},n}(x) \le k$ for each $n \in \mathbb{N}$ and $x \in X$. Thus $L^H_{\alpha_{B_m},n}(X) \le k$ for each $n \in \mathbb{N}$.

If (X, f) is not uniformly (α, H)-equicontinuous, then there exist such $m \in \mathbb{N}$, that for each $k \in \mathbb{N}$, we have that for some $x \in X$ there exists $y \in \alpha_{B_k B_m H^{-1}}(x)$ such that $f^i(y) \notin \alpha_{B_m H^{-1}}(f^i(x))$ for some $i \in \mathbb{N}$. Therefore for each $k \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $L^H_{\alpha_{B_m},i}(X) > k$. The claim follows. □

Lemma 5.4.9. Let (X, f) be a G-equivariant dynamical system. Let $H \subseteq G$ be such that $H^{-1}B_i = B_i H^{-1}$ for each $i \in \mathbb{N}$. Let α be topologically G-generating, strongly irreducible, open partition of X. Then $(L^H_{\alpha,n}(X))$ is bounded over $n \in \mathbb{N}$ if and only if (X, f) is uniformly (α, H)-equicontinuous.

Proof. By definition

$$L^H_{\alpha,n}(x) = \min\{k \in \mathbb{N} \mid f^i(\alpha_{B_k H^{-1}}(x)) \subseteq \alpha_{H^{-1}}(f^i(x)) \text{ for each } i < n\}$$

and

$$L^H_{\alpha,n}(X) = \max\{L^H_{\alpha,n}(x) \mid x \in X\}.$$

If (X, f) is not uniformly (α, H)-equicontinuous, then there exist such $m \in \mathbb{N}$, that for each $k > m$, we have that for some $x \in X$ there exists $y \in \alpha_{B_k H^{-1}}(x)$ such that $f^i(y) \notin \alpha_{B_m H^{-1}}(f^i(x)) = \alpha_{H^{-1}B_m}(f^i(x))$ for some $i \in \mathbb{N}$. Therefore there exists such $g \in B_m$, that

$$y \in \alpha_{B_k H^{-1}}(x) = \alpha_{H^{-1}B_k}(x) \subseteq \alpha_{H^{-1}B_j g}(x) = \alpha_{H^{-1}B_j}(gx) = \alpha_{B_j H^{-1}}(gx),$$

where $j = k - m$, but $f^i(y) \notin \alpha_{H^{-1}g}(f^i(x)) = \alpha_{H^{-1}}(f^i(gx))$. Hence for each $k \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $L^H_{\alpha,i}(X) > j = k - m$. The other direction follows from Lemma 5.4.8 and so the claim follows. □

Lemma 5.4.10. Let (X, f) be a G-equivariant dynamical system. Let α be topologically G-generating partition of X. If f is expansive then f is (α, H)-expansive for any $H \subseteq G$.

Proof. Since f is expansive there exists such $\epsilon > 0$, that for each $x \in X$ and $y \in X$ such that $x \neq y$, there exists such $i \in \mathbb{N}$, that $f^i(y) \notin B_\epsilon(f^i(x))$. Since α is G-generating, there exists such $n \in \mathbb{N}$, that $\alpha_{B_nH^{-1}}(x) \subseteq \alpha_{B_n}(x) \subseteq B_\epsilon(x)$ for each $x \in X$. Therefore for any $y \in \alpha_{B_nH^{-1}}(x) \setminus \{x\}$, there exists such $i \in \mathbb{N}$, that $f^i(y) \notin \alpha_{B_nH^{-1}}(f^i(x))$. □

In summary we have the following implications: (X, f) is expansive $\implies$ (X, f) is (α, H)-expansive $\implies$ (X, f) is (α, H)-sensitive $\implies$ (X, f) is not (α, H)-equicontinuous $\implies$ (X, f) is not equicontinuous.

5.5 Upper Bound: Relating the Entropy to the Lyapunov Exponents

In this chapter we develop an upper bound relation between our entropy and Lyapunov exponents. We show that if we are dealing with an amenable group that has a normal subgroup N such that the quotient G/N is isomorphic to the additive group of integers, then the entropy h^N_μ is bounded from above by the product of the measure-theoretic entropy of the group action and a pair of suitable Lyapunov exponents. The values in this product are finite and therefore so is the entropy, thus satisfying one of the properties we were after. This is a generalization of the analogous result for one dimensional cellular automata proved in [75].

***Theorem* 5.5.1.** *Let $(X, \mathcal{B}, \mu, f)$ be a measure-preserving, G-equivariant dynamical system, where X is a compact metric space, f is continuous and $\mathcal{B}$ is the Borel σ-algebra of X. Let $N \trianglelefteq G$ such that $G/N \cong \mathbb{Z}$. Let $g \in G$ be such that $\langle gN \rangle = G/N$. Let α be topologically G-generating, strongly irreducible, open partition. Then*

$$h^N_\mu \leq \bar{h}^G_\mu(\lambda^{H_+}_\alpha + \lambda^{H_-}_\alpha),$$

where $H_+^{-1} = J_+N$, $H_-^{-1} = J_-N$, $J_+ = \{g^i \mid i \geq 0\}$ and $J_- = \{g^i \mid i \leq 0\}$.

Proof. Let $H' = \{1_G\}$. Choose $\mathcal{H} = \{H_+, H_-, H'\}$. Clearly $1_G = \bigcap_{H\in\mathcal{H}} H^{-1}$. For H_+^{-1}, we have that $B_iH_+^{-1} = J_{i,+}N$, where $J_{i,+} = \{g^k \mid k \geq -i\}$ and for H_-^{-1}, we have that $B_iH_-^{-1} = J_{i,-}N$, where $J_{i,-} = \{g^k \mid k \leq i\}$. Therefore $B_iH_+^{-1} \cap B_jH_-^{-1} = J_{i,j}N$, where $J_{i,j} = \{g^k \mid -i \leq k \leq j\}$. Furthermore for some mapping $\phi : \mathbb{N} \to \mathbb{N}$, we have that $J_{i,j}N \cap B_k \subseteq J_{i,j}B^N_{\phi(k)}$, which is finite.

Recall that by definition

$$L^H_{\alpha,n}(x) = \min\{k \mid f^m(\alpha_{B_k H^{-1}}(x)) \subseteq \alpha_{H^{-1}}(f^m(x)) \text{ for each } m \leq n\}$$

and

$$L^H_{\alpha,n}(X) = \max\{L^H_{\alpha,n}(x) \mid x \in X\}.$$

Let us use the shorthand $L_H(n) = L^H_{\alpha,n}(X)$. Then by the above reasoning

$$A_n = \bigcap_{H\in\mathcal{H}} B_{L_H(n)}H^{-1} \subseteq K_n B^N_{\phi(L_{H'(n)})},$$

where $K_n = J_{L_{H_+(n)},L_{H_-(n)}}$. Especially A_n is finite for every $n \in \mathbb{N}$. By Lemma 5.3.11, we have that

$$f^m(\alpha_{A_n}(x)) \subseteq \alpha(f^m(x)),$$

for every $m < n$. Therefore α_{A_n} is a refinement of α^n and thus $\alpha_{A_n F}$ is a refinement of $(\alpha^n)_F$ for any $F \subseteq G$. Let then $\varphi_i : \mathbb{N} \to \mathbb{N}$ be such that $B^N_{\phi(k)} J_{i,i} \subseteq J_{i,i} B^N_{\varphi_i(k)}$ for each $k \in \mathbb{N}$ and denote $A^i_n = J_{i,i} K_n B^N_{\varphi_i(L_{H'(n)})}$. Then $A_n J_{i,i} \subseteq A^i_n$ and so $\alpha_{A^i_n}$ is a refinement of $\alpha_{A_n J_{i,i}}$, which in turn is a refinement of $(\alpha^n)_{J_{i,i}}$. We have that $\alpha_{A^i_n F}$ is a refinement of $(\alpha^n)_{J_{i,i}F} = (\alpha_{J_{i,i}F})^n = \alpha^n_{J_{i,i}F}$, for any finite $F \subseteq G$. Here the equalities are due to commutativity as seen in Lemma 5.2.6. Since α is topologically G-generating, the sequence $\alpha_i = \alpha_{J_{i,i}B^N_i}$ is generating in the measure-theoretic sense and by Lemma 5.2.5 $h^N_\mu = \lim_{i\to\infty} h^N_{\mu,\alpha_i}$. On the other hand by Lemma 5.2.7 we have that $h^N_{\mu,\alpha_i} = h^N_{\mu,\alpha_{J_{i,i}}}$ for each $i \in \mathbb{N}$. Then for any Følner sequence (F_n) of N such that $\lim_{n\to\infty} \frac{|B^N_{\varphi_i(L_{H'(n)})}F_n|}{|F_n|} = 1$ holds, we have that:

$$\begin{aligned}
h^N_{\mu,\alpha_i} &= h^N_{\mu,\alpha_{J_{i,i}}} \\
&= \lim_{n\to\infty} \tfrac{1}{|F_n|} \lim_{m\to\infty} \frac{H_\mu(\alpha^m_{J_{i,i}F_n})}{m} \\
&= \lim_{n\to\infty} \tfrac{1}{|F_n|} \inf_{m\in\mathbb{N}} \frac{H_\mu(\alpha^m_{J_{i,i}F_n})}{m} \\
&\leq \lim_{n\to\infty} \tfrac{1}{|F_n|} \frac{H_\mu(\alpha^n_{J_{i,i}F_n})}{n} \\
&\leq \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{n|F_n|} \\
&= \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{|A^i_n F_n|} \frac{|A^i_n F_n|}{n|F_n|} \\
&= \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{|A^i_n F_n|} \frac{|B^N_{\varphi_i(L_{H'(n)})}F_n||J_{i,i}K_n|}{n|F_n|} \\
&= \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{|A^i_n F_n|} \frac{|B^N_{\varphi_i(L_{H'(n)})}F_n||2i+1+L_{H_+(n)}+L_{H_-(n)}|}{n|F_n|} \\
&= \lim_{n\to\infty} \frac{|B^N_{\varphi_i(L_{H'(n)})}F_n|}{|F_n|} \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{|A^i_n F_n|} \frac{|2i+1+L_{H_+(n)}+L_{H_-(n)}|}{n} \\
&\leq \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{|A^i_n F_n|} \frac{(|2i+1|+|L_{H_+(n)}+L_{H_-(n)}|)}{n} \\
&= \lim_{n\to\infty} \frac{H_\mu(\alpha_{A^i_n F_n})}{|A^i_n F_n|} \frac{|L_{H_+(n)}+L_{H_-(n)}|}{n}.
\end{aligned}$$

Strictly speaking in the above we made some assumptions about convergence in the last three steps. But we will see in the following that each of the sequences in the products do indeed converge.

Now either (X, f) is both uniformly (α, H_-)- and (α, H_+)-equicontinuous or it is not. In the former case, we have that $(L_{H_+}(n))$ and $(L_{H_-}(n))$ are bounded and in the latter case at least one of them is not. Let us look at the former case first.

If $(L_{H_+}(n))$ and $(L_{H_-}(n))$ are both bounded from above, then we have that $A_n^i F_n = J_{k_1,k_2} B_{m_n}^N F_n$ for large enough n, where $k_1 \in \mathbb{N}$, $k_2 \in \mathbb{N}$ and $m_n \in \mathbb{N}$. Now by Lemma 5.1.2 we can choose such a Følner sequence of N that $B_{k_n}^N F_n$ is a Følner sequence. Hence $\lim_{n\to\infty} \frac{H_\mu(\alpha_{A_n^i F_n})}{|A_n^i F_n|} = \frac{h_{\mu,\alpha_{J_{k_1,k_2}}}}{|J_{k_1,k_2}|} \in \mathbb{R}$. On the other hand $\lim_{n\to\infty} \frac{|L_{H_+(n)} + L_{H_-(n)}|}{n} = 0$. Therefore as a product of two converging sequences we have that $h_{\mu,\alpha_i}^N = 0$ for each $i \in \mathbb{N}$ and so $h_\mu^N = 0$.

Suppose then that at least one of $(L_{H_+}(n))$ or $(L_{H_-}(n))$ is unbounded. Then $(J_{i,i}K_n)$ maps to a Følner sequence of G/N by the canonical epimorphism. Then we can choose such a Følner sequence (F_n) of N, that $(B_{\varphi_i(L_{H'(n)})}^N F_n)$ is a Følner sequence of N and $(A_n^i F_n)$ is a Følner sequence of G and $\lim_{n\to\infty} \frac{|B_{\varphi_i(L_{H'(n)})}^N F_n|}{|F_n|} = 1$. Such a choice exists by Lemmas 5.1.2 and 5.1.3. Therefore we have that

$$\begin{aligned} h_\mu^N &= \lim_{i\to\infty} h_{\mu,\alpha_i}^N \\ &\le \lim_{i\to\infty} \lim_{n\to\infty} \frac{H_\mu(\alpha_{A_n^i F_n})}{|A_n^i F_n|} \frac{|L_{H_+(n)} + L_{H_-(n)}|}{n} \\ &= \lim_{i\to\infty} \bar{h}_\mu^G(\lambda_\alpha^{H_+} + \lambda_\alpha^{H_-}) \\ &= \bar{h}_\mu^G(\lambda_\alpha^{H_+} + \lambda_\alpha^{H_-}). \end{aligned}$$

□

Theorem 5.5.2. *Let $\mathcal{B}$ be the Borel σ-algebra of Σ^G and μ be the uniform measure on $\mathcal{B}$. Let $(\Sigma^G, \mathcal{B}, \mu, f)$ be a left permutive cellullar automaton with respect to $h \in G$. Let $g \in G$ be such that $\langle gN \rangle = G/N$. Let $hN = g^k N$. Suppose that the neighbourhood V is contained in the set $H_-^{-1} = J_- N$, where $J_- = \{g^i \mid i \le 0\}$. Then $h_\mu^N = |k| \log |\Sigma|$.*

Proof. Let α be a topologically G-generating, strongly irreducible, open partition of Σ^G. Then it is easy to see that $\lambda_\alpha^{H_-} \le |k|$. This follows because the neighbourhood is contained in KN, where $K = \{g^i \mid -k \le i \le 0\}$. On the other hand we have that $\lambda_\alpha^{H_+} = 0$. Hence by Theorem 5.5.1 we have that $h_\mu^N \le \bar{h}\mu^G(\lambda_\alpha^{H_+} + \lambda_\alpha^{H_+}) \le |k| \log |\Sigma|$.

On the other hand we saw in Lemma 5.2.12 that $h_\mu^N \ge |k| \log |\Sigma|$ and so the claim follows. □

An analogous result holds for the right permutive cellular automata also.

6 Open Problems and Future Directions

Here we have gathered some open conjectures from literature and some questions of our own. First one is due to Xavier Bressaud and Pierre Tisseur as stated in [9].

Conjecture 6.0.1. [9] Let X be an irreducible SFT. Let (X, f) be a sensitive and surjective cellular automaton. If μ is the Parry measure of X, then $I^+_\mu + I^-_\mu > 0$.

The following conjecture is from T.K. Subrahmonian Moothathu in [66] and can be also found restated in [55]. As all transitive CA are surjective by Theorem 2.7.12 it would suffice to find such transitive cellular automaton whose both average Lyapunov exponents (with respect to the uniform measure) are zero to disprove it. Or to show that the maximal Lyapunov exponents are zero.

Conjecture 6.0.2. [66] Let $(\Sigma^{\mathbb{Z}}, f)$ be a transitive cellular automaton. Then $h_f > 0$.

We could ask several related questions here. If a given cellular automaton is either 1) surjective and sensitive, 2) bijective and sensitive, 3) transitive, 4) mixing, then does there necessarily exist a configuration with a positive pointwise Lyapunov exponent? Can the average Lyapunov exponents be zero in any of these cases?

The decision problem that asks whether two given cellular automata are conjugate was shown undecidable in [38] by Joonatan Jalonen and Jarkko Kari. The proof utilized the fact that conjugate cellular automata (and dynamical systems in general) have the same topological entropy. As the entropy can be calculated exactly in the framework of linear cellular automata it is then natural to ask if the question is decidable in such setting. Although non-conjugate linear cellular automata can have the same entropy, so the fact that the entropy can be calculated might not indicate anything in itself. To see this consider a radius-0 neighbourhood cellular automata. Suppose one has 2 states and another one has 3 states and the local rule just adds 1 modulo the amount of states. Then each configuration has period 2 in one cellular automaton and period 3 in another one and so they are not conjugate. But the entropies are zero for both. Naturally we can ask the same question in the setting of the group cellular automata also.

CONJUGACY OF LCA: Given two linear cellular automata, decide if they are topologically conjugate.

CONJUGACY OF GCA: Given two group cellular automata, decide if they are topologically conjugate.

Question 6.0.3. Is the CONJUGACY OF LCA decidable?

Question 6.0.4. Is the CONJUGACY OF GCA decidable?

In Chapter 5 we generalized the measure-theoretic entropy for such measure-preserving dynamical systems whose function is G-equivariant for some amenable group G. Analogously we could define the generalized topological entropy and by slightly altering the proofs we could probably show that all the equivalent properties hold. Does there then exist some kind of variational principle between these two entropies? The entropies were taken with respect to some subgroup of G, we could define the set of entropies over all subsets. Would such set imply something meaningful about the system that a single entropy does not? We did not define the average generalized Lyapunov exponents, but one can easily do so analogous to the one-dimensional ones. Could our upper bound then be slightly improved with them? It is likely that our proof would not need to be edited that much, one could probably apply the generalized Shannon-McMillan-Breiman Theorem for amenable groups.

List of References

[1] Stål O. Aanderaa and Harry R. Lewis. Linear sampling and the ∀∃∀ case of the decision problem. *Journal of Symbolic Logic*, 39(3):519–548, 1974. doi: https://doi.org/10.2307/2272894.

[2] Serafino Amoroso and Yale N. Patt. Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures. *Journal of Computer and System Sciences*, 6(5):448–464, 1972. doi: https://doi.org/10.1016/S0022-0000(72)80013-8.

[3] Laurent Bartholdi. A converse to Moore's theorem on cellular automata. *Preprint arXiv 0709.4280*, 2007.

[4] Laurent Bartholdi. Gardens of Eden and amenability on cellular automata. *Journal of the European Mathematical Society*, 012(1):241–248, 2010. doi: https://doi.org/10.4171/JEMS/196.

[5] Laurent Bartholdi. Amenability of groups is characterized by Myhill's theorem. *Journal of the European Mathematical Society*, 21(10):3191–3197, 2019. doi: https://doi.org/10.4171/JEMS/900.

[6] Pierre Béaur and Jarkko Kari. Decidability in Group Shifts and Group Cellular Automata. In Javier Esparza and Daniel Král̆, editors, *45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020)*, volume 170 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 12:1–12:13. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2020. ISBN 978-3-95977-159-7. doi: https://doi.org/10.4230/LIPIcs.MFCS.2020.12.

[7] François Blanchard and Pierre Tisseur. Entropy rate of higher-dimensional cellular automata. *Preprint arXiv 1206.6765*, 2012.

[8] Vincent D. Blondel, Julien Cassaigne, and Codrin Nichitiu. On the presence of periodic configurations in Turing machines and in counter machines. *Theoretical Computer Science*, 289(1): 573–590, 2002. doi: https://doi.org/10.1016/S0304-3975(01)00382-6.

[9] Xavier Bressaud and Pierre Tisseur. On a zero speed sensitive cellular automaton. *Nonlinearity*, 20(1):1–19, 2006. doi: https://doi.org/10.1088/0951-7715/20/1/002.

[10] Michael Brin and Anatole Katok. On local entropy. In J. Palis, editor, *Geometric Dynamics*, pages 30–38. Springer Berlin Heidelberg, 1983. ISBN 978-3-540-40969-4.

[11] Julien Cassaigne, Nicolas Ollinger, and Rodrigo Torres-Avilés. A small minimal aperiodic reversible Turing machine. *Journal of Computer and System Sciences*, 84:288–301, 2017. doi: https://doi.org/10.1016/j.jcss.2016.10.004.

[12] Gianpiero Cattaneo, Enrico Formenti, Giovanni Manzini, and Luciano Margara. Ergodicity, transitivity, and regularity for linear cellular automata over zm. *Theoretical Computer Science*, 233 (1):147–164, 2000. doi: https://doi.org/10.1016/S0304-3975(98)00005-X.

[13] Tullio G. Ceccherini-Silberstein, Antonio Machì, and Fabio Scarabotti. Amenable groups and cellular automata. *Annales de l'Institut Fourier*, 49(2):673–685, 1999. doi: https://doi.org/10.5802/aif.1686.

[14] Karel Culik II and Sheng Yu. Undecidability of CA classification schemes. *Complex Systems*, 2: 177–190, 1988.

[15] Michele D'amico, Giovanni Manzini, and Luciano Margara. On computing the entropy of cellular automata. *Theoretical Computer Science*, 290(3):1629–1646, 2003. doi: https://doi.org/10.1016/S0304-3975(02)00071-3.

[16] Efim I. Dinaburg. The relation between topological entropy and metric entropy. *Soviet Mathematics. Doklady*, 11:13–16, 1970.

[17] Bruno Durand, Enrico Formenti, and Georges Varouchas. On undecidability of equicontinuity classification for cellular automata. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AB, Discrete Models for Complex Systems (DMCS'03), 2003. doi: https://doi.org/10.46298/dmtcs.2302.

[18] Michael Fekete. Über die verteilung der wurzeln bei gewissen algebraischen gleichungen mit ganzzahligen koeffizienten. *Mathematische Zeitschrift*, 17:228–249.

[19] Michele Finelli, Giovanni Manzini, and Luciano Margara. Lyapunov exponents versus expansivity and sensitivity in cellular automata. *Journal of Complexity*, 14(2):210–233, 1998. doi: https://doi.org/10.1006/JCOM.1998.0474.

[20] Erling Følner. On groups with full Banach mean value. *Mathematica Scandinavica*, 3:243–254, 1955.

[21] Anahí Gajardo, Nicolas Ollinger, and Rodrigo Torres-Avilés. Some undecidable problems about the trace-subshift associated to a Turing machine. *Discrete Mathematics and Theoretical Computer Science*, Vol. 17 no.2(2):267–284, 2015. doi: https://doi.org/10.46298/dmtcs.2137.

[22] Anahí Gajardo, Nicolas Ollinger, and Rodrigo Torres-Avilés. The Transitivity Problem of Turing Machines. In Giuseppe F Italiano, Giovanni Pighizzini, and Donald T. Sannella, editors, *Mathematical Foundations of Computer Science 2015*, pages 231–242. Springer Berlin Heidelberg, 2015. ISBN 978-3-662-48057-1.

[23] Anahí Gajardo, Nicolas Ollinger, and Rodrigo Torres-Avilés. Some undecidable problems about the trace-subshift associated to a Turing machine. *Discrete Mathematics & Theoretical Computer Science*, Vol. 17 no.2, 2015. doi: https://doi.org/10.46298/dmtcs.2137.

[24] Tim N. T. Goodman. Relating topological entropy and measure entropy. *Bulletin of the London Mathematical Society*, 3(2):176–180, 1971. doi: https://doi.org/10.1112/blms/3.2.176.

[25] L. Wayne Goodwyn. Topological entropy bounds measure-theoretic entropy. *Proceedings of the American Mathematical Society*, 23(3):679–688, 1969.

[26] Walter Greiner. *Lyapunov Exponents and Chaos*, pages 503–516. Springer Berlin Heidelberg, 2010. ISBN 978-3-642-03434-3.

[27] Jules Grolous. Un théoreme sur les fonctions. *L'Institut, Journal Universel des Sciences et des Sociétés Savantes en France. Première section, Sciences Mathématiques (2)*, 3:401, 1875.

[28] Pierre Guillon and Ville Salo. Distortion in one-head machines and cellular automata. In Alberto Dennunzio, Enrico Formenti, Luca Manzoni, and Antonio E. Porreca, editors, *Cellular Automata and Discrete Complex Systems*, pages 120–138. Springer International Publishing, 2017. ISBN 978-3-319-58631-1.

[29] Gustav A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Mathematical systems theory*, 3(4):320–375, 1969. doi: https://doi.org/10.1007/BF01691062.

[30] Toni Hotanen. Everywhere zero pointwise Lyapunov exponents for sensitive cellular automata. In Hector Zenil, editor, *AUTOMATA 2020: Lecture Notes in Computer Science. Cellular Automata and Discrete Complex Systems*, volume 12286, pages 71–85. Springer International Publishing, 2020. ISBN 978-3-030-61588-8.

[31] Toni Hotanen. Generalized measure-theoretic entropy and Lyapunov exponents for g-equivariant functions. In Anni Hakanen, Vesa Halava, Pyry Herva, Jarkko Kari, Tero Laihonen, Ion Petre, and Aleksi Saarela, editors, *Proceedings of the Sixth Russian-Finnish Symposium on Discrete Mathematics*, pages 71–82, 2021. ISBN 978-952-12-4113-0.

[32] Toni Hotanen. Undecidability of the topological entropy of reversible cellular automata and related problems. In *Unconventional Computation and Natural Computation: 20th International Conference, UCNC 2023, Jacksonville, FL, USA, March 13–17, 2023, Proceedings*, page 108–123. Springer-Verlag, 2023. ISBN 978-3-031-34033-8. doi: https://doi.org/10.1007/978-3-031-34034-5_8.

[33] Toni Hotanen. Decision problems related to topological entropy for specific classes of cellular automata and Turing machines. Submitted in 2024.

[34] Toni Hotanen. Generalized measure-theoretic entropy and Lyapunov exponents for equivariant dynamical systems. Submitted in 2024.

[35] Lyman P. Hurd, Jarkko Kari, and Karel Culik. The topological entropy of cellular automata is uncomputable. *Ergodic Theory and Dynamical Systems*, 12(2):255–265, 1992. doi: https://doi.org/10.1017/S0143385700006738.

[36] Otto Hölder. Ueber einen mittelwerthabsatz. *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, 1889:38–47, 1889.

[37] Masanobu Itô, Nobuyasu Ôsato, and Masakazu Nasu. Linear cellular automata over zm. *Journal of Computer and System Sciences*, 27(1):125–140, 1983. doi: https://doi.org/10.1016/0022-0000(83)90033-8.

[38] Joonatan Jalonen and Jarkko Kari. On the conjugacy problem of cellular automata. *Information and Computation*, 274:104531, 2020. doi: https://doi.org/10.1016/j.ic.2020.104531.

[39] Emmanuel Jeandel. Computability of the entropy of one-tape Turing machines. *Leibniz International Proceedings in Informatics, LIPIcs*, 25, 2013. doi: https://doi.org/10.4230/LIPIcs.STACS.2014.421.

[40] Johan L. W. V. Jensen. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Mathematica*, 30:175–193, 1906.

[41] Kate Juschenko. *Amenability of Discrete Groups by Examples*. Mathematical Surveys and Monographs. American Mathematical Society, 2022. ISBN 9781470470326.

[42] Irving Kaplansky. *Set Theory and Metric Spaces*. AMS Chelsea Publishing. American Mathematical Society, 2020. ISBN 9781470463847.

[43] Jarkko Kari. Reversibility of 2d cellular automata is undecidable. *Physica D: Nonlinear Phenomena*, 45(1):379–385, 1990. doi: https://doi.org/10.1016/0167-2789(90)90195-U.

[44] Jarkko Kari. The nilpotency problem of one-dimensional cellular automata. *SIAM Journal on Computing*, 21(3):571–586, 1992. doi: https://doi.org/10.1137/0221036.

[45] Jarkko Kari. Reversibility and surjectivity problems of cellular automata. *Journal of Computer and System Sciences*, 48(1):149–182, 1994. doi: https://doi.org/10.1016/S0022-0000(05)80025-X.

[46] Jarkko Kari and Nicolas Ollinger. Periodicity and immortality in reversible computing. In Edward Ochmański and Jerzy Tyszkiewicz, editors, *Mathematical Foundations of Computer Science 2008*, pages 419–430. Springer Berlin Heidelberg, 2008. ISBN 978-3-540-85238-4.

[47] Anatole Katok. Fifty years of entropy in dynamics: 1958 – 2007. *Journal of Modern Dynamics*, 1, 2007. doi: https://doi.org/10.3934/jmd.2007.1.545.

[48] John L. Kelley. *General Topology*. Graduate Texts in Mathematics. Springer New York, 1975. ISBN 9780387901251.

[49] John F. C. Kingman. The ergodic theory of subadditive stochastic processes. *Journal of the Royal Statistical Society: Series B (Methodological)*, 30(3):499–510, 1968. doi: https://doi.org/10.1111/j.2517-6161.1968.tb00749.x.

[50] Bruce Kitchens and Klaus Schmidt. Automorphisms of compact groups. *Ergodic Theory and Dynamical Systems*, 9(4):691–735, 1989. doi: https://doi.org/10.1017/S0143385700005290.

[51] Achim Klenke. *Probability Theory: A Comprehensive Course*. Universitext. Springer London, 2007. ISBN 9781848000483.

[52] Johan Kopra. On computing the Lyapunov exponents of reversible cellular automata. *Natural Computing*, 20(2):273–286, 2021. doi: https://doi.org/10.1007/s11047-020-09821-3.

[53] Petr Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 17(2):417–433, 1997. doi: https://doi.org/10.1017/S014338579706985X.

[54] Petr Kůrka. *Topological and Symbolic Dynamics*. Collection SMF. Société mathématique de France, 2003. ISBN 9782856291436.

[55] Petr Kůrka. Topological dynamics of cellular automata. In Robert A. Meyers, editor, *Encyclopedia of Complexity and Systems Science*, pages 9246–9268. Springer New York, 2009. ISBN 978-0-387-30440-3.

[56] Petr Kůrka. On topological dynamics of Turing machines. *Theoretical Computer Science*, 174(1):203 – 216, 1997. doi: https://doi.org/10.1016/S0304-3975(96)00025-4.

[57] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Inventiones mathematicae*, 146(2): 259–295, 2001. doi: https://doi.org/10.1007/s002220100162.

[58] Elon Lindenstrauss and Benjamin Weiss. Mean topological dimension. *Israel Journal of Mathematics*, 115(1):1–24, 2000. doi: https://doi.org/10.1007/BF02810577.

[59] Ville Lukkarila. Sensitivity and topological mixing are undecidable for reversible one-dimensional cellular automata. *Journal of Cellular Automata*, 5(3), 2010.

[60] Alexandr M. Lyapunov. *General Problem of the Stability Of Motion*. Control Theory and Applications Series. Taylor & Francis, 1992. ISBN 9780748400621.

[61] Giovanni Manzini and Luciano Margara. Invertible linear cellular automata over zm: Algorithmic and dynamical aspects. In Igor Prívara and Peter Ružička, editors, *Mathematical Foundations of Computer Science 1997*, pages 378–387. Springer Berlin Heidelberg, 1997. ISBN 978-3-540-69547-9.

[62] Giovanni Manzini and Luciano Margara. A complete and efficiently computable topological classification of d-dimensional linear cellular automata over zm. *Theoretical Computer Science*, 221 (1):157–177, 1999. doi: https://doi.org/10.1016/S0304-3975(99)00031-6.

[63] Tom Meyerovitch. Finite entropy for multidimensional cellular automata. *Ergodic Theory and Dynamical Systems*, 28(4):1243–1260, 2008. doi: https://doi.org/10.1017/S0143385707000855.

[64] Cristopher Moore. Generalized shifts: unpredictability and undecidability in dynamical systems. *Nonlinearity*, 4(2):199, 1991. doi: https://doi.org/10.1088/0951-7715/4/2/002.

[65] Edward F. Moore. Machine models of self-reproduction. In Richard E. Bellman, editor, *Proceedings of Symposia in Applied Mathematics: Mathematical Problems in the Biological Sciences*, volume 14, pages 17 – 34, 1962. doi: https://doi.org/10.1090/psapm/014.

[66] T.K. Subrahmonian Moothathu. Homogeneity of surjective cellular automata. *Discrete and Continuous Dynamical Systems*, 13:195–202, 2005. doi: https://doi.org/10.3934/dcds.2005.13.195.

[67] Gary Morris and Thomas Ward. Entropy bounds for endomorphisms commuting with k actions. *Israel Journal of Mathematics*, 106:1–12, 1997.

[68] James R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000. ISBN 9780131816299.

[69] John R. Myhill. The converse of Moore's Garden-of-Eden theorem. In *Proceedings of the American Mathematical Society*, volume 14, pages 685 – 686, 1963. doi: https://doi.org/10.1090/S0002-9939-1963-0155764-9.

[70] John von Neumann. Zur allgemeinen theorie des masses. *Fundamenta Mathematicae*, 13(1): 73–116, 1929.

[71] Donald Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Advances in Mathematics*, 4(3):337–352, 1970. doi: https://doi.org/10.1016/0001-8708(70)90029-0.

[72] Donald Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *Journal d'Analyse Mathématique*, 48:1–141, 1987.

[73] William Parry. Intrinsic Markov chains. *Transactions of the American Mathematical Society*, 112 (1):55–66, 1964.

[74] Ville Salo. Conjugacy of reversible cellular automata and one-head machines. *Preprint arXiv 2011.07827*, 2022.

[75] Mark A. Shereshevsky. Lyapunov exponents for one-dimensional cellular automata. *Journal of Nonlinear Science*, 2(1):1–8, 1992. doi: https://doi.org/10.1007/BF02429850.

[76] Mark A. Shereshevsky. Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms. *Indagationes Mathematicae*, 4(2):203–210, 1993. doi: https://doi.org/10.1016/0019-3577(93)90040-6.

[77] Terence Tao. *Analysis II*. Texts and readings in mathematics. Hindustan Book Agency, 2022. ISBN 9788195196128.

[78] Terence Tao. *Analysis I: Fourth Edition*. Texts and Readings in Mathematics. Springer Nature Singapore, 2023. ISBN 9789811972614.

[79] Pierre Tisseur. Cellular automata and Lyapunov exponents. *Nonlinearity*, 13(5):1547–1560, 2000. doi: https://doi.org/10.1088/0951-7715/13/5/308.

[80] Rodrigo Torres-Avilés. Undecidability of the speed positiveness problem in reversible and complete Turing machines. *Journal of Computer and System Sciences*, 121, 2021. doi: https://doi.org/10.1016/j.jcss.2021.04.007.

[81] Alan M. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, s2-42(1):230–265, 1937. doi: https://doi.org/10.1112/plms/s2-42.1.230.

[82] Peter Walters. *An Introduction to Ergodic Theory*. Graduate texts in mathematics. Springer-Verlag, 1982. ISBN 9783540905998.

TURUN
YLIOPISTO
UNIVERSITY
OF TURKU

Painosalama, Turku, Finland 2024

ISBN 978-951-29-9806-7 (PRINT)
ISBN 978-951-29-9807-4 (PDF)
ISSN 0082-7002 (PRINT)
ISSN 2343-3175 (ONLINE)