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UNIVERSITY OF TURKU

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Wiljami Sillanpää

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ABSTRACT

The focus of this thesis is in analyzing optimal stopping (OSP) and impulse control problems (ICP) of linear diffusions under regime switching and Poisson constraints. Regime switching means that the problem's parameters undergo a change at a random instant. Regime switching models are useful for modelling problems where time horizon is long compared to the expected duration of relevant exogenous conditions such as business cycles or macroeconomic policies. We study OSPs and ICPs in the presence of a regime switch in a single paper. The other three papers are devoted to various timing problems under a Poisson constraint.

The Poisson constraint can be explained as follows. In the standard formulations of various stochastic control problems, controls can be exercised at any time. This is not always a realistic assumption as in practice there may exist liquidity and information constraints that pose significant limitations to the available controlling opportunities. We model these limitations by constraining the admissible control times to be the set of jump times of an independent time-homogeneous Poisson process.

The underlying processes of the problems are assumed to be linear diffusions. The reason for this is that linear diffusions admit a particularly rich analytical theory which allows us to obtain explicit information about the structure of solutions in addition to the usual abstract existence and uniqueness results. We illustrate the general theory with examples ranging from economics and mathematical finance to operations research and optimal harvesting of renewable resources.

KEYWORDS: Optimal stopping, Impulse control, Linear diffusions, Poisson process, Regime switching

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TIIVISTELMÄ

Tämän väitöskirjan aiheena ovat lineaaristen diffuusioiden regiimiä vaihtavien ja Poisson-rajoitteellisten optimaalisen pysäytyksen (OSP) ja impulssikontrollion-gelmien (ICP) analysointi. Regiiminvaihdolla tarkoitetaan tilannetta, jossa ongelman parametrin muuttuvat satunnaisella hetkellä. Regiiminvaihtomallit ovat käteviä mallinnettaessa ongelmia, joiden aikahorisontti on pitkä eksogeenisten olosuhteiden, kuten taloussuhdanteiden tai makroekonomisten poliittisten linjauksien odotettuun kestoon verrattuna. Tutkimme yhden regiiminvaihdon OSP:tä ja ICP:tä yhdessä artikkelissa. Loput kolme artikkelia on omistettu erilaisille Poisson-rajoitteellisille ajoitusongelmille.

Poisson-rajoite voidaan selittää seuraavalla tavalla. Stokastisten kontrolliongelmien tavanomaisissa muotoiluissa kontrollointi on mahdollista kaikkina ajanhetkinä. Tämä oletus ei aina ole realistinen, sillä käytännössä likviditeetti- ja informaatiorajoitteet voivat merkittävästi rajoittaa kontrollointimahdollisuuksia. Mallinnamme näitä rajoitteita rajaamalla sallitut kontrollointihetket riippumattoman aikahomogeenisen Poisson-prosessin hyppyhetkiksi.

Ongelmien perustana olevat prosessit oletetaan lineaarisiksi diffuusioiksi. Tämä siksi, että lineaarisilla diffuusioilla on poikkeuksellisen rikas analyyttinen teoria, jonka avulla ongelmien ratkaisuista on mahdollista saada eksplisiittistä informaatiota tavanomaisten abstraktien olemassaolo- ja yksikäsitteisyystulosten lisäksi. Havainnollistamme yleistä teoriaa esimerkkien avulla, jotka ulottuvat taloustieteestä ja matemaattisesta rahoituksesta operaatioanalyysiin ja uusiutuvien luonnonvarojen optimaaliseen käyttöön asti.

ASIASANAT: Optimaalinen pysäytys, Impulssikontrolli, Lineaariset diffuusiot, Poisson-prosessi, Regiiminvaihto

Acknowledgements

I began working on this thesis in January 2021, which was one of the many peaks of the covid-19 epidemic in Finland. Having already completed more than seven years' worth of degrees and courses in four and half years, mainly working in solitude at home, in the Quantum lobby or in the former Quantum library and spending almost a year at home due to corona regulations, barely even going to the grocery store, getting a personal office room was definitely a welcome change. I thought I would keep up the pace I had been accustomed to, write the required number of papers in no time and graduate after about two years. Now, four years later I have to admit that research isn't a linear process. Sometimes the results fall right into your hands and sometimes things take more time than expected.

I am most grateful to my supervisors, Professors Jukka Lempa and Luis H.R. Álvarez Esteban. I also want to thank PostDoctoral Researcher Harto Saarinen for his input in our joint papers and for always emphasizing the 'applied' in applied mathematics. Support from the Foundation of Economic Education (Liikesivistysrahasto) is gratefully acknowledged. I furthermore wish to express my gratitude towards all the people who had a say in granting me the maximum number of months in a salaried doctoral candidate position as well as Professor Henri Nyberg for including me in the Econometrics project funded by Liikesivistysrahasto. I am also grateful for being given a personal office room, which I take has become the exception rather than the norm. As I understand, the somewhat privileged circumstances I've enjoyed for the past four years are becoming increasingly rare in the university's deteriorating financial climate. Last but not least, I want to thank my family for forcing me to have a work-life balance instead of a work-sleep balance.

6.9.2024

Wiljami Sillanpää

Table of Contents

List of Original Publications

This dissertation is based on the following original publications, which are referred to in the text by their Roman numerals:

I Alvarez E, L. H., Sillanpää , W. (2023). Optimal stopping and impulse control in the presence of an anticipated regime switch. Mathematical Methods of Operations Research, 98(2), 205-230.

II Lempa, J., Saarinen, H., Sillanpää, W. (2024). Optimal Stopping with Variable Attention. Manuscript.

III Alvarez E, L. H., Lempa, J., Saarinen, H., Sillanpää, W. (2024). Solutions for Poissonian Stopping Problems of Linear Diffusions via Extremal Processes. Stochastic Processes and their Applications, 172, 104351

IV Alvarez E, L. H., Lempa, J., Saarinen, H., Sillanpää, W. (2024). On the Impact of Poisson Timing Constraints on Impulse Control Policies. Manuscript.

1 Introduction

Problems of optimal timing are present in probably everyone's lives in one way or the other. When do I put my shoes on if I want to catch the bus but minimize the waiting time? Can I risk missing it and take the next one? Do I have to clear the snow from my yard today or can I wait until tomorrow? What if it snows more during the night? What if the temperature goes above zero during the day and drops again, so the yard will be coated with a thick layer of ice instead? What if the weather forecast if wrong? Or if I have a spot-priced electricity contract, what is the optimal time for heating the oven or doing laundry if want to minimize my bills? A common theme in these kind of questions is that they often include some element that evolves randomly in time. These time-driven random elements are more technically known as stochastic processes, stochastic being derived from the Ancient Greek word $\sigma\tau\acute{o}\chi o\varsigma$ (stokhos) and meaning ”aim/guess”.

While we can never know the future values of a stochastic process for certain, we can make predictions, or guesses as the name suggests, if we know the dynamics according to which the process evolves. Thus even though the optimality of a timing strategy is uncertain for a single instance of a given timing problem, such as the electricity consumption example for the next weekend, we can usually find a policy that is optimal *on average*. What this means is that we can usually find a policy that yields the best result when the problem is repeated over and over again. For the electricity example, an optimal policy may be one for which the costs for each individual day remain uncertain but which minimizes the yearly average of electricity bills.

Despite the aforementioned examples, the pursuit of using specialised and technical analysis for solving stochastic optimization problems usually becomes relevant when the problems are being addressed on an industrial or macroeconomic scale. Some canonical examples of these kind of problems include the valuation of options (real or financial), various other questions related to finance and operations research, and the management and harvesting of renewable resources, whether by a resource we mean trees or rabbits or fish or a combination of them.

Conceptually the simplest of form a timing problem is arguably the optimal stopping problem (OSP). Optimal stopping problems usually aim at modelling scenarios where a decision maker, or an agent as they are often called in the literature, is trying to find an optimal time to make a single irreversible decision such as investing capital in a new project or abandoning an ongoing one. The theory of optimal stopping problems has its roots in statistics, particularly in sequential analysis (see e.g. the classic papers by Wald & Wolfowitz [1] and Arrow, Blackwell & Girschick [2]). The diffusion theoretic framework was pioneered by Dynkin (see e.g. [3], [4] and [5]) in the 1960s.

Another important class of timing problems is formed by the impulse control problems (ICPs). In an impulse control problem, the agent is allowed to exert a controlling decision multiple times. The set of these control times can be and in many cases is infinite but the times can't be "infinitely close" to each other. A classic example of an impulse control problem is the so called Faustmann problem in forestry. In the problem, an agent wants to maximize the net present value of a forestland by constructing an optimal rotation plan. Trees are a renewable resource but a cut forest takes a long time to regrow so the problem is clearly of the impulse control type. The Faustmann problem is also perhaps the oldest known example of an impulse control problem, being originally formulated and solved by the German forester Martin Faustmann in 1849 [6], roughly 80 years before the advent of axiomatic probability theory and around 120 years before stochastic calculus became a mainstream tool in solving these types of optimization problems. We refer the interested reader to [7] and the references therein for a more thorough discussion on the Faustmann problem and related topics.

So far we have seen examples of timing problems that have been solved a long time ago and since then have been generalized and extended into myriad new directions. Solving and otherwise analysing specific extensions of these well-known problems is also the subject of this thesis. In [I] we study OSPs and ICPs in the presence of a regime switch. This means that the problems' parameters (payoff and dynamics) may change at a random instant. In general regime switching is useful for modelling randomly occurring exogenous changes that have an effect on the controllable system, such as business cycles.

However the main focus has been on a feature known as the Poisson constraint. The name comes from the fact that even though the problem may evolve in continuous time, the agent is allowed to make the control decision only at arrival times of randomly occurring signals and these signals are modelled as the jump times of an independent Poisson process. This implies that the waiting times between consecutive signals form a sequence of independent, identically and exponentially distributed random variables. The use of exponentially distributed random variables gives rise to convenient technical properties that usually allow the problems to be solved in a manner that is similar to the unrestricted case. The significance of the Poisson constraint is that it can be used to model and solve timing problems in which the agent is facing liquidity or information constraints. These constraints and their applications are explored in more detail in Chapter 4.

The rest of this dissertation is organized as follows. Chapters 2 and 3 contain the standard mathematical tools required to understand the technical framework of the papers [I - IV]. Chapter 2 collects basic facts on martingales and linear diffusions and Chapter 3 introduces the reader to the standard definitions of stochastic optimal stopping and impulse control problems as well as the methods we employ in solving them. Regime switching problems are also discussed in Chapter 3. In Chapter 4 we introduce the Poisson constraint and discuss its various interpretations as well as related literature on Poisson constrained timing problems. Chapter 5 contains short technical summaries of the papers [I - IV] which are included as appendices.

One final remark is in place before closing this chapter. As a compilation dissertation, the text is practically divided in two parts. Chapters 1-5 comprise the first half of this dissertation while the original publications [I - IV] form the second half and contain the actual research that we have done during this project. The topics connected to this thesis contain the mathematical theory of diffusions, optimal stopping and impulse control problems, regime-switching problems and Poisson constraints. So much has been written about each of these topics and their applications over the years that entire books could be (and indeed have been) written about them. I have therefore deliberately refrained from making these first chapters a definitive textbook introduction to the world of stochastic control theory. Instead, Chapters 1-5 will be a thorough but short introduction to the particular questions analysed in the papers [I - IV] and the methods that are utilized therein. My aim is that this first half should make the papers accessible on a technical level to a grad student with a background in analysis or probability. I have cut some corners in terms of rigorous details in places where it doesn't compromise the understanding our work. For example, stochastic differential equations and the various types of stochastic integration (Ito/Stratonovich/Skorokhod) have been completely left out because they are not used in the papers [I - IV] aside from a single assumption which is only needed in order to guarantee that our processes are always uniquely specified in terms of probability laws.

2 Martingales and Linear Diffusions

In this chapter we present the mathematical tools from stochastic analysis that are necessary in order to understand the technical side of this thesis. The reader is assumed to have a basic level of knowledge on topology, measure theory and real analysis. Some familiarity with Markov processes and Brownian motion is assumed as well. We begin by recalling certain elementary definitions and results from martingale theory and then briefly overview key concepts from the classical theory of diffusions, which forms the basis for most of the analysis carried out in the papers [I - IV]. The definitions and theorems of this chapter are based on Chapters 1 and 2 in Borodin and Salminen [8], unless explicitly stated otherwise. [8] also contains a much more thorough exposition of martingale theory and linear diffusions as well as references for proofs. We also point out the classic textbooks on the subject by Blumenthal and Getoor [9], Ito and McKean [10] and Stroock and Varadhan [11] for the interested reader. Throughout this chapter we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

***Definition* 1.** *A filtration on $(\Omega, \mathcal{F})$ is a collection of σ-algebras $\mathbb{F} = \{\mathcal{F}_t\}_{i\in J}$ such that $\mathcal{F}_i \subseteq \mathcal{F}_j \subseteq \mathcal{F}$ for all $i \leq j$. The filtration $\mathbb{F}$ is said to be right-continuous if $\mathcal{F}_i = \mathcal{F}_{i+} = \bigcap_{j>i} \mathcal{F}_j$ for all $i \in J$.*

Note that in the above definition, the index set J may be discrete or continuous. The most common choices for J are the set of non-negative integers $\mathbb{N}_0$ and the set of non-negative real numbers $\mathbb{R}_{\geq 0}$. We will assume that $J = \mathbb{N}_0$ or $J = \mathbb{R}_{\geq 0}$ for the rest of this chapter. The collection $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a filtered probability space.

***Definition* 2.** *The filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is said to satisfy the usual conditions if $\mathcal{F}$ is $\mathbb{P}$-complete, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of Ω and $\mathbb{F}$ is right-continuous.*

***Definition* 3.** *A stopping time with respect to the filtration $\mathbb{F}$ is a mapping $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq i\} \in \mathcal{F}_i$ for all $i \in J$. The σ-algebra $\mathcal{F}_\tau$ corresponding to the stopping time τ is defined as*

$$A \in \mathcal{F}_\tau \Leftrightarrow A \in \mathcal{F} \text{ and } A \cap \{\tau \leq i\} \in \mathcal{F}_i \text{ for all } i \in J$$

***Definition* 4.** *Let $(E, \mathcal{E})$ be a Polish space. A stochastic process X on E is a family $X = \{X_i,\ i \in J\}$ of random variables $x_i : \Omega \to E$. X is said to be adapted to the* *filtration* $\mathbb{F} = \{\mathcal{F}_t\}_{i \in J}$ *if* X_i *is* $\mathcal{F}_i$*-measurable for all* $i \in J$. *If the filtration is known,* X *is simply said to be adapted.*

***Definition* 5.** *A stochastic process* X *is uniformly integrable if for all* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that* $\mathbb{E}[|X_i| \mid |X_i| > c(\varepsilon)] < \varepsilon$ *for all* $i \in J$.

We end this subsection by recalling the definition of a Markov process.

***Definition* 6.** *A time-homogeneous Markov process on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_{\geq 0}}, \mathbb{P})$ *taking values in the state space* $(E, \mathcal{E})$ *is an adapted stochastic process such that for all* $t \in \mathbb{R}_{\geq 0}$ *there exist a transition function* $P_t : E \times \mathcal{E} \to [0,1]$ *such that*

(i) $\mathbb{P}(X_{s+t} \in A | \mathcal{F}_t) = P_t(X_t, A)$ almost surely for all $A \in \mathcal{E}$ and $s, t \in \mathbb{R}_{\geq 0}$

(ii) $P_t(\cdot, A)$ and $P_t(x, \cdot)$ are measurable for all $A \in \mathcal{E}$ and $x \in E$

(iii) $P_0(x, A) = 1_{\{x\}}(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ for all $x \in E, A \in \mathcal{E}$

(iv) The Chapman $-$ Kolmogorov relation

$$P_{s+t}(x, A) = \int_E P_t(x, dy) P_s(y, A)$$

holds for all $x \in E, A \in \mathcal{E}$ and $s, t \in \mathbb{R}_{\geq 0}$

2.1 Martingales

Martingales form an important class of stochastic processes. They are very general objects but still possess convenient analytical properties that make them suitable for proving general results in stochastic analysis. We only need their general definition and the optional stopping theorem but due to their significance they have been included as a separate section.

***Definition* 7.** *An adapted stochastic process* $M = \{M_t : t \geq 0\}$ *on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_{\geq 0}}, \mathbb{P})$ *is a submartingale if* $\mathbb{E}[M_t] < \infty$ *and* $\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s$ *for all* $s \leq t$. M *is a supermartingale if the conditions hold with the exception that* $\mathbb{E}[M_t | \mathcal{F}_s] \geq M_s$ *for all* $s \leq t$. M *is a martingale if it is both a sub- and a supermartingale.*

For a discrete time filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$ *the definitions are similar with the exception that the conditional expectation inequalities are* $\mathbb{E}[M_{n+1} | \mathcal{F}_n] \leq M_n$ *and* $\mathbb{E}[M_{n+1} | \mathcal{F}_n] \geq M_n$.

***Definition* 8.** *An adapted stochastic process* $M = \{M_i : i \in J\}$ *on* $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_{i \in J}, \mathbb{P})$ *is a local martingale if there exists a sequence of stopping times* $(\tau_n)_{n \geq 0}$ *such that* $\tau_n \uparrow \infty$ *and the stopped processes* $\{M_{\tau_n \wedge i} : i \in J\}$ *are martingales.*

It can be seen from the Definition 7 that if M is a supermartingale, then $-M$ is a submartingale and vice versa. The next theorem is a basic tool in martingale theory. It shows that under suitable conditions the martingale inequalities carry over to pairs of arbitrary stopping times.

***Theorem* 1** (Optional stopping)**.** *Let M be a uniformly integrable or a non-negative supermartingale with $M_\infty = \lim_{t\uparrow\infty}$ and let τ, σ be stopping times such that $\tau \leq \sigma$ almost surely. Then $\mathbb{E}[M_\sigma|\mathcal{F}_\tau] \leq M_\tau$ almost surely. If M is a martingale, the inequality is replaced by an equality.*

2.2 Linear Diffusions

Linear diffusions are arguably one of the most well understood and widely studied class of stochastic processes. They are a subclass of one-dimensional Markov processes that admit a particularly thorough real analytic interpretation. It is for this reason that many stochastic optimization problems that are formulated in terms of linear diffusions can be converted into calculations in real analysis. The theory of linear diffusions provides the basic framework for the analysis carried out the papers [I - IV] as well. As Definition 9 demonstrates, linear diffusions are loosely speaking one-dimensional strong Markov processes with continuous sample paths. Since we will be working solely with linear diffusions, we will simply refer to them as diffusions from now on.

***Definition* 9.** *Let $I \in \mathbb{R}$ be an interval with left endpoint $\mathfrak{l} \geq -\infty$ and a right endpoint $\mathfrak{r} \leq \infty$ and let $\mathcal{B}(I)$ be its Borel σ-algebra. A linear diffusion is a time-homogeneous Markov process X on $(I, \mathcal{B}(I))$ such that*

$$\begin{aligned}(i)t \mapsto X_t(\omega) \text{ is continuous on } [0,\zeta)\ \mathbb{P}_x - \text{a.s.}\\ (ii)\mathbb{E}_x[\eta \circ \theta_\tau|\mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[\eta]\ \mathbb{P}_x - \text{a.s.}\end{aligned}$$

where $\zeta = \inf\{t \geq 0 : \ X_t \notin I\}$ is the lifetime X, η is a $\mathcal{F}_\infty$-measurable random variable and θ is the shift operator, defined on Ω by $\theta_t(\omega) = \omega'$ with $\omega'_s = \omega_{s+t}$.

Diffusions have three fundamental characteristics. They are the speed measure m, scale function S and the killing measure k. We shall be working exclusively in the case where all three are absolutely continuous w.r.t. the Lebesgue measure of $\mathbb{R}$, so in order to make this introduction less top heavy we omit general definitions and instead express the characteristics only in terms of the infinitesimal parameters which are defined below.

***Definition* 10.** *Let X be a linear diffusion. The infinitesimal parameters of X, which are the infinitesimal mean μ, infinitesimal variance σ^2 and the infinitesimal killing* *rate c are defined as*

$$
\begin{aligned}
\mu(x) &= \lim_{t\downarrow 0} t^{-1}\mathbb{E}_x\left[X_t - x\right] \\
\sigma^2(x) &= \lim_{t\downarrow 0} t^{-1}\mathbb{E}_x\left[X_t - x\right]^2 \\
c(x) &= \lim_{t\downarrow 0} t^{-1}\left(1 - \mathbb{P}_x\left[\zeta > t\right]\right)
\end{aligned}
$$

***Theorem* 2.** *If the characteristics (m, S, k) of a linear diffusion are absolutely continuous w.r.t. the Lebesgue measure of the state space, they can be expressed as*

$$
m'(x) = \frac{2e^{B(x)}}{\sigma^2(x)},\ S'(x) = e^{-B(x)},\ k'(x) = \frac{2c(x)e^{B(x)}}{\sigma^2(x)},\ B(x) = \int_{\mathfrak{l}}^{x} \frac{2\mu(z)}{\sigma^2(z)}dz
$$

The killing measure is related to the distribution of a diffusion at its lifetime ([8], Section 2.1.4) by

$$
\mathbb{P}_x(X_{\zeta-} \in A|\zeta < t) = \int_0^t \int_I k'(y)p(s;x,y)dyds
$$

where p is a certain positive function, continuous in all variables. In this dissertation we only consider diffusions that do not die in the interior I° of the state space, so for these processes the above relation directly implies $k' \equiv 0$ and hence $c \equiv 0$.

2.2.1 Resolvents and Generators

It was stated above that linear diffusions admit a particularly thorough real analytic interpretation. In this subsection and the next we elaborate on this notion. The transition function of a linear diffusion gives rise to an operator semigroup, the generator (Definition 11) of which contains important information about the diffusion. Another fundamental linear operator associated to diffusions - the resolvent, is defined in Definition 12.

***Definition* 11.** *The generator of a linear diffusion X is a linear operator $\mathcal{A}$ defined as*

$$
(\mathcal{A}f)(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}_x\left[f(X_t)\right] - f(x)}{t}
$$

The domain $\mathcal{D}(\mathcal{A})$ consist of functions $f \in \mathcal{C}_b(I)$ for which the above limit exists for all $x \in I$, is in $\mathcal{C}_b(I)$ and for which

$$
\sup_{t>0} t^{-1} \parallel \mathbb{E}_x\left[f(X_t)\right] - f(x) \parallel < \infty
$$

***Definition* 12.** *Let $\alpha > 0$ and $B(I)$ be the set of bounded (measurable) functions on I. The α-resolvent of X is a linear operator $R_\alpha : B(I) \to B(I)$ defined by*

$$(R_\alpha f)(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha} f(X_t) dt \right]$$

Definition 12 presents the resolvent as a linear operator acting on bounded functions on I. In [I - IV] we will usually relax this definition a bit and consider domains in which the defining property is closer to absolute integrability. That is, we take the domain to be

$$L_r^1(I) = \{ f : I \to \mathbb{R},\ (R_\alpha |f|)(x) < \infty \; for\ all\ x \in I \}$$

***Theorem* 3** (The resolvent equation)**.** *Let $\alpha, \beta > 0$. Then the resolvents R_α, R_β satisfy the resolvent equation $R_\alpha - R_\beta = (\beta - \alpha) R_\beta R_\alpha$.*

The next theorem collects together various useful ways of expressing the generator of a diffusion as well as the fundamental relation between resolvents and generators.

***Theorem* 4.** *Assuming $k' \equiv 0$, the infinitesimal generator $\mathcal{A}$ can be expressed in terms of the fundamental characteristics or infinitesimal parameters as*

$$\mathcal{A} = \frac{d}{dm} \frac{d}{dS} = \frac{\sigma^2(x)}{2} \frac{d^2}{dx^2} + \mu(x) \frac{d}{dx}$$

Moreover, the α-resolvent and the generator satisfy the relation $R_\alpha = (\alpha - \mathcal{A})^{-1}$ whenever both sides of the equation are well-defined.

2.2.2 Harmonic and Excessive Functions

We close this chapter by presenting some important facts about excessive functions, which are roughly speaking a probabilistic analogue of superharmonic functions. Most of the actual analysis done in the papers [I - IV] is about doing calculations with excessive functions.

***Definition* 13.** *A non-negative measurable function f is called α-excessive if*

$$\mathbb{E}_x \left[e^{-\alpha t} f(X_t) \right] \leq f(x)$$
$$\lim_{t \downarrow 0} \mathbb{E}_x \left[e^{-\alpha t} f(X_t) \right] = f(x)$$

for all $t \geq 0$ and $x \in I$. f is called α-invariant if equality holds in the first condition.

α-invariant functions are often called α-harmonic in the literature (see e.g. [9] and [10]). This is a minor abuse of language since α-harmonic functions are not required to be non-negative in general. However, those harmonic functions that one usually encounters in the diffusion theoretic framework of stochastic control theory (especially the minimal harmonic functions of Theorem 6) are positive, so this will not lead to confusion.

***Theorem* 5.** *Let f be α-excessive. Then $\mathcal{A}f \leq 0$. If f is α-harmonic, then $(\mathcal{A} - \alpha)f = 0$.*

***Theorem* 6.** *The equation $(\mathcal{A} - \alpha)f = 0$ admits two fundamental solutions. One of them is strictly increasing while the other one is strictly decreasing.*

The increasing fundamental solution in the above theorem is often denoted as ψ_α and the decreasing as φ_α. These are commonly referred to as the minimal α-harmonic functions. In fact, they are often also called the minimal α-excessive functions, even though excessivity is a more general notion than harmonicity. Usually this would not lead to confusion, but in this subsection we make the distinction because we will go over the Riesz representation theorem for excessive functions by Salminen ([12], pp. 91-93), which explicitly uses the full set of minimal α-excessive functions.

***Theorem* 7.** *Let $x, y, z \in I$ with $x < y < z$. Laplace transforms of the first hitting times $\tau_x = \inf\{t \geq 0 : X_t \leq x\}$ and $\tau^z = \inf\{t \geq 0 : X_t \geq z\}$ can be given as*

$$\mathbb{E}_y[e^{-\alpha\tau_x}] = \frac{\varphi_\alpha(y)}{\varphi_\alpha(x)},\ \mathbb{E}_y[e^{-\alpha\tau^z}] = \frac{\psi_\alpha(y)}{\psi_\alpha(z)}$$

***Theorem* 8.** *The resolvent R_α can be expressed in terms of the minimal α-harmonic functions as*

$$(R_\alpha f)(x) = B_\alpha^{-1}\left(\varphi_\alpha(x)\int_{\mathfrak{l}}^{x}\psi_\alpha(z)f(z)m'(z)dz + \psi_\alpha(x)\int_{x}^{\mathfrak{r}}\varphi_\alpha(z)f(z)m'(z)dz\right)$$

Here $B_\alpha = S'^{-1}(\varphi_\alpha\psi'_\alpha - \varphi'_\alpha\psi_\alpha)$ is the Wronskian of ψ_α and φ_α which is also a constant.

***Theorem* 9.** *Let $x_0 \in I$ and suppose that the function f is continuous w.r.t. the scale function S. Then f is α-excessive s.t. $f(x_0) = 1$ if and only if there exists a probability measure $\sigma_{x_0}^f$ on I s.t. $f(x) = \int_I K_y(x, x_0)\sigma_{x_0}^f(dy)$, where*

$$K_y(x, x_0) = \frac{k_y(x)}{k_y(x_0)},\ k_y(x) = \begin{cases} \frac{\psi_\alpha(x)}{\psi_\alpha(y)} & x \leq y \\ \frac{\varphi_\alpha(x)}{\varphi_\alpha(y)} & x \geq y \end{cases}$$

and the measure $\sigma_{x_0}^f$ is given by

$$\sigma_{x_0}^f\left((x,\mathfrak{r}]\right) = \frac{\psi_\alpha(x_0)}{B_\alpha S'(x)}\left(\varphi_\alpha(x)f'(x) - \varphi_\alpha'(x)f(x)\right), \text{ for } x \geq x_0$$
$$\sigma_{x_0}^f\left([\mathfrak{l},x)\right) = \frac{\varphi_\alpha(x_0)}{B_\alpha S'(x)}\left(\psi_\alpha'(x)f(x) - \psi_\alpha(x)f'(x)\right), \text{ for } x \leq x_0$$

The behaviour of the minimal harmonic functions near the boundaries of I naturally depends on the diffusion's behaviour near the said boundaries and hence on the fundamental characteristics. However, we do not need to use the classification in its full generality in the papers [I - IV] so we will omit it here as well. Instead, in Definition 14 we present the effects natural and killing boundaries have on the minimal harmonic functions. We will not discuss other boundary types since these two are the only ones used in the examples in [I - IV].

***Definition* 14.** *If the lower boundary $\mathfrak{l}$ of the state space I is*

$$(i) \text{ natural, then } \lim_{x\downarrow\mathfrak{l}}\psi_\alpha(x) = \lim_{x\downarrow\mathfrak{l}}\frac{\psi_\alpha'(x)}{S'(x)} = 0,\ \lim_{x\downarrow\mathfrak{l}}\varphi_\alpha(x) = -\lim_{x\downarrow\mathfrak{l}}\frac{\varphi_\alpha'(x)}{S'(x)} = \infty$$
$$(ii) \text{ killing, then } \lim_{x\downarrow\mathfrak{l}}\psi_\alpha(x) = 0$$

Similarly, if the upper boundary $\mathfrak{r}$ of the state space I is

$$(i) \text{ natural, then } \lim_{x\uparrow\mathfrak{r}}\psi_\alpha(x) = \lim_{x\uparrow\mathfrak{r}}\frac{\psi_\alpha'(x)}{S'(x)} = \infty,\ \lim_{x\uparrow\mathfrak{r}}\varphi_\alpha(x) = \lim_{x\uparrow\mathfrak{r}}\frac{\varphi_\alpha'(x)}{S'(x)} = 0$$
$$(ii) \text{ killing, then } \lim_{x\uparrow\mathfrak{r}}\varphi_\alpha(x) = 0$$

***Theorem* 10.** *Let $f \in C^2(I)$, $F = (\mathcal{A}-\alpha)f \in L_r^1(I)$ and suppose that $\mathfrak{r}$ is natural. If $\lim_{x\downarrow\mathfrak{l}}|f(x)| < \infty$, then*

$$\frac{f'(x)}{S'(x)}\psi_\alpha(x) - f(x)\frac{\psi_\alpha'(x)}{S'(x)} = \int_{\mathfrak{l}}^x \psi_\alpha(z)F(z)m'(z)dz - \delta$$
$$\frac{f''(x)}{S'(x)}\psi_\alpha(x) - f(x)\frac{\psi_\alpha''(x)}{S'(x)} = \frac{2}{\sigma^2(x)}\left(F(x)\frac{\psi_\alpha'(x)}{S'(x)} - \alpha\int_{\mathfrak{l}}^x \psi_\alpha(z)F(z)m'(z)dz + \alpha\delta\right)$$

where $\delta = 0$ if $\mathfrak{l}$ is natural and $\delta = B_\alpha f(0)/\varphi_\alpha(0)$ if $\mathfrak{l}$ is killing. If $\lim_{x\uparrow\infty} f(x)/\psi_\alpha(x) = 0$, then

$$\frac{f'(x)}{S'(x)}\varphi_\alpha(x) - f(x)\frac{\varphi_\alpha'(x)}{S'(x)} = -\int_x^{\mathfrak{r}} \psi_\alpha(z)F(z)m'(z)dz$$
$$\frac{f''(x)}{S'(x)}\varphi_\alpha(x) - f(x)\frac{\varphi_\alpha''(x)}{S'(x)} = \frac{2\alpha}{\sigma^2(x)}\int_x^{\mathfrak{r}} \varphi_\alpha(z)(F(z) - F(x))m'(z)dz$$

3 Some Problems of Optimal Timing

In this chapter we present rudimentary facts about the classes of stochastic control problems that are studied in the papers [I - IV], namely optimal stopping and impulse control problems. In our framework, optimal stopping problems are optimization problems in which the decision maker tries to maximize a given reward (or minimize a cost) that depends on the value of a linear diffusion. The only control decision available is to stop the underlying process and receive the reward (or pay the cost) indicated by the payoff at the stopping time. Impulse control problems on the other hand allow for a possibly infinite sequence of controls exerted instantaneously at a discrete set of control times. These control times are also referred to as intervention dates. Due to the strong Markov property of linear diffusions, impulse control problems ”restart” at each control time and hence they can be seen as a sequence of optimal stopping problems. They are in this sense generalizations of optimal stopping problems.

Direct analysis of OSPs and ICPs as defined in this chapter is in general feasible on an abstract level only. The canonical method for proving the existence and uniqueness of solutions to these problems is based on variational (VI) and quasi-variational inequalities (QVI) as well as studying the viscosity solutions to the associated Hamilton-Jacobi-Bellman (HJB) equations. However we do not employ this approach so we will not review it further here. The interested reader is referred to [13] and [14] for a comprehensive overview on the topic. OSPs are also closely related to certain boundary value problems (BVPs), in particular Stefan problems which are also known as free boundary problems [15]. These are basically boundary value problems for which the location and even the shape of the boundary may be unknown a priori. It is no surprise then that general explicit information regarding the structure of solutions to OSPs and ICPs is considered to be intractable when the underlying process has more than one dimension.

For one dimensional processes and linear diffusions in particular the situation becomes manageable. We may first use control theoretic arguments to make a 'sophisticated guess' on how the optimal strategy and value function might look like. We then use the excessive characterisation for the OSP value function (Theorem 11) or some other verification theorem in order to show that the constructed solution candidate is indeed optimal. This approach is particularly convenient for linear diffusions because the associated Martin kernels admit a simple description in terms of the minimal harmonic functions as noted in Theorem 9.

While the main purpose of this chapter is to give a general overview of OSPs and ICPs and to present important technical arguments we use in the papers [I - IV], we have included a discussion on regime switching timing problems as Chapter 3.3. This is because we analyze regime switching problems only the paper [I] so writing an entire chapter on the topic would have diverted attention away from the main focus this work, which is the Poisson constraint (see Chapter 4). Lastly, we use the same notation and terminology as in Chapter 2 throughout this chapter.

3.1 Optimal Stopping

The study of optimal stopping problems has its roots in statistics, as noted in the introduction. The concept of optimal stopping is quite simple so it has natural applications in many other areas of research. In this thesis we will be mainly focusing on applications that are found in economics, mathematical finance and operations research.

A classical application of OSPs in economics and operations research is the optimal timing of an irreversible investment. It is a problem in which an agent may at any time choose either to invest or to continue waiting for a better investment opportunity. Investing is allowed at any decision time but the investment decision cannot be undone after it has been made. This fundamental asymmetry in the agent's available options implies that the optimal net present value of the investment must exceed costs by a positive amount, as noted in McDonald and Siegel [16]. McDonald and Siegel also discuss potential real-world applications of this investment problem, such as the investment decision of a firm in a competitive industry with stochastic entries.

Let us also mention the classic reference on stochastic investment problems by Dixit and Pindyck [17] for the interested reader. In [17] the authors present a comprehensive overview of stochastic investment timing problems and techniques that may be used to analyze them. The topics include the basic model of McDonald and Siegel [16] but also equilibria in competitive industries and sequential and incremental investment problems.

In mathematical finance, the theory of OSPs is usually applied on the valuation of various assets and options. Well-known references in this area include e.g. Bensoussan [18], Karatzas [19], Jacka [20] and Guo and Shepp [21]. [18] and [19] mainly focus on the valuation of American options i.e. options which can be exercised at a given maturity date or at any time prior to it. Jacka [20] focuses exclusively on American puts and [21] contains analyzes on more exotic options, such as the perpetual lookback American option which is basically a combination of an American and a Russian option (Russian options were introduced by Shepp and Shiryaev [22]).

The diffusion theoretic framework of optimal stopping was pioneered by Dynkin in the 1960s (see e.g. [3], [4] and [5]) and later expanded by Shiryaev [23] and Peskir and Shiryaev [15] among others. We now present some of the most important results concerning the existence and uniqueness of solutions as well as the characterization of the value functions. The value function of an optimal stopping problem is defined as follows.

***Definition* 15.** *Let $g \in C(I)$, $r > 0$ and denote by $\mathcal{S}$ the set of all stopping times for the linear diffusion X. The value function $\tilde{V} : I \to \mathbb{R}_{\geq 0}$ of an optimal stopping problem is defined as*

$$\tilde{V}(x) = \sup_{\tau \in \mathcal{S}} E_x \left[e^{-r\tau} g(X_\tau) \right] \tag{1}$$

We often use the terms optimal stopping problem and the value function of an optimal stopping problem interchangeably. A classic result in the theory of OSPs is the excessive characterization of the value function which is due to Dynkin [3].

***Theorem* 11** ([3], Thrm. 1)**.** *The value function of an optimal stopping problem (in the sense of Definition 15) is the smallest r-excessive majorant of the payoff function.*

Theorem 11 is our main tool in verifying the optimality of candidate strategies for OSPs. There does exist another closely related characterization, expressed in terms of minimal harmonic functions and concavity properties of the value function ([24], Prop. 4.2). We do not use the result in the papers [I - IV] but it should be nevertheless mentioned here because it fits into our technical framework and is equivalent to Dynkin's original characterization (Theorem 11). In fact, this equivalence result is also due to Dynkin ([5], Thrm. 12.4).

***Definition* 16** ([24], pp. 6)**.** *Let $F : I \to \mathbb{R}$ be a strictly increasing function. A function $u : I \to \mathbb{R}$ is said to be F-concave if*

$$u(x) \geq u(\mathfrak{l}) \frac{F(\mathfrak{r}) - F(x)}{F(\mathfrak{r}) - F(\mathfrak{l})} + u(\mathfrak{r}) \frac{F(x) - F(\mathfrak{l})}{F(\mathfrak{r}) - F(\mathfrak{l})}$$

***Theorem* 12** ([24], Prop. 4.2)**.** *Let $F = \psi/\varphi$ and $G = -\varphi/\psi$. The value function $\tilde{V}$ is the smallest non-negative majorant of g such that $\tilde{V}/\varphi$ is F-concave (or equivalently, $\tilde{V}/\psi$ is G-concave).*

Value function characterizations are useful in verifying optimality of candidate strategies. However they say nothing about how these candidates should look like. In their book on optimal stopping and associated boundary value problems [15], Peskir and Shiryaev present a useful general result that describes the general structure of the optimal stopping times whenever they exist.

***Theorem* 13.** *Let $\tilde{V}$ be as in Definition 15. Suppose that τ^* is an optimal stopping time for the problem* (1) *i.e. $\tau^* \in \mathcal{S}$ and $\tilde{V}(x) = E_x \left[e^{-r\tau^*} g(X_{\tau^*}) \right]$. Then τ^* is the first hitting time to the stopping region $E = \{x \in I : \tilde{V}(x) = g(x)\}$ i.e. $\tau^* = \inf\{t \geq 0 : X_t \in E\}$.*

Theorem 13 essentially says that if an optimal stopping strategy exists, it must be a hitting time to a unique set which is determined by comparing the value function with the payoff obtained by instantaneous stopping at each state. This comparison is of course impractical when the value function is written as in (1). The trick is to choose a set of general assumptions for the payoff and the diffusion such that $\tilde{V}$ can be expressed in a purely analytical form with the help of Theorem 7. Optimization in (1) can then be turned into a conceptually simple (but sometimes tedious) exercise in real analysis. As a result, the examples that are studied explicitly in the literature are usually one- or two-boundary problems i.e. problems where the continuation region is characterized as an interval with one or two variable endpoints. For example, all the problems in [21] are one-boundary problems. An example of a two-boundary problem is given by Gapeev and Lerche [25].

On a related note, the necessary optimality criteria that characterize the boundary points of the continuation region often involve derivatives of the value function candidate. As such, smoothness properties of the value function are of great importance. The most commonly required property is the so called *smooth fit*, which means that the value function candidate is continuously differentiable across the optimal boundaries. Villeneuve [26] studies conditions which for a linear diffusion OSP guarantee the optimality of a one-boundary strategy and under which smooth fit holds.

3.2 Impulse Control

Impulse control problems are another well-known class of stochastic control problems. The difference between them and optimal stopping problems lies in the structure of available controlling decisions. OSPs only admit a single irreversible stopping decision whereas the controls in an ICP consist of a sequence of control times (or intervention dates) and the actual controls, known as impulses. At each control time, the system is instantaneously driven to a new state which is determined by the impulse and the state of the system just before the control time. The underlying process is then immediately restarted from this new state. Due to the greater flexibility of the controls, ICPs may be used to build models that are more realistic and intricate than what can be achieved with OSPs. The downside is that obtaining explicit information about these models may become even more difficult.

Literature on ICP applications tends to focus on problems in which the repetitive nature of the decisions plays a key role. In the areas of economics, finance and operations research these applications include e.g. portfolio management and consumption ([27], [28], [29], [30]), storage problems ([31], [32]), dividend optimization ([33], [34], [35], [36]) and the management of renewable resources ([37], [38], [39], [40]). A common theme in these problems is that controlling the system incurs a cost whenever controls are exerted or that controlling the system continuously is in some other way infeasible.

Out of the mentioned topics, portfolio management has received perhaps the most attention. Notable early references in this area include Eastham and Hastings [27] and Duffie and Sun [28], in which the authors study portfolio management with fixed transaction costs combined with utility maximization from consumption. The results were later refined by e.g. Morton and Pliska [29] and Korn [30]. On a related note, we point out for the interested reader the paper [41], in which Korn provides further applications of ICPs in mathematical finance.

Harrison and Taylor [31] and Harrison, Sellke and Taylor [32] study a storage problem for a Brownian motion. "Storage" is understood in a very broad sense: the authors simply assume that the process is constrained to remain non-negative. The optimality of impulse controls is again the result of fixed costs incurred from controlling the system. The authors also describe certain specific applications of this class of problems which include stochastic cash management and the controlling of inventory/production systems.

The earliest mentioned papers on portfolio management and storage problems are from the 1980s. On the other hand, dividend optimization in the presence of fixed costs wasn't given a mathematically rigorous treatment until 1995 (see Jeanblanc-Picqué and Shiryaev [33]). These results were later extended by Cadenillas et. al. [34] for an insurance company that can control both its dividend policy and cash reserve dynamics. Generalizations for arbitrary diffusions can be found in e.g. Paulsen [35] and Bai and Paulsen [36].

Despite the Faustmann example mentioned in Chapter 1, the management of renewable resources (also known as optimal harvesting) represents a newer trend in the literature on stochastic ICPs. In fact, Willassen [37] studies precisely the Faustmann problem for a stochastically growing forest and presents explicit results for cases where the growth follows a geometric Brownian motion or a logistic diffusion. The case of a general linear diffusion is tackled by Alvarez [38]. In a more recent paper, Kharroubi, Lim and Vath [39] analyze the effect delayed renewal has on optimal harvesting policies while Liu and Zervos [40] deal with more complicated performance criteria.

The particular ICP value function we analyze in the papers [I] and [IV] can be given in general as

$$V(x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}_x \left[\sum_{i=0}^{N} e^{-r\tau_i} g(X_{\tau_i -}, \zeta_i) \right] \tag{2}$$

Here $\mathcal{V}$ is the set of admissible impulse control strategies i.e. sequences $((\tau_i, \zeta_i))_{i=0}^{N}$ where $N \in \mathbb{N}_0 \cup \{\infty\}$, $\zeta_i \geq 0$ and $\tau_{i+1} > \tau_i$ almost surely. τ_i are the intervention dates and ζ_i are the impulse sizes. Controls are exercised instantaneously so that at time τ_i the process will be immediately restarted from the state $X_{\tau_i} = X_{\tau_i -} - \zeta_i$.

Impulse control problems of the form (2) are closely related to optimal stopping problems of the form (1). Indeed, the strong Markov property of linear diffusions implies that an ICP ”restarts” at each control time. Thus an ICP can be seen as a possibly infinite sequence of OSPs. (1) is also known as the OSP associated to (2). As the formal similarity suggests, there does exist an excessive characterization for the ICP value function (2) in the spirit of Theorem 11 (see e.g. [42] and [43]). The characterization works by identifying the value function (2) as a pointwise minimum of a suitably chosen set of superharmonic functions. The precise formulation of the associated verification theorem and its proof use QVIs and viscosity solution techniques as described by Belak, Christensen and Seifried [44]. As such we do not present it here in detail. Instead, we make do with more elementary verification techniques as outlined in Theorems 14 and 15.

***Theorem* 14.** *Suppose that the payoff is of the form $g(x,\zeta) = g(x)$. Suppose that f is a non-negative and r-excessive function that satisfies the inequality $f(x) \geq g(x) + f(x-\zeta)$ for all $x \in I$. Then $f \geq V$.*

***Theorem* 15.** *Suppose that the payoff is of the form $g(x,\zeta) = \zeta - c$ where $c > 0$. Suppose that f is a non-negative and r-excessive function that satisfies the inequality $f(x) \geq \sup_{\zeta\in[0,x-\mathfrak{l}]}(\zeta - c + f(x-\zeta))$ for all $x \in I$. Then $f \geq V$.*

3.3 Problems with Regime Switching

The term ”regime switching” refers to a mechanism where the exogenously determined parameters of a problem undergo a change at some (usually random) instant. These regime switching models are useful for analyzing problems where the time horizon is long compared to the expected duration of exogenous conditions such as business cycles, market structure and public policies. Among other things, incorporating regime switching structures into timing problems provides a way for studying the effect anticipation of a future switch has on the optimal strategy as we have done in [I]. This line of thought is motivated in part by the works of Drazen and Helpman ([45], [46]) who discuss the effects that anticipating changes in macroeconomic policies have on the rational behaviour of individual consumers.

There has been increasing interest in studying various timing problems in a regime switching environment during the past two decades for the reasons mentioned above. Examples of such problems include the investment/consumption problem ([47], [48], [49], [50], [51]), dividend optimization ([52], [53], [54], [55], [56], [57]), portfolio selection ([58], [59]), option valuation ([60], [61], [62], [63], [64], [65], [66]), capital structure ([67], [68]) and the designing of monetary policies ([69], [70]). There are also several papers which address regime switching OSPs and ICPs on a general level, such as Guo [71], Le and Wang [72], Zhang and Zhang [73], Korn, Melnyk and Seifried [74] and Zhu [75].

The papers mentioned thus far involve regime switching as an additional feature in an otherwise well-understood problem. This is not always the case in the literature as there has been a growing interest in problems in which the switching itself is incorporated into the available controlling decisions. Research in this direction includes [76], [77], [78], [79], [80] to name a few.

The regimes and the switches between them are often modelled as continuous time Markov chains i.e. time-homogeneous Markov processes with discrete state spaces as the reader can verify from the supplied references. The switching mechanism is usually assumed to be quite general, but the problem's underlying process is often either a Brownian motion or a geometric Brownian motion (see e.g. [71]). In [I] we flip this assumption on its head. Our switching structure is very simple since we only allow a single switch to happen but the underlying process can be any linear diffusion. Moreover, in our setting the payoffs and diffusions do not have to be related to each other in any way in different regimes. We thus allow substantial changes in the problems' dynamics. The regime-specific (i.e. non-switching) OSPs and ICPs are defined as in Chapters 3.1 and 3.2.

The anticipative OSP and ICP value function are

$$\tilde{V}_1(x) = \sup_{\tau \in \mathcal{S}_1} \mathbb{E}_x \left[e^{-r\tau} g_1(X^{\nu}_{1,\tau}) 1_{\{\tau < T\}} + e^{-rT} \tilde{V}_2(X_{1,T}) 1_{\{T \leq \tau\}} \right]$$

and

$$V_1(x) = \sup_{\nu \in \mathcal{V}_1} \mathbb{E}_x \left[\sum_{i=0}^{N} \left(e^{-r\tau_i} g_1(X^{\nu}_{1,\tau_i -}) 1_{\{\tau_i < T\}} + e^{-rT} V_2(X^{\nu}_{1,T}) 1_{\{T \leq \tau_i\}} \right) \right]$$

The verification theorems we use for these anticipative problems in [I] differ slightly from their non-switching counterparts.

***Theorem* 16.** *$\tilde{V}_1$ is the smallest majorant of g_1 s.t. $\tilde{V}_1 - \lambda(R_{r+\lambda}\tilde{V}_2)$ is $r+\lambda$-excessive w.r.t. X_1.*

***Theorem* 17.** *Let $f : I \to \mathbb{R}_+$ be a function satisfying $f(x) \geq g_1(x) + f(x_1)$ for every $x \in I$ and suppose that $f - \lambda(R_{r+\lambda}\tilde{V}_2)$ is $r + \lambda$-excessive w.r.t. X_1. Then $f(x) \geq V_1(x)$ for every $x \in I$.*

4 The Poisson Constraint

The previous two chapters outlined the mathematical theory and presented the starting point for the subject of this thesis. In this chapter we delve into the setting of the problems analyzed in the papers [II - IV]. We begin by discussing the motivation and the mathematical nature of the Poisson constraint as well as its applications to optimal stopping and impulse control problems through a short literature review. Then we discuss problems in which the Poisson constraint itself can be controlled in some way.

4.1 Literature on Timing Constraints

The study of optimal stopping problems under timing constraints began some 20 years ago with the paper by Dupuis and Wang [81]. In [81], the authors solve the perpetual American option-pricing problem with the additional assumption that stopping is allowed only at the jump times of an independent Poisson process. This idea of restricting possible control times to arrival times of some randomly occurring signals has since been expanded into a number of directions. In [82] the problem is generalized for continuous payoffs and linear diffusions and the general analytical properties of the value function are studied in [83]. [84] provides a further regime-switching generalization. In [85], [86] and [87] the setup is even more general and the authors also consider ICPs and ICPs with ergodic performance criteria. In particular, the signal waiting times are assumed to form a general IID sequence and thus do not have to be exponentially distributed. It is also worth noting that the Poisson constraint can sometimes be used as a technical tool for studying more complicated problems such as in [88] where the method is applied for the analysis of randomized stopping times. The argument is that the Poisson constraint effectively discretizes time in a way which works well with other analytical tools available for linear diffusions. The solution in continuous time is then obtained as a limit where the intensity of the Poisson process goes to infinity.

The mentioned timing constraints are usually interpreted as a model for liquidity constraints (see e.g. [89], [90] and [91] in addition to the other references mentioned in this section) and they have been applied to numerous problems in economics and finance. Examples of these applications include Dynkin games ([92], [93]), switching problems [94], optimal investment ([95], [96]), the investment/consumption problem ([97], [98]), portfolio management [99], dividend optimization [100] and option valuation ([101], [102], [103], [104]). Sometimes the Poisson constraint is used to model a limited information processing capacity, such as in [105]. We discuss this topic further in Chapter 4.2.1.

4.2 Optimal Timing under the Poisson Constraint

In [II - IV], a Poisson constrained version of a stochastic control problem is a problem where observing and controlling the underlying process is possible only at the jump times of an independent Poisson point process. The waiting times between these jumps are known to form an IID sequence of exponentially distributed random variables. We may label the set of these times as $\{T_n : n \in \mathbb{N}_0\}$ where $T_0 = 0$ and $T_{n+1} - T_n \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. The corresponding set of stopping times is given by

$$\mathcal{S}_0^\lambda = \{\tau : \text{ for all } \omega \in \Omega,\ \tau(\omega) = T_n(\omega) \text{ for some } n \in \mathbb{N}_0\} \tag{3}$$

and the set of admissible impulse strategies $\mathcal{V}_0^\lambda$ is constructed from $\mathcal{S}_0^\lambda$ as in Chapter 3. The Poisson constrained OSP and ICP value functions are

$$\tilde{V}_0^\lambda(x) = \sup_{\tau \in \mathcal{S}_0^\lambda} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right] \tag{4}$$

and

$$V_0^\lambda(x) = \sup_{\nu \in \mathcal{V}_0^\lambda} \mathbb{E}_x \left[\sum_{i=0}^{N} e^{-r\tau_i} g(X_{\tau_i -}, \zeta_i) \right] \tag{5}$$

The verification procedure we use for the OSP (4), originally developed in [82], is somewhat different from Dynkin's characterization (Theorem 11) in Chapter 3. Denoting the value function candidate by G_0, the idea is to show that $(e^{-rT_n} G_0(X_{T_n}))_{n \geq 0}$ forms a non-negative uniformly integrable supermartingale so that $G_0 \geq \tilde{V}_0^\lambda$ follows by optional stopping theorem. The inequality is then shown to hold as an equality. This is done by proving that the stopped process $(e^{-rT_{n\wedge N^*}} G_0(X_{T_{n\wedge N^*}}))_{n\geq 0}$ (where N^* is the candidate stopping time) is actually a martingale.

The verification theorem we use in [IV] for the ICP (5) is a Poissonized analogue to Theorem 15.

***Theorem* 18.** *Suppose that the payoff is of the form* $g(x, \zeta) = \zeta - c$ *where* $c > 0$. *Suppose that* $f \in L^1_{r+\lambda}(I)$ *is non-negative and satisfies* $\lambda(R_{r+\lambda} f)(x) \leq f(x)$ *and* $f(x) \geq \sup_{\zeta \in [0, x-\mathfrak{l}]}(\zeta - c + f(x - \zeta))$ *for all* $x \in I$. *Then* $f \geq V_0^\lambda$.

4.2.1 Rational inattention and controllable Poisson constraint

With the exception of [91], the Poisson constraint is assumed to be determined by a fixed, exogenously given constant in the papers and problems mentioned thus far. However in [II] we study a problem where this parameter is controllable as well. The motivation for this dynamic lies in a phenomenon known as *rational inattention*.

Rational inattention is a term originating in the economics literature (see [106] for a comprehensive overview). It means that it is sometimes optimal for agents to remain inattentive to the problem at hand i.e. to update their information and decisions only sporadically, rather than continuously. This is due to the fact that in reality all agents have finite resources and capacities for acquiring and processing information, so continuously and perfectly monitoring the system under study may be too costly or downright impossible.

The rational inattention literature can be roughly divided into two categories. The first consists of papers which focus on the particular nature of information constraints and allocation the mechanisms they give rise to. Examples of these include [107], [108] in which information is acquired through channels with finite Shannon capacity. The other category consists of papers in which the main focus is on timing problems themselves and information constraints are modelled through more simple means such as discrete time with fixed intervals ([109], [110], [111]), the Poisson constraint ([105]) or some other randomization mechanism ([112], [113]). In [112] the available decision times are given by a strictly increasing sequence of stopping times while in [113] the mechanism is essentially a generalization of the Poisson constraint.

In [II] the main focus is on the timing problem. We assume that the agent is solving an OSP of the form (4) but due to limitations in the agents' information processing capacity, observations are possible only at discrete signal arrival times, which we model as the jump times of two independent Poisson processes with parameters $\lambda_2 > \lambda_1 > 0$. Here λ_1 is the attention rate at which the agent may observe and control the system for free, and λ_2 represents a higher information rate which requires extra effort from the agent. The effort is in turn modelled as a possibly state-dependent cost c which the agent must pay in order to receive information at the λ_2-rate. Moreover, switching the attention rate is possible only at the signal times and the agent does not observe the system between consecutive signals.

The setting in [II] is similar to that of [91], but there are fundamental technical differences. In [91] the system is monitored continuously and the Poisson parameter can be controlled in continuous time whereas we allow observations and controls only at the signal times. There are also differences in motivation; [91] uses the Poisson constraint to model liquidity effects while we aim at modelling information constraints.

5 Article Summaries

5.1 Article I: Optimal Stopping and Impulse Control in the Presence of an Anticipated Regime Switch

We solve an OSP and an ICP (as defined in Chapter 3) in which a regime switch is assumed to occur at an $\mathrm{Exp}(\lambda)$-distributed random time. The underlying processes and payoffs are assumed to be regime-dependent and this dependency is denoted by additional subscripts 1 and 2. The regime-specific OSP value functions are

$$\tilde{V}_0(x) = \sup_{\tau \in \mathcal{S}_1} \mathbb{E}_x \left[e^{-r\tau} g_1(X_{1,\tau}) \right], \; \tilde{V}_2(x) = \sup_{\tau \in \mathcal{S}_2} \mathbb{E}_x \left[e^{-r\tau} g_2(X_{2,\tau}) \right]$$

and the ICP value functions are

$$V_0(x) = \sup_{\nu \in \mathcal{V}_1} \mathbb{E}_x \left[\sum_{i=0}^{N} e^{-r\tau_i} g_1(X^{\nu}_{1,\tau_i-}) \right], \; V_2(x) = \sup_{\nu \in \mathcal{V}_2} \mathbb{E}_x \left[\sum_{i=0}^{N} e^{-r\tau_i} g_2(X^{\nu}_{2,\tau_i-}) \right]$$

The anticipative value function is

$$\tilde{V}_1(x) = \sup_{\tau \in \mathcal{S}_1} \mathbb{E}_x \left[e^{-r\tau} g_1(X^{\nu}_{1,\tau}) 1_{\{\tau < T\}} + e^{-rT} \tilde{V}_2(X_{1,T}) 1_{\{T \leq \tau\}} \right]$$

for the OSP and

$$V_1(x) = \sup_{\nu \in \mathcal{V}_1} \mathbb{E}_x \left[\sum_{i=0}^{N} \left(e^{-r\tau_i} g_1(X^{\nu}_{1,\tau_i-}) 1_{\{\tau_i < T\}} + e^{-rT} V_2(X^{\nu}_{1,T}) 1_{\{T \leq \tau_i\}} \right) \right]$$

for the ICP. We show that under fairly general monotonicity assumptions related to the payoffs and the minimal r-harmonic functions all of the above problems admit one-sided threshold solutions. In particular, the anticipative OSP value function is of the form

$$\tilde{V}_1(x) = \begin{cases} g(x) & x \geq \tilde{y}_1 \\ \lambda(R_{r+\lambda}\tilde{V}_2)(x) + \frac{g(\tilde{y}_1) - \lambda(R_{r+\lambda}\tilde{V}_2)(\tilde{y}_1)}{\psi_{r+\lambda}(\tilde{y}_1)} \psi_{r+\lambda}(x) & x < \tilde{y}_1 \end{cases}$$

where $\tilde{y}_1 \in \mathrm{argmax}_{x \in I} \left\{ \frac{g(x) - \lambda(R_{r+\lambda}\tilde{V}_2)(x)}{\psi_{r+\lambda}(x)} \right\}$ is unique and the anticipative ICP value function is

$$V_1(x) = \begin{cases} \lambda(R_{r+\lambda}V_2)(\hat{y}_1) + \frac{g(\hat{y}_1) - \lambda(R_{r+\lambda}V_2)(\hat{y}_1) + \lambda(R_{r+\lambda}V_2)(x_1)}{\psi_{r+\lambda}(\hat{y}_1) - \psi_{r+\lambda}(x_1)} \psi_{r+\lambda}(x_1) + g(x) & x \geq \hat{y}_1 \\ \lambda(R_{r+\lambda}V_2)(x) + \frac{g(\hat{y}_1) - \lambda(R_{r+\lambda}V_2)(\hat{y}_1) + \lambda(R_{r+\lambda}V_2)(x_1)}{\psi_{r+\lambda}(\hat{y}_1) - \psi_{r+\lambda}(x_1)} \psi_{r+\lambda}(x) & x < \hat{y}_1 \end{cases}$$

where $\hat{y}_1 \in \operatorname{argmax}_{x\in(x_1,\mathfrak{r})} \left\{ \frac{g(x)-\lambda(R_{r+\lambda}V_2)(x)+\lambda(R_{r+\lambda}V_2)(x_1)}{\psi_{r+\lambda}(x)-\psi_{r+\lambda}(x_1)} \right\}$ is unique. After solving the problems, we compare the various stopping thresholds and value functions and illustrate the results with two examples. The first example deals with a switching cash flow tax rate. The optimal stopping thresholds are found to be equal in both regimes but the anticipation of a switch may result in a different threshold. In the second example we study total neutrality for a switching GBM i.e. non-trivial GBM regime switches such that the anticipative thresholds and value functions coincide with both regime-specific counterparts.

5.2 Article II: Optimal Stopping with Variable Attention

In this paper we solve a Poisson OSP in which an agent can increase the intensity of the Poisson process from λ_1 to λ_2 but doing so incurs a possibly state-dependent cost c. Moreover, controlling and stopping are allowed only at the signal times. The value function reads as

$$V_0(x) = \sup_{A} \sup_{\tau\in\mathcal{S}_0^A} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) - \sum_{n=0}^{\infty} e^{-rT_n^A} c(X_{T_n^A}) 1_{\{A_n=2\}} 1_{\{T_n^A<\tau\}} \right]$$

where A is what we call an attention sequence. It is a sequence $(A_n)_{n\geq 0}$ such that $A_n \in \{\lambda_1, \lambda_2\}$ and it represents the waiting decisions made by the agent. $\mathcal{S}_0^A$ is the set of stopping times that are admissible w.r.t. the filtration generated by the signal times associated to A.

Due to the length of the calculations we introduce in this paper the shorthand notation

$$(\Phi_\alpha f)(x) = \int_x^{\mathfrak{r}} \varphi_\alpha(z) f(z) m'(z) dz,$$
$$(\Psi_\alpha f)(x) = \int_{\mathfrak{l}}^{x} \psi_\alpha(z) f(z) m'(z) dz$$

for $\alpha \geq 0$ and $f:\ I \to \mathbb{R}$ such that the expressions are well-defined.

We show that under certain fairly general assumptions the problem admits a unique solution that is of a threshold form i.e. there exist thresholds $x^* < y^*$ such that if a signal arrives when the process is in the set $(\mathfrak{l}, x^*)$ it is optimal to wait for the next signal with λ_1 rate, choosing to wait with the costly rate λ_2 is optimal on $[x^*, y^*)$ and on $[y^*, \mathfrak{r})$ it is always optimal to stop. In particular, we find an explicit condition on the problem's parameters that determines whether $x^* < y^{\lambda_1} < y^*$ or $y^* = y^{\lambda_1}$ in which case the problem reduces to a standard Poisson stopping problem (in the sense of Chapter 4.2) with information rate λ_1. In the case where this reduction does not happen we find the value function V_0 to be

$$V_0(x) = \begin{cases} g(x) & x \geq y^* \\ C_2\psi_r(x) + C_3\varphi_r(x) - p(x) & x^* \leq x < y^* \\ \frac{C_2\psi_r(x^*)+C_3\varphi_r(x^*)-c(x^*)-\lambda_2(R_r c)(x^*)}{\psi_r(x^*)}\psi_r(x) & x < x^* \end{cases}$$

where $p(x) = c(x) + \lambda_2(R_r c)(x)$,

$$C_2 = \frac{\varphi_r(y^*)(\Phi_{r+\lambda_2}P)(y^*) - P(y^*)(\Phi_{r+\lambda_2}\varphi_r)(y^*)}{\varphi_r(y^*)(\Phi_{r+\lambda_2}\psi_r)(y^*) - \psi_r(y^*)(\Phi_{r+\lambda_2}\varphi_r)(y^*)},$$
$$C_3 = \frac{P(y^*)(\Phi_{r+\lambda_2}\psi_r)(y^*) - \psi_r(y^*)(\Phi_{r+\lambda_2}P)(y^*)}{\varphi_r(y^*)(\Phi_{r+\lambda_2}\psi_r)(y^*) - \psi_r(y^*)(\Phi_{r+\lambda_2}\varphi_r)(y^*)}$$

and $P(x) = g(x)+p(x)$. (x^*, y^*) is a solution (unique in a certain region of interest) to

$$\begin{cases} H_1(x^*) = H_2(y^*) \\ K_1(x^*) = K_2(y^*) \end{cases}$$

where

$$\begin{aligned} H_1(x) =& \frac{p(x)(\Psi_{r+\lambda_2}\psi_r)(x) - \psi_r(x)(\Psi_{r+\lambda_2}p)(x)}{\varphi_r(x)(\Psi_{r+\lambda_2}\psi_r)(x) - \psi_r(x)(\Psi_{r+\lambda_2}\varphi_r)(x)}, \\ H_2(x) =& \frac{P(x)(\Phi_{r+\lambda_2}\psi_r)(x) - \psi_r(x)(\Phi_{r+\lambda_2}P)(x)}{\varphi_r(x)(\Phi_{r+\lambda_2}\psi_r)(x) - \psi_r(x)(\Phi_{r+\lambda_2}\varphi_r)(x)}, \\ K_1(x) =& \frac{p(x)}{\psi_r(x)}(\Phi_{r+\lambda_1}\psi_r)(x) - (\Phi_{r+\lambda_1}p)(x) \\ &- H_1(x)\left(\frac{\varphi_r(x)}{\psi_r(x)}(\Phi_{r+\lambda_1}\psi_r)(x) - (\Phi_{r+\lambda_1}\varphi_r)(x)\right), \\ K_2(x) =& \frac{(\Phi_{r+\lambda_2}P)(x)}{(\Phi_{r+\lambda_2}\psi_r)(x)}(\Phi_{r+\lambda_1}\psi_r)(x) - (\Phi_{r+\lambda_1}P)(x) \\ &- H_2(x)\left(\frac{(\Phi_{r+\lambda_2}\varphi_r)(x)}{(\Phi_{r+\lambda_2}\psi_r)(x)}(\Phi_{r+\lambda_1}\psi_r)(x) - (\Phi_{r+\lambda_1}\varphi_r)(x)\right). \end{aligned}$$

The value function V_0 is also seen to satisfy the dynamic programming equation $V_0(x) = \max\{g_(x), \lambda_1(R_{r+\lambda_1}V_0)(x), \lambda_2(R_{r+\lambda}V_0)(x) - c(x)\}$.

We illustrate the general results with two examples. In the first example the period cost is proportional ($c(x) = kx,\ k > 0$) and the diffusion is a GBM while in the second the cost is fixed ($c(x) = k > 0$) and the underlying process is a logistic diffusion. In both examples, we find the critical value for the cost parameter which dictates whether the problem reduces to a standard Poisson OSP with information rate λ_1. For a fixed cost this value is

$$k_f^* = \frac{\lambda_2}{B_{r+\lambda_2}}\psi_{r+\lambda_2}(y^{\lambda_1})\left(\Phi_{r+\lambda_2}\left(g - \frac{g(y^{\lambda_1})}{\psi_r(y^{\lambda_1})}\psi_r\right)\right)(y^{\lambda_1})$$

and the critical proportional cost k_p^* satisfies $k_p^* = k_f^* y^{\lambda_1}$. In both examples we find that there exists a critical threshold $\hat{\lambda}$ for λ_1 such that $x^* < y^{\lambda_1} < y^*$ if $\lambda_1 < \hat{\lambda}$ and if $\lambda_1 \geq \hat{\lambda}$, then the problem reduces to the standard Poisson OSP. In particular this implies that when $\lambda_1 < \hat{\lambda}$, the size of the region where costly waiting is optimal is strictly decreasing with respect to the "free" attention rate λ_1.

5.3 Article III: Solutions for Poissonian Stopping Problems of Linear Diffusions via Extremal Processes

We develop a way for expressing the expected discounted payoff at the time of the first exit from an open interval in terms of the infimum $I_t = \inf_{0 \leq s \leq t} X_s$ and supremum $M_t = \sup_{0 \leq s \leq t} X_s$ processes. Similar results for one-sided hitting times are obtained as special cases. We then present general sufficient conditions under which the Poisson stopping problem

$$\tilde{V}_0^{\lambda}(x) = \sup_{\tau \in \mathcal{S}_0^{\lambda}} \mathbb{E}_x[e^{r\tau} g(X_\tau)]$$

(see Chapter 4.2) admits a unique one- or two-sided threshold solution. The method is then illustrated with numerous explicit examples.

More specifically, denoting the Poisson times by $\{T_n\}_{n \in \mathbb{N}_0}$ as in Chapter 4 and letting $y < x < z$, $T^z = \inf\{T_n : n \in \mathbb{N}_0, X_{T_n} \geq z\}$ and $T_y = \inf\{T_n : n \in \mathbb{N}_0, X_{T_n} \leq y\}$, we show that the expected payoff can be written as

$$\begin{aligned}
\mathbb{E}_x\left[e^{-rT^z \wedge T_y} g(X_{T^z \wedge T_y})\right] = &\left(\frac{\frac{\psi_r(x)}{\varphi_r(x)} - \frac{\mathbb{E}_x[\psi_r(X_{\bar{T}})|I_{\bar{T}}=z]}{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|I_{\bar{T}}=z]}}{\frac{\mathbb{E}_x[\psi_r(X_{\bar{T}})|M_{\bar{T}}=y]}{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|M_{\bar{T}}=y]} - \frac{\mathbb{E}_x[\psi_r(X_{\bar{T}})|I_{\bar{T}}=z]}{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|I_{\bar{T}}=z]}} \right) \\
&\times \frac{\mathbb{E}_x[g(X_{\bar{T}})|M_{\bar{T}} = y]}{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|M_{\bar{T}} = y]} \varphi_r(x) \\
&+ \left(\frac{\frac{\varphi_r(x)}{\psi_r(x)} - \frac{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|M_{\bar{T}}=y]}{\mathbb{E}_x[\psi_r(X_{\bar{T}})|M_{\bar{T}}=y]}}{\frac{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|I_{\bar{T}}=z]}{\mathbb{E}_x[\psi_r(X_{\bar{T}})|I_{\bar{T}}=z]} - \frac{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|M_{\bar{T}}=y]}{\mathbb{E}_x[\psi_r(X_{\bar{T}})|M_{\bar{T}}=y]}} \right) \\
&\times \frac{\mathbb{E}_x[g(X_{\bar{T}})|I_{\bar{T}} = z]}{\mathbb{E}_x[\psi_r(X_{\bar{T}})|I_{\bar{T}} = z]} \psi_r(x)
\end{aligned}$$

where $\bar{T} \sim \mathrm{Exp}(r + \lambda)$. If $\mathfrak{r}$ is a natural boundary, then the expected payoff corresponding to T^z is just

$$\mathbb{E}_x\left[e^{-rT^z} g(X_{T^z})\right] = \frac{\mathbb{E}_x[g(X_{\bar{T}})|I_{\bar{T}} = z]}{\mathbb{E}_x[\psi_r(X_{\bar{T}})|I_{\bar{T}} = z]} \psi_r(x)$$

An analogous result hold for T_y as well. If $\mathfrak{l}$ is a natural boundary, then

$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = \frac{\mathbb{E}_x[g(X_{\bar{T}})|M_{\bar{T}}=y]}{\mathbb{E}_x[\varphi_r(X_{\bar{T}})|M_{\bar{T}}=y]}\varphi_r(x)$$

Our analysis gives rise to an interesting path decomposition result: we may also write the one-sided expected payoffs as

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \mathbb{E}_x\left[e^{-r\tau^z}\right]\mathbb{E}_z\left[e^{-rT^z}\right]\mathbb{E}_x\left[g(X_{\bar{T}})|I_{\bar{T}}=z\right],$$
$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = \mathbb{E}_x\left[e^{-r\tau_y}\right]\mathbb{E}_y\left[e^{-rT_y}\right]\mathbb{E}_x\left[g(X_{\bar{T}})|M_{\bar{T}}=y\right]$$

where $\tau^z = \inf\{t \geq 0 : X_t \geq z\}, \tau_y = \inf\{t \geq 0 : X_t \leq y\}$ are hitting times in continuous time. The significance of this result is that the expected payoff can be decomposed into three independent factors. The first factor represents the expected discount up to the first time the given state is hit in continuous time. The second factor quantifies the expected discount up to the first boundary crossing that is observed at the Poisson times $\{T_n\}_{n\geq 1}$ when the process is started from the boundary. The third factor represent the actual payoff that is received at the observed Poisson hitting time.

Lastly, in the case where X is a twice continuously differentiable monotone function of a Brownian motion the derived formulae simplify a step further because

$$\mathbb{E}_x\left[g(X_{\bar{T}})|I_{\bar{T}}=z\right] = \mathbb{E}_z\left[g(M_{\bar{T}})\right]$$

and

$$\mathbb{E}_x\left[g(X_{\bar{T}})|M_{\bar{T}}=y\right] = \mathbb{E}_y\left[g(I_{\bar{T}})\right].$$

5.4 Article IV: On the Impact of Poisson Timing Constraints on Impulse Control Policies

In this paper, we study a Poisson constrained (with parameter $\lambda > 0$) impulse control problem with a linear payoff structure. The value function is given by

$$V_\lambda(x) = \sup_{\nu\in\mathcal{V}_0^\lambda}\mathbb{E}_x\left[\sum_{k=0}^{N}e^{-r\tau_k}(\zeta_k - c)\right]$$

We show that under certain fairly general conditions the problem admits a unique optimal strategy which is of a one-sided threshold form. For low information rates λ and high fixed costs c it is shown to reduce to a stopping strategy. In that case the value function becomes

$$V(x) = \begin{cases} x - c & x \geq \tilde{y}_\lambda \\ \frac{\tilde{y}_\lambda - c}{\psi_r(\tilde{y}_\lambda)}\psi_r(x) & x < \tilde{y}_\lambda \end{cases}$$

where $\tilde{y}_\lambda$ is the optimal stopping threshold for the associated OSP

$$\tilde{V}_\lambda(x) = \sup_{\tau \in \mathcal{S}_0^\lambda} \mathbb{E}_x \left[e^{-r\tau}(X_\tau - c) \right]$$

Otherwise the value function is

$$V_\lambda(x) = \begin{cases} x - y_\lambda^* + \frac{\int_{y_\lambda^*}^\infty \varphi_\theta(t)(t - z_\lambda^* - c)m'(t)dt}{\int_{y_\lambda^*}^\infty \varphi_\theta(t)(\psi_r(t) - \psi_r(z_\lambda^*))m'(t)dt} \psi_r(y_\lambda^*) & x \geq y_\lambda^* \\ \frac{\int_{y_\lambda^*}^\infty \varphi_\theta(t)(t - z_\lambda^* - c)m'(t)dt}{\int_{y_\lambda^*}^\infty \varphi_\theta(t)(\psi_r(t) - \psi_r(z_\lambda^*))m'(t)dt} \psi_r(x) & x < y_\lambda^* \end{cases}$$

where $z_\lambda^* = y_\lambda^* - \zeta_\lambda^*$ is the state to which the underlying process is instantaneously driven at each control time and $(z_\lambda^*, y_\lambda^*)$ is the unique solution to a certain non-linear pair of equations. We prove that as $\lambda \uparrow \infty$, the value function V_λ and the thresholds converge to their continuous time counterparts studied in [114].

We illustrate the theory with two numerical examples. In the first example we assume that the underlying process is a Brownian motion killed at the origin. This can be seen as a model for dividend optimization under positive ruin probability. The second example models the optimal harvesting of a mean reverting renewable resource stock. In both examples it is found that as the information rate λ increases, the optimal thresholds z_λ^*, y_λ^* increase and approach their continuous time counterparts. On the contrary, increased volatility of the underlying diffusion leads to thresholds which are farther away from the continuous time values. The optimal impulse sizes are also increasing with respect to the information rate and volatility but the dependence on volatility is stronger in the continuous time problem.

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