TURUN YLIOPISTON JULKAISUJA
ANNALES UNIVERSITATIS TURKUENSIS

SARJA - SER. A I OSA - TOM. 413

ASTRONOMICA - CHEMICA - PHYSICA - MATHEMATICA

NON-MARKOVIANITY AND QUANTUM CORRELATIONS IN QUBIT-SYSTEMS

by

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TURUN YLIOPISTO
UNIVERSITY OF TURKU
Turku 2010

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ISBN 978-951-29-4420-0 (Print)
ISBN 978-951-29-4419-4 (PDF)
ISSN 0082-7002
Painosalama Oy - Turku, Finland 2010

Acknowledgments

The work presented in this Thesis has been carried out under the careful guidance of Dr. Sabrina Maniscalco and Dr. Jyrki Piilo in the years 2008-2010 at the University of Turku.

I want to express my sincere gratitude to my supervisors. I am very thankful that they wanted to share their knowledge of physics and their enthusiasm for research with me. I am glad they created such a pleasant and friendly atmosphere in their groups. I want to thank them from the bottom of my heart for their unconditional support, and comprehension especially in recent hard times.

I would like to thank Prof. Kalle-Antti Suominen for inviting me to carry out my doctoral studies here in Turku and for following my progress. I also would like to thank Dr. Sergey Vasiliev for the collaboration at the beginning of the PhD.

I am deeply grateful to my friends and colleagues of "the corridor". I enjoyed very much working with them and I am happy we shared so many nice moments and fun times.

I want to thank my friends in Palermo, Berlin, and Edinburgh and a bunch of people I met in schools and conferences, for the amazing time spent together. I wish I could see them more often.

A word of thanks to Ruggero for being, for better or for worse, a very important person in my life.

I want to thank my biggest supporters: Mamma, Papi, Stefania, Marco and Cicci, for being always by my side.

Finally, I gratefully acknowledge Magnus Ehrnrooth Foundation, Väisälä Foundation, Emil Aaltonen Foundation, Finnish Cultural Foundation, and CIMO for financial support over the last three years.

Contents

Abstract

In this Thesis I discuss the exact dynamics of simple non-Markovian systems. I focus on fundamental questions at the core of non-Markovian theory and investigate the dynamics of quantum correlations under non-Markovian decoherence.

In the first context I present the connection between two different non-Markovian approaches, and compare two distinct definitions of non-Markovianity. The general aim is to characterize in exemplary cases which part of the environment is responsible for the feedback of information typical of non-Markovian dynamics. I also show how such a feedback of information is not always described by certain types of master equations commonly used to tackle non-Markovian dynamics.

In the second context I characterize the dynamics of two qubits in a common non-Markovian reservoir, and introduce a new dynamical effect in a well-known model, i.e., two qubits under depolarizing channels. In the first model the exact solution of the dynamics is found, and the entanglement behavior is extensively studied. The non-Markovianity of the reservoir and reservoir-mediated-interaction between the qubits cause non-trivial dynamical features. The dynamical interplay between different types of correlations is also investigated. In the second model the study of quantum and classical correlations demonstrates the existence of a new effect: the sudden transition between classical and quantum decoherence. This phenomenon involves the complete preservation of the initial quantum correlations for long intervals of time of the order of the relaxation time of the system.

List of publications

This Thesis consists of an introductory part, followed by nine research publications.

I *Pseudomodes as an effective description of memory: non-Markovian dynamis of two-state systems in structured reservoirs*
L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway
Physical Review A **80** (2009) 012104 (4 pages).

II *Phenomenological memory-kernel master equation and time-dependent Markovian processes*
L. Mazzola, E.-M. Laine, H.-P. Breuer, S. Maniscalco, and J. Piilo
Physical Review A **81** (2010) 062120 (5 pages).

III *Sudden death and sudden birth of entanglement in common structured reservoirs*
L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway
Physical Review A **79** (2009) 042302 (4 pages).

IV *Interplay between entanglement and entropy in two-qubit systems*
L. Mazzola, S. Maniscalco, J. Piilo, and K.-A. Suominen
Journal of Physics B: Atomic, Molecular and Optical Physics **43** (2010) 085505 (10 pages).

V *Connection among entanglement, mixedness, and nonlocality in a dynamical context*
L. Mazzola, B. Bellomo, R. Lo Franco, and G. Compagno
Physical Review A **81** (2010) 052116 (8 pages).

VI *Reservoir cross-over in entanglement dynamics*
L. Mazzola, S. Maniscalco, K.-A. Suominen, and B. M. Garraway
Quantum Information Process **8** (2009) 577-585 (9 pages).

VII *Sudden violation of the CHSH inequality in a two qubits system*
L. Mazzola
Phyica Scripta **T140** (2010) 014055, Topical articles: The 16th Central European Workshop on Quantum Optics (5 pages).

VIII *Sudden transition between classical and quantum decoherence*
L. Mazzola, J. Piilo, and S. Maniscalco
Physical Review Letters **104** (2010) 200401 (4 pages).

IX *Frozen discord in non-Markovian depolarizing channels*
L. Mazzola, J. Piilo, and S. Maniscalco
arXiv:1006.1805v1 submitted for publication (6 pages).

1 Introduction

If we aim at describing Nature in a quantum mechanical way we need to consider that realistic quantum systems are open. Quantum systems, in fact, live in environments often affecting their properties in a non-negligible way. The interaction with the environment is source of decoherence, causing destruction of quantum properties and disappearance of quantum superpositions [1]. This means that, in order to describe this type of systems, a different formalism than the Schrödinger equation is generally needed. This new description is offered by the theory of open quantum systems [2, 3].

The theory of open quantum systems plays a major role in many applications of quantum physics (quantum optics [4], condensed matter physics [5], quantum chemistry [6], to name a few) and allows to investigate fundamental aspects of quantum mechanics, such as the quantum measurement problem.

Open quantum systems are in general difficult to tackle, their equations of motion can be in a form which cannot be solved, and therefore very often one needs to simplify the problem and make approximations. The theory of Markovian open quantum systems relies on two approximations: the Born or weak coupling approximation, and the Markov approximation. Markovian quantum systems are certainly the most studied and characterized open quantum systems [7, 8]. There are, however, cases in which these two assumptions are simply not admissible. Systems strongly interacting with their environments or building massive correlations with it have to be handled with more complicated approaches. In order to deal with these systems we need a theory of non-Markovian open quantum systems.

Despite the existence of many different techniques and the achievement of many important results [9–13] non-Markovian quantum systems are not yet fully understood. Fundamental questions at the heart of the theory do not have yet a clear answer. What are non-Markovian systems? Which is the general form of their equation of motion?

Driven by the will of solving this conundrum, great deal of effort has been invested in the investigation of open systems beyond the Markov approximation. Based on the properties that many quantum systems show when the Markov approximation is released, different definitions and measures of non-Markovianity have been proposed [14–16]. One of them is based on the observation of flow-back of information from the environment into the system [14, 17].

When open systems are constituted of many parts their dynamical features become even more interesting. In fact multipartite quantum systems can share peculiar types of correlations having no classical counterpart. Certainly

the most famous of them is entanglement [18]. Entanglement is a fundamental concept at the foundations of quantum mechanics. Moreover it is of key importance in many applications of quantum information science [19].

As other quantum properties, entanglement is very sensitive to the presence of the environment. The study of the open dynamics of entanglement reveals that it can be lost in a finite time, phenomenon known as entanglement sudden death [20]. This dynamical effect, first found in a Markovian scenario, acquires much richer features when moving to the non-Markovian context [21]. In fact the so-called memory effects characterizing non-Markovian systems can cause the reappearance of entanglement after its death.

The general aim of my research is to contribute to the understanding of non-Markovian systems, in particular, when considering quantum correlations. In this Thesis I will review the results obtained during my PhD studies and focus more specifically on the connections between two different approaches and definitions of non-Markovianity, and on where the memory of the non-Markovian environments resides.

I will present the *exact* solutions of simple but fundamental system-environment interaction models, namely two-state systems or, in the jargon of quantum information, qubits. Studying the exact dynamics of models such as one or two two-level atoms in a cavity gives us the chance to treat fundamental problems and gain insight on the non-Markovian dynamics.

Our exact study of two atoms inside a cavity shows that despite the presence of the environment it is possible to have residual entanglement trapped inside the system. This is due to the reservoir mediated interaction between the atoms [22–24]. An even more surprising result is the discovery of a sudden transition between classical and quantum decoherence. In fact we have shown that for certain types of environment, there exists a quantum correlation more general than entanglement [25, 26] which is completely unaffected by the presence of the environment. This is the first evidence of a quantum property fully robust against decoherence. There are indications that this quantum correlation, known as discord, might be a resource in certain protocols of quantum technology [27]. This gives an applicative importance to our result.

Here is a guideline to the structure of the Thesis. In Chapter 2 I introduce the basic concepts of the theory of open quantum systems, focusing in particular on the different definitions and techniques to study non-Markovian dynamics. In Chapter 3 I present our effective description of memory and the comparison between different definitions of non-Markovianity. In Chapter 4 I introduce the concept of entanglement and present the results on the model

we have solved exactly: two qubits in a common non-Markovian reservoir. In Chapter 5 I introduce the concept of discord and discuss the phenomenon we have discovered, the sudden transition between quantum and classical decoherence. Finally in the Conclusions I will briefly summarize the main results of the Thesis.

2 Open quantum systems

The theory of open quantum systems aims at describing the dynamics of quantum mechanical systems interacting with their surroundings [2,3]. The importance of this theory stems from the fact that the assumption of a quantum system completely isolated from the rest of the universe is only a simplification. Indeed it is precisely the action of the environment which causes phenomena such as decoherence and dissipation, threatening the quantumness of the state of the system. Furthermore, the theory of open quantum systems allows to address the quantum measurement problem in terms of decoherence induced by the environment [1, 28]. Understanding and describing decoherence is also a key requirement for the development of all quantum technologies [29].

Unfortunately, tackling the dynamics of open quantum systems is not always an easy task. For this reason one needs to perform a series of approximations in order to derive equations of motion for the state of the system, possibly solve them, and thus gain insights into the dynamics [30].

Two of the main approximations commonly done to derive such an equation of motion, also known as the *master equation*, are the Born and the Markov approximations. The first one amounts to assuming that the interaction between the system and the environment is sufficiently weak. The second one neglects short-time correlations between system and environment. When both these assumptions are satisfied we say that the open quantum system is Markovian.

A milestone in the theory of open quantum systems is the Lindblad theorem providing a full characterization of the Markovian dynamics [7, 8]. Master equations describing Markovian processes have a well defined mathematical reference structure, called the Lindblad form. This structure guarantees that the solution of the master equation represents a physical state at all times.

However the Born and the Markov approximations are not always valid. There exist systems, e.g., strongly interacting with their environments [31,32], for which the Born approximation breaks down. In other cases the structure of the modes of the environment induces long lasting system-enviroment correlations invalidating the Markov approximation [33]. For treating these systems it becomes necessary to have at hand alternative theoretical techniques leading to non-Markovian master equations which can be analytically or numerically solved. When going beyond the Markov approximation a series of fundamental questions pop up: what are the most distinguishing features of *non-Markovian* systems? How do we characterize the degree of Markovianity or non-Markovianity of a system? And most of all, does a generalization of the Lindblad theorem to non-Markovian systems exist? Most of the research

presented in this Thesis deals with these questions.

Let us begin with a brief review of the basic concepts and properties of Markovian open quantum systems.

2.1 Markovian dynamics

Markovian systems are from both physical and mathematical points of view the best characterized open quantum systems. The dynamics of the density matrix of the system is described by the dynamical map $V(t) : \rho(t) = V(t)\rho(0)$, satisfying the positivity, complete positivity (CP) and semigroup properties [See, e.g., Ref. [2, 30]]. The positivity condition ensures that a positive operator is mapped into a positive operator. The CP condition guarantees that density operators in all possible extended Hilbert spaces are mapped into density operators in the same extended Hilbert space. Finally the semigroup property is given by $V(t_1)V(t_2) = V(t_1 + t_2)$.

The equation of motion ruling the dynamics of Markovian open quantum systems is the celebrated Lindblad master equation[1]:

$$\frac{d}{dt}\rho_S(t) = -\imath[H_S, \rho_S(t)] + \sum_k \gamma_k \left(A_k \rho_S(t) A_k^\dagger - \frac{1}{2} A_k^\dagger A_k \rho_S(t) - \frac{1}{2} \rho_S(t) A_k^\dagger A_k \right). \tag{1}$$

The equation above is a linear differential equation for the density matrix of the system $\rho_S(t)$. The first term in the right hand side (r.h.s.) represents the unitary part of the evolution induced by the Hamiltonian of the system H_S. The second part is often called the *dissipator* and it describes the effect of the environment on the system. The operators A_k appearing in the dissipator are known as *jump operators*. The non-negative coefficients γ_k, having the dimension of the inverse of time, play the role of relaxation rates for the different channels of the open dynamics.

One of the most widely used numerical approaches to solve Eq. (1) is the Monte Carlo Wave Function Method (MCWF) [34]. The key idea of the method is the generation of a large number of single wave function histories $|\Psi(t)\rangle$ including stochastic quantum jumps mathematically described by the jump operators A_k. In a time interval δt quantum jumps occur in the k-th decay channels with a probability $P_k = \delta t \gamma_k \langle \Psi(t)|A_k^\dagger A_k|\Psi(t)\rangle$, proportional to the decay rate of the channel. The dynamics of the density matrix of the

[1] Throughout the Thesis we set $\hbar = 1$.

system $\rho_S(t)$ can then be calculated as ensemble averages of single-wave function histories. The MCWF technique is not simply a numerical method but allows also to grasp the dynamics of individual quantum systems such as single trapped ions and photons in cavities [35]. In these systems indeed the occurrence of quantum jumps has been experimentally verified [36–38].

Master equation in the Lindblad form can be derived microscopically starting from the von Neumann equation of the total closed system comprising of the system of interest and the reservoir. This approach sheds light into the physical meaning of the approximations on which Eq. (1) relies.

As already mentioned a first approximation is the Born or weak coupling assumption. Other two key assumptions are factorized initial conditions (the state of the total closed system is the product of the state of the system and the state of the environment), and the stationarity of the environment (the state of the environment does not change during the evolution). As a consequence of the previous assumptions, the correlations between system and environment, established during the time evolution, do not sensibly affect the reduced dynamics of the open system [39].

Finally, the Markov approximation corresponds to assuming that the time scale τ_S over which the state of the system varies appreciably is large compared to the time scale τ_R over which the reservoir correlation function decays. Performing the Markov approximation therefore amounts at a coarse-graining on time scales of the order of τ_R.

The reservoir correlation time and the reservoir correlation function are usually linked to the shape of the spectrum of the environment, or spectral distribution. By environmental spectral distribution we mean the function describing how the open system couples to each degree of freedom of the environment. For simple structures of the environmental spectrum, τ_R can be directly related to the inverse of the width of the spectrum. In fact, it turns out that the Markov approximation holds when the environment has an almost flat spectral distribution. On the contrary, whenever narrow structures appear in the spectrum, the Markov approximation cannot usually be done. Structured environments therefore often lead to non-Markovian dynamics, in this sense we refer to them as non-Markovian environments.

2.2 Non-Markovian dynamics

As mentioned above, the Markovian dynamics is always an approximation of the time evolution of an open system. The study of the exact dynamics of open quantum system is, therefore, of fundamental importance. The Markovian

approximation, indeed, can hide crucially quantum effects such as the quantum Zeno effect [40, 41].

Moreover, while certain physical systems are characterized by very short reservoir correlation times, and therefore are naturally consistent with a Markovian description, other systems are intrinsically non-Markovian. Well-known examples are photonic band gap materials or photonic crystals [33], where highly non-Markovian phenomena such as population and entanglement trapping occur [42]. It is worth noting that effects such as entanglement trapping could be a resource in the battle against decoherence, since they counter the loss of entanglement caused by the environment. For this reason very recently reservoir engineering techniques have received a huge deal of attention [43].

Non-Markovian short-time dynamics seem to play a crucial role also in the time evolution of certain quantum biological systems. Indeed in Ref. [44], e.g., a new non-Markovian approach [10] was used to describe excitonic energy transfer in large photosynthetic complexes.

Finally, recently, the internal inconsistency of the Markovian theory of fault-tolerant quantum computation has been demonstrated [45]. Such claims, together with results on non-Markovian quantum computation [46, 47], further stress the importance of studies on non-Markovian dynamics also for quantum technologies. Indeed, the need to have faster and faster quantum gates, in order to reduce the effects of decoherence, requires the understanding of the short-time dynamics, where non-Markovian effects are non-negligible.

2.2.1 Definitions of non-Markovianity

Due to the difficulties in dealing with non-Markovian systems, the theoretical methods to study their dynamics strongly depend on both the given physical system and the specific aspect of the dynamics under consideration. Even the very definition of a non-Markovian process has very recently given rise to a vivid debate in the scientific community [14–16, 48].

Historically, Markovian time evolutions were associated to the Lindblad form and, as a consequence, all master equations which could not be recast in the Lindblad form were called non-Markovian. Following this line of thought a measure of non-Markovian behaviour based on the deviation of the master equation from the Lindblad form has recently been presented in the literature [48].

Following this first attempt at least three measures of non-Markovianity have been proposed in the last two years. In this Thesis I focus on the measure by Breuer *et al.* [14], which has the advantage of being independent of

the specific mathematical structure of the master equation. This measure is based on the definition of non-Markovian systems as those for which there exists at least one pair of states whose distinguishability increases for some intervals of time. This increase indicates that the system experiences *in toto* a feedback of information from the environment into the open system. The measure proposed by Rivas *et al.* [15], on the contrary, seems to detect non-Markovianity even when the feedback of information occurs in only one of the decay channels, while the overall information flow goes from the system to the environment. Finally in Ref. [16] the concept of quantum Fisher information is used to quantify the degree of non-Markovianity.

The connection between different definitions is an important open question lying at the heart of non-Markovian theory. In Chapter 3 I present one of the result of the Thesis, dealing with the investigation of this connection.

2.2.2 Non-Markovian techniques

I now briefly review three main classes of approaches to the dynamics of open quantum systems. Other methods exist in the literature, however I focus here only on the non-Markovian techniques used in this Thesis.

1. **Exact approaches**: Some simple but fundamental models of open quantum systems are amenable to an exact solution. A very relevant example is a two-level atom interacting with a zero temperature bosonic structured reservoir. For such a system it is possible to derive an infinite set of differential equations, for the degrees of freedom of both the system and the environment, which can be formally solved. From the dynamics of the total closed system one then derives the non-Markovian atomic evolution. The exact solution also allows to derive *a posteriori* a master equation for the reduced system. This is important because the form of the master equation may be useful to single out the main physical processes characterizing the dynamics.

One of the approaches discussed in this Thesis, namely the *pseudomode approach* [9, 49, 50], falls into this category. This method allows to solve the dynamics of a non-Markovian system by enlarging the Hilbert space of the system in a way to include a "part" of the reservoir: the pseudomodes.

Consider for example the case of a two-level atom in a structured reservoir. Starting from the *infinite* set of differential equations for the total closed system, a finite number of auxiliary variables, the pseudomodes, can be isolated. This allows to map the original problem into a simpler one requiring the solution of a *finite* set of differential equations for the atom and the pseudomodes. The number and the properties of the pseudomodes depend on the

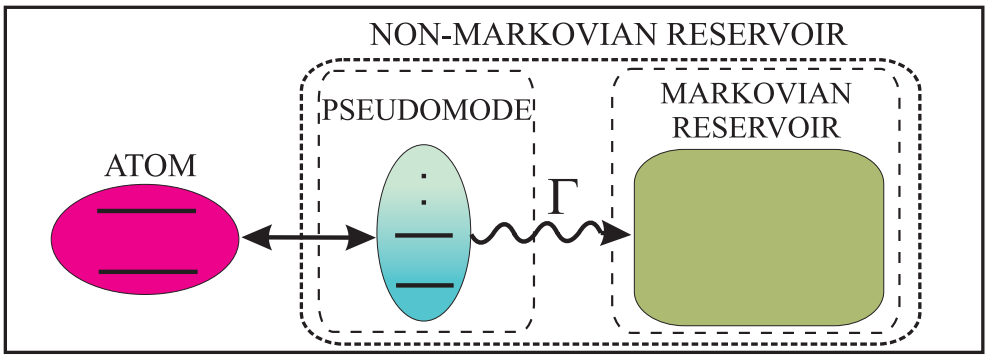

Figure 1: Diagramatic representation of the open system-pseudomode dynamics: the system interacts with a single pseudomode which leaks with constant decay rate Γ into a Markovian reservoir.

form of the spectral distribution. By manipulating this new set of equations it is possible to derive a master equation. Surprisingly such a master equation, describing the time evolution of the enlarged system (atom and pseudomodes), is Markovian.

From the form of the Markovian master equation one sees that the atom interacts coherently with the pseudomodes, while the pseudomodes decay into independent Markovian reservoirs (see Figure 1). From a fundamental point of view this method establishes an equivalence between a non-Markovian system (e.g., the atom) described by a non-Markovian master equation and the Markovian extended system comprising a part of the reservoir. Therefore some non-Markovian systems can be studied using Markovian master equations provided one pays the price of an enlargement of the dimension of the Hilbert space.

2. **Time-local master equations**: A wide class of non-Markovian master equations has a local-in-time form. This means that the time derivative of the reduced density matrix at time t depends only on the state of the system at that time, and not on the previous history of the system.

The most general form for this class of master equations, also known as time-convolutionless (TCL) master equations, is given in Refs. [2, 13]. Here I consider the subclass of TCL equations having the form

$$\frac{d}{dt}\rho(t) = -\imath[H, \rho(t)] + \sum_k \gamma_k(t) \left(A_k \rho(t) A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho(t)\} \right). \qquad (2)$$

In Eq. (2) the specific time-dependence of the decay rates $\gamma_k(t)$ is related to the form of the spectral density. If $\gamma_k(t) > 0$ for any time and any decay channel one speaks of time-dependent Markovian master equation, in the sense that at each time instant the master equation is in the Lindblad form. In this case one can use the MCWF approach, with the only difference that the jump rates now depend on time. For certain reservoir structures, however, $\gamma_k(t)$ take temporarily negative values for certain time intervals. An example is the photonic band gap case [33]. Since in the MCWF method the probability of a quantum jump in the decay channel k is directly proportional to the decay rate as $P_k = \delta t \gamma_k(t) \langle \Psi(t)|A_k^\dagger A_k|\Psi(t)\rangle$, a negative value for $\gamma_k(t)$ would lead to a negative probability of occurrence of a quantum jump, having no physical meaning.

The *Non Markovian Quantum Jumps Technique* (NMQJ) generalizes the MCWF method to non-Markovian systems described by master equations in the form of Eq. (2) for which negative decay rates can occur [10, 51, 52]. The NMQJ approach coincides with the MCWF method when the decay rates are positive. When $\gamma_k(t) < 0$ it has been shown that the negative probability of occurrence of a quantum jump corresponds to a positive probability for the occurrence of a "reverse" jump. A reverse jump basically cancels a quantum jump which occurred at a previous time (in the region of positivity of the coefficients).

Since quantum jumps cause decoherence/dissipation in the open system dynamics, the reverse jumps partly cancel the previously occurred decoherence and can be seen as the result of the reservoir finite memory. Another way of seeing the action of the reverse jumps is by considering a jump-reverse jump pair as describing a virtual process.

Interestingly, the exact non-Markovian master equation for a two-level atom in a zero temperature structured environment has exactly a local-in-time form

$$\frac{d\rho_S}{dt} = \frac{S(t)}{2i}[\sigma_+\sigma_-, \rho_S] + \gamma(t)\left[\sigma_-\rho_S\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S\}\right], \tag{3}$$

where ρ_S is the atomic density operator. The time dependent term $S(t)$ descibes the Lamb shift, while $\gamma(t)$ is a time-dependent decay rate. The analytic expression of $S(t)$ and $\gamma(t)$ are given in Ref. [2]. In the Markovian limit $\lambda \to \infty$ (flat spectrum) Eq. (3) reduces to the Markovian master equation describing spontaneous emission in free space.

3. **Memory-kernel master equation**. The third class of non-Markovian approaches uses master equations having integro-differential form as the fol-

lowing

$$\frac{d}{dt}\rho_S(t) = \int_{t_0}^{t} ds \mathcal{K}(t,s)\rho_S(s), \tag{4}$$

where $\mathcal{K}(t,s)$ is the memory-kernel superoperator [11, 12]. In Eq. (4) the time derivative of the density operator, and therefore the future dynamics of the state, depends on the past history of the system via the memory-integral. Memory kernel master equations can be either derived microscopically or introduced phenomenologically. One should note that, since a non-Markovian generalization of the Lindblad theorem does not exist, phenomenological master equations can lead to unphysical conditions indicated by the violation of positivity and complete positivity [53, 54]. The presence of the integral over the past history of the system is normally associated to memory effects in the dynamics and hence has been considered as synonymous of non-Markovianity. We have seen however that there exists exact (and therefore non-Markovian) master equations having local in time form.

Can the memory effects due to the memory kernel be always associated to the feedback of information from the environment into the system? The answer to this question, which will be addressed in Chapter 3, will shed light on very general features of non-Markovian systems. There I will present our effective description of environmental memory and our results on the comparison between two different definitions of non-Markovian dynamics.

3 Understanding non-Markovianity

In the previous Chapter we have presented the basic ingredients of the description of quantum systems interacting with their surroundings. In several physical contexts such as quantum optics, solid state physics, and quantum chemistry, the correlations between the system and the environment persist for very long times and therefore non-Markovian approaches are needed. The theory of non-Markovian open quantum systems however is, from many aspects, still in its infancy.

As we showed in the previous section, many analytical methods and numerical techniques have been developed to treat non-Markovian dynamics. In general the choice of a certain non-Markovian approach rather than another is motivated by the properties of the physical system under consideration. Comparisons between different approaches, however, are highly desirable but still rare, due to the enormous difficulties in treating even the simplest non-Markovian systems. In the following I will discuss an example in which, by comparing two different non-Markovian approaches, one can gain new insight in the physical mechanisms governing the physics of a paradigmatic model of open quantum system.

3.1 Pseudomode as effective description of memory

Let us consider a two-state system interacting with a zero temperature bosonic environment with Lorentzian structure. This system may model, for example, an atom interacting almost resonantly with a mode of an optical cavity [31]. The pseudomode master equation for such a system is [9]

$$\frac{d\rho}{dt} = -i[H_0, \rho] - \frac{\Gamma}{2}\left(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a\right), \tag{5}$$

where

$$H_0 = \omega_0\sigma_+\sigma_- + \omega_c a^\dagger a + \Omega_0\left(a^\dagger\sigma_- + a\sigma_+\right), \tag{6}$$

ρ is the density operator for the atom and the pseudomode, and a ($a^\dagger$) is the annihilation (creation) operator of the pseudomode. Since the Lorentzian spectral distribution,

$$J(\omega) = \frac{\Omega_0^2}{\pi}\frac{\Gamma}{(\omega-\omega_c)^2 + (\Gamma/2)^2}, \tag{7}$$

has only one pole in the lower half complex plane, the atom interacts with one pseudomode only. The constants ω_c, Γ and Ω_0 are the oscillation frequency,

the decay rate, and the coupling constant of the pseudomode, respectively. As Eq. (5) shows the qubit is coherently coupled to the pseudomode which is in turn leaking into a Markovian reservoir.

In paper I we compare the pseudomode approach and the TCL description of this system given by Eq. (3). We verify the equivalence between the two approaches by taking the partial trace of the pseudomode in Eq. (5). In this way we obtain a time-local master equation for the qubit, having the same coefficients of Eq. (3). The proof of the equivalence of the approaches singles out the existence of a relation connecting parameters and quantities belonging to the two different descriptions. In particular, let us denote with $|c_1(t)|^2$ the atomic excited state probability, and with $|b_1(t)|^2$ the pseudomode population. We recall that $\gamma(t)$ is the time-dependent decay rate in Eq. (3), which can take negative values for certain intervals of times, and Γ is the pseudomode decay rate of Eq. (5). The following relation holds

$$\frac{d|b_1(t)|^2}{dt} + \Gamma|b_1(t)|^2 = \gamma(t)|c_1(t)|^2. \tag{8}$$

The equation above and its interpretation are the main results of this section. Equation (8) links the *compensated rate of change* of the pseudomode population (i.e., the variation in time of the pseudomode population after removing the effect of losses) with the decay rate of the qubit in the non-Markovian reservoir.

According to the NMQJ method when $\gamma(t)$ is negative some of the coherence lost due to the action of the environment is restored by reversed quantum jumps. Stated another way, due to the reservoir memory effects, a part of the information that leaked from the system to the environment feeds back into the system [10, 51, 52]. When $\gamma(t) < 0$ the r.h.s. of Eq. (8) is negative. Since the pseudomode losses $\Gamma|b_1(t)|^2$ always give a positive contribution, the time derivative of the pseudomode population must be negative and dominant compared to the losses. Therefore, the pseudomode population decreases not only because the pseudomode leaks into the Markovian reservoir but also as a consequence of the recovery of energy and coherence by the atom.

Equation (8) establishes a link between the restoration of coherence, caused by the reverse jumps, and the pseudomode depletion. This observation suggests an interpretation of the pseudomode as that part of the reservoir from which the atom receives back information and probability due to memory effects. So, the pseudomode behaves as a storage place for the reservoir memory. This effective description of memory can be derived also for more complicated spectral distribution, e.g., a toy-model photonic band gap, as described at the end

of paper I. Concluding, the comparison between two different non-Markovian approaches allows us to understand where the "reservoir memory", responsible for the non-Markovian dynamics of the system, resides.

3.2 Memory kernel master equation without memory

While in the previous section we compared two different non-Markovian techniques, in this section we make a comparison between two different "definitions" of non-Markovianity. A wide part of the literature on non-Markovian open quantum systems defines a non-Markovian system in terms of the specific form of the master equation ruling its dynamics. More precisely many identify non-Markovianity with master equations containing an integral of memory. More recently definitions of non-Markovian dynamics based on intuitive physical concepts, such as the flow-back of information from the environment into the system, have been proposed [14–16, 48]. Therefore it is important to investigate the differences and analogies between these two ways of defining non-Markovian dynamics. Do all memory kernel master equations describe a flow-back of information as defined in Ref. [14]? In paper II we answer to this question.

In Ref. [14] a Markovian quantum process is defined as one that continuously reduces the distinguishability of quantum states. One can interpret this loss of distinguishability as a flow of information from the open system to its environment. By contrast, in a non-Markovian process there exists at least one pair of states, in the Hilbert space of the system, the distinguishability of which grows for certain times. This growth of distinguishability can be interpreted as a reverse flow of information from the environment to the open system. This backflow of information is considered the essential feature of non-Markovian dynamics [14, 17].

Following Ref. [14] we measure the distinguishability between two quantum states, given by density matrices ρ_1 and ρ_2, by means of the trace distance [19] $D(\rho_1, \rho_2) = \frac{1}{2}\mathrm{Tr}|\rho_1 - \rho_2|$ where $|A| = \sqrt{A^\dagger A}$. The trace distance represents a metric on the space of physical states and it has the important property that all completely positive and trace preserving maps are contractions for this metric. Given a pair of initial states $\rho_{1,2}(0)$ the rate of change of the trace distance under the time evolution is defined by

$$\sigma(t, \rho_{1,2}(0)) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)). \tag{9}$$

According to Eq. (9) a given process is said to be Markovian or time-dependent

Markovian if for all pairs of initial states the rate of change of the trace distance is smaller than zero for all times, i.e., $\sigma(t, \rho_{1,2}(t)) \leq 0$. Thus, a process is defined as non-Markovian if there exists a pair of initial states $\rho_{1,2}(0)$ and a certain time t at which the trace distance increases, $\sigma(t, \rho_{1,2}(t)) > 0$.

We consider again a fundamental model of open quantum systems, i.e., a two-state system in a bosonic reservoir. The simplest memory-kernel master equation for this system is

$$\frac{d\rho_S(t)}{dt} = \int_0^t k(t')\mathcal{L}\rho_S(t-t')dt'. \tag{10}$$

Here, $\rho_S(t)$ represents the reduced density matrix of the qubit, $k(t)$ is a memory kernel function containing information about the properties of the reservoir, and $\mathcal{L}$ is the Markovian Liouvillian superoperator. A common choice for the memory kernel function is $k(t) = \gamma e^{-\gamma t}$, describing the case in which the reservoir memory decays exponentially in time. In this case the reservoir correlation time quantifying the non-Markovian scale is $\tau_R = 1/\gamma$. For the system considered the Liouvillian is given by

$$\begin{aligned}\mathcal{L}\rho_S = \quad & \tfrac{\gamma_0(N+1)}{2}(2\sigma_-\rho_S\sigma_+ - \sigma_+\sigma_-\rho_S - \rho_S\sigma_+\sigma_-) \\ & + \tfrac{\gamma_0 N}{2}(2\sigma_+\rho_S\sigma_- - \sigma_-\sigma_+\rho_S - \rho_S\sigma_-\sigma_+),\end{aligned} \tag{11}$$

with N average number of thermal excitations in the reservoir. The master equation (10) and (11) was solved in Ref. [55] and the condition of positivity and complete positivity of the dynamical map were also studied [56]. It turns out that complete positivity is guaranteed only for moderate and high temperatures of the reservoir. Only in this case the master equation (10) is physical. In paper II we use the exact solution to calculate the rate of change of the trace distance, given by Eq. (9). We demonstrate that, within the limit of validity of the master equation, the trace distance of any couple of states always decreases, i.e., $\sigma(t) \leq 0$, $\forall t$. Hence, the information that flows from the system to the environment never comes back to the system during the dynamics. In this sense the two definitions of non-Markovianity do not coincide.

Our analysis demonstrates that the common belief that certain forms of master equations, in particular memory kernel equations, always capture aspects of non-Markovianity, such as the flow-back of information, is wrong. In other words the presence of an integral over the past history of the system does not necessarily guarantee by itself a reverse flow of information from the environment into the open system.

Recently the exact memory-kernel master equation for a qubit in a zero-temperature reservoir has been derived from the exact solution of the dynamics [57, 58]. The structure of the Liouvillian $\mathcal{L}$ in the exact memory kernel master equation differs from the one appearing in the phenomenological one, given by Eq. (10). In particular the exact superoperator $\mathcal{L}$ contains an additional dephasing channel. Interestingly this additional channel appears only in the memory-kernel form and it is not present in the generator of the exact TCL master equation.

The memory kernel master equation obtained from the exact solution is by definition always positive and completely positive. Moreover, it does describe non-Markovian feedback of information, for certain values of the parameters, so in this case the memory kernel master equation is non-Markovian in the sense of Ref. [14]. In general, however, the two definitions of non-Markovianity do not coincide.

4 Loss of quantum correlations in structured environments

Entanglement is an exquisitely quantum type of correlation marking the departure of quantum from classical physics. The state of a multipartite system is said to be entangled if it cannot be written as the product of the states associated to each single parts [59]. As a consequence, the global description given by the total entangled state of the system contains information that the single reduced states cannot contain.

Entanglement is at the heart of quantum information theory and represents an essential ingredient in the success of a number of different protocols. For example, in quantum cryptography entanglement guarantees fundamentally secure communication [60]. In Shor's quantum algorithm it is responsible of the exponential speed up with respect to any classical algorithm [61]. In teleportation it enables the transmission of an unknown quantum state without actually sending any particle through a channel [19].

The two paradigmes of quantum theory are entangled states and Schrödinger's cat states. With the latter term we indicate here quantum superpositions of macroscopically distinguishable quantum states. Entanglement and quantum superpositions are usually threatened by the presence of the environment. In particular the environment kills the Schrödinger's cat, transforming the quantum superposition into a classical statistical mixture. While the death of the Schrödinger's cat always occurs exponentially in time, the loss of entanglement may present different features.

One of the key results in the study of entanglement dynamics in presence of the environment is the phenomenon known as early-stage disentanglement, or entanglement sudden death (ESD) [62]. Yu and Eberly showed that entanglement shared by two uncoupled qubits interacting with two independent Markovian reservoirs can die in a finite time. For such Markovian systems this behaviour is in striking contrast with the exponential decay of quantum coherences.

This fundamental difference is intriguing: if the coherences disappear asymptotically, why entanglement may be completely lost after a finite time? Finite time disentanglement could have important consequences also in quantum technologies, where one would like the survival time of entanglement to be longer than the time needed for information processing.

Since the seminal work of Yu and Eberly entanglement dynamics has been extensively investigated in a a number of models in different physical scenarios.

Research has touched on issues ranging from quantum foundations [63,64], to coherence control [65, 66], error correction [67] and entanglement generation [68], to name a few. Besides, ESD has been examined in several distinctive model situations involving pairs of atomic, photonic and spin qubits [69, 70], continuous Gaussian states [72–74], multiple qubits [75] and spin chains [76–78]. ESD is also studied for different environments including random matrix environments [79, 80] and thermal noise [81–83]. The open dynamics of entanglement has been investigated also in the case of initial correlations between the system and the environment [84].

All these results rely on Markovian approaches. Due to its fundamental nature it is important to understand whether ESD is a consequence of an approximation or rather characterizes also the exact dynamics. The first study of non-Markovian entanglement dynamics dealt with two qubits in two independent structured reservoirs (i.e., in which narrow structures appear in the spectrum) specifically having Lorentzian spectral distribution [21]. This model, in a cavity-QED scenario, describes two two-level atoms inside two independent lossy cavities. As a result of the non-Markovianity of the environment revivals of entanglement after sudden death can occur.

When, instead, the two atoms are placed inside the same cavity reservoir-mediated-interaction between the atoms takes place. This interaction, absent in the model of Ref. [21], amplifies the revivals of entanglement caused by the reservoir memory. In this Chapter I review our results on this more complicated model.

Our predictions apply to cavity QED experiments with trapped ions, and to circuit QED experiments. In the first context [85, 86], entanglement between two remotely located trapped atomic ions has been demonstrated [87] and multiparticle entangled states can be generated and fully characterized via state tomography [88]. In the second context, field coupling and coherent quantum state storage between two Josephson phase qubits have been achieved through a microwave cavity on chip [89, 90].

4.1 Entanglement and nonlocality

Let us consider a bipartite quantum system and indicate with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ the Hilbert space of the composite system, $\mathcal{H}_A$ and $\mathcal{H}_B$ being the Hilbert spaces of part A and part B. A pure quantum mechanical state $|\Psi_{AB}\rangle \in \mathcal{H}$ is entangled if it cannot be written in factorized form $|\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$ [18, 19]. For

pure bipartite states the von Neumann entropy of the reduced systems,

$$S(\rho_i) = -\mathrm{Tr}(\rho_i \log_2 \rho_i), \qquad i = A, B, \tag{12}$$

unambiguously quantifies entanglement: $E(|\Psi\rangle\langle\Psi|) = S(\rho_A) = S(\rho_B)$. The estimation of entanglement becomes much more complicated when the state under consideration is mixed. A mixed state is separable (not entangled) when it can be written as a convex combination of product states [59],

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B. \tag{13}$$

Since this decomposition is not unique, different measures of entanglement with different operational meanings exist. In general the choice of the most convenient measure depends on the properties of the state we want to study. Even if different measures quantify differently the amount of entanglement contained in the state, they all agree on the zero (separable state) and the maximum (maximally entangled state) values.

Let us introduce the first measure of entanglement used in the Thesis, namely the relative entropy of entanglement [18]

$$E_R(\rho) = \min_{\sigma \in \mathcal{D}(\mathcal{H})} S(\rho||\sigma) = \mathrm{Tr}\rho(\log_2 \rho - \log_2 \sigma), \tag{14}$$

where σ is a separable state, and the minimization is performed over the set of separable states $\mathcal{D}(\mathcal{H})$. Calculating the relative entropy of entanglement is generally difficult because the form of the closest separable state is known only for a few simple cases.

Another measure of entanglement is the entanglement of formation defined as [18]

$$E_F(\rho) = \min \sum_i \mu_i E(|\Psi_i\rangle\langle\Psi_i|), \tag{15}$$

where the minimum is taken over all the possible decompositions of $\rho = \sum_i \mu_i |\Psi_i\rangle\langle\Psi_i|$ in terms of pure states. For two qubits a closed analytical formula for $E_F(\rho)$ has been derived by Wootters [91]

$$E_F(\rho) = H\left\{\frac{1}{2}\left[1 + \sqrt{1 - C^2(\rho)}\right]\right\}, \tag{16}$$

with $H\{x\} = -x \log_2 x - (1-x)\log_2(1-x)$. The function $C(\rho)$, called *concurrence*, is defined in the following way:

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{17}$$

where $\{\lambda_i\}$ are the eigenvalues of the matrix

$$R = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B), \tag{18}$$

with ρ^* the complex conjugate of ρ and $\sigma_y^{A/B}$ the Pauli matrices for qubits A and B. This quantity attains its maximum value 1 for maximally entangled states and vanishes for separable states. Since $E_F(\rho)$ is monotonically related to the concurrence C, the latter quantity is often used to quantify entanglement.

Quantum correlations described by entanglement are often related to the nonlocal nature of quantum mechanics. In 1964 John Bell proved a theorem considered a cornerstone of quantum theory. Bell's theorem states that no local hidden variable theory can reproduce all the predictions of quantum mechanics [92]. Starting from the assumptions of realism and locality, Bell found an inequality for correlations between measurements performed on two distant systems.

Since the introduction of the first inequality, derived by Bell himself, other inequalities going generically under the name of "Bell inequalities" have been proposed. Probably the most popular among them is the Clauser-Horne-Shimony-Holt (CHSH) inequality [93]. The importance of Bell inequalities is that they allow to test experimentally the predictions of quantum mechanics. Several experiments have been performed to detect violations of Bell inequalities [94–96]. These experiments have always confirmed quantum theory, thus shedding light on its nonlocal character.

It is worth mentioning that not all entangled states violate Bell inequalities, entanglement and nonlocality coincide only for pure states. Nonlocality of quantum states is even more sensitive to the presence of the environment than entanglement. Interestingly, however, in non-Markovian systems the reservoir memory may temporarily restore nonlocality after its disappearance [97]. In this Chapter we also summarize results elucidating the nonlocality revivals in a specific physical system.

4.2 The model: two qubits in a common structured reservoir

In this section we introduce the system studied in papers III-VII of this Thesis, namely two qubits in a common zero-temperature Lorentzian structured reservoir. For the sake of concreteness we consider the specific case of two atoms in a lossy cavity. The Hamiltonian of the system under investigation, in the

rotating wave approximation, is given by $H = H_0 + H_{int}$, where

$$H_0 = \omega_0(\sigma_+^A\sigma_-^A + \sigma_+^B\sigma_-^B) + \sum_k \omega_k a_k^\dagger a_k, \tag{19}$$

$$H_{int} = (\sigma_+^A + \sigma_+^B)\sum_k g_k a_k + \text{h.c.} \tag{20}$$

Here, $\sigma_\pm^A$ and $\sigma_\pm^B$ are, respectively, the Pauli raising and lowering operators for the atoms A and B, ω_0 is the Bohr frequency of the two atoms, a_k ($a_k^\dagger$) is the annihilation (creation) operator for the field mode k, and the mode k is characterized by the frequency ω_k and the coupling constant g_k. Under these conditions the dynamics of the two atoms can be effectively described by a four state system in which three states are coupled to the vacuum in a ladder configuration with equally spaced energetic levels, and one state is completely decoupled from the other states and from the field. This can be easily understood by writing the Hamiltonian in the new basis $\{|0\rangle = |00\rangle,\ |+\rangle = (|10\rangle + |01\rangle)/\sqrt{2}, |-\rangle = (|10\rangle - |01\rangle)/\sqrt{2},\ |2\rangle = |11\rangle\}$:

$$H_0 = 2\omega_0|2\rangle\langle 2| + \omega_0(|+\rangle\langle +| + |-\rangle\langle -|) + \sum_k \omega_k a_k^\dagger a_k, \tag{21}$$

$$H_{int} = \sum_k \sqrt{2} g_k a_k(|+\rangle\langle 0| + |2\rangle\langle +|) + \text{h.c.} \tag{22}$$

The states $|+\rangle$ and $|-\rangle$ are known as the superradiant and subradiant states, respectively. The total Hamiltonian consists of two parts, a first part describing the free dynamics of the $|-\rangle$ state, and the remaining terms describing a three-state ladder system $\{|0\rangle, |+\rangle, |2\rangle\}$ with the transitions $|0\rangle \leftrightarrow |+\rangle$ and $|+\rangle \leftrightarrow |2\rangle$.

In paper III we solved this system exactly (without performing any Markov or weak coupling approximation) by means of the pseudomode approach for a Lorentzian spectral density as in Eq. (7). Notice that writing the total Hamiltonian in the new basis is essential for solving the dynamics exactly. Indeed it allows us to use the pseudomode method for multiple excitations [49]. The master equation describing the extended system, comprising of the two atoms and the pseudomode, is

$$\frac{d\rho}{dt} = -i[V, \rho] - \frac{\Gamma}{2}(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger), \tag{23}$$

with

$$V = \sqrt{2}\Omega_0(a|+\rangle\langle 0| + a^\dagger|0\rangle\langle +| + a|2\rangle\langle +| + a^\dagger|+\rangle\langle 2|), \tag{24}$$

where ρ is the density matrix of the extended system in the interaction picture and V is the effective coupling between the atoms and the pseudomode. We recall that the parameters Γ and Ω_0 in Eq. (24) represent the pseudomode decay rate and the coupling constant between the pseudomode and the ladder system, respectively. We solve the master equation in Eqs. (23) and(24) using the Laplace transform method as shown in paper III.

4.3 Non-Markovian dynamics of entanglement

In papers III and IV we study the dynamics of entanglement for a wide class of states having the following X form[2]

$$\rho(t) = \begin{pmatrix} a(t) & 0 & 0 & w(t) \\ 0 & b(t) & z(t) & 0 \\ 0 & z^*(t) & c(t) & 0 \\ w^*(t) & 0 & 0 & d(t) \end{pmatrix}. \tag{25}$$

For this type of states concurrence attains a simple expression,

$$C(t) = \max\{0, C_1(t), C_2(t)\}, \tag{26}$$

where

$$C_1(t) = 2|w(t)| - 2\sqrt{b(t)c(t)}, \tag{27}$$
$$C_2(t) = 2|z(t)| - 2\sqrt{a(t)d(t)}. \tag{28}$$

Note that the Hamiltonian model considered, given by Eqs. (19) and (20), preserves the X form during the time evolution.

We start our analysis considering a particularly relevant class of states, called Bell-like states, defined as

$$|\Psi\rangle = \alpha|00\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|11\rangle, \tag{29}$$

$$|\Phi\rangle = \alpha|01\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|10\rangle. \tag{30}$$

This class includes also the celebrated maximally entangled Bell states $|\Psi_\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Phi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. The dynamics of entanglement for Bell-like states with one excitation was studied in Ref. [22]. There it was shown that entanglement exhibits collapses and revivals and that non-zero asymptotic entanglement can be present.

[2]The density matrix is written in the computational basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$.

The persistence of entanglement despite the presence of the environment is due to the existence of the subradiant state $|-\rangle$ which is maximally entangled and completely decoupled from the dynamics. Therefore if the state of the two atoms has initially a non zero $|-\rangle$ component some entanglement will be trapped in the system.

The exact solution of the model shows that, as in the Markovian case, when only one excitation is present ESD never occurs [22], as one can see from the mathematical structure of concurrence. Hence, to investigate the presence of ESD in this exact model it is necessary to extend the analytical approach to the more complicated case in which two excitations are present in the system.

In paper III of this Thesis we presented our results on dynamics of entanglement for two atoms initially prepared in Bell-like states with two excitations, as given by Eq. (29), placed inside an empty cavity. We are mainly interested in the dynamics in the strong coupling regime, i.e., for $\Gamma/\Omega_0 < 4$. The dynamics of entanglement for this class of states shows that ESD occurs also in the exact model, and therefore it is not a consequence of the Markovian approximation.

Figure 2 (a) shows the behavior of entanglement measured by concurrence as a function of the parameter α^2 and of the scaled time $\gamma_0 t$ (with $\gamma_0 = 4\Omega_0^2/\Gamma$). Looking at this figure one sees immediately that the non-monotic decay of entanglement, typical of non-Markovian systems [74], occurs also in the two excitations case. From the plot we can clearly identify two different dynamical regions. For $\alpha^2 \geq 1/4$ concurrence oscillates but, contrarily to the one excitation case, never reaches zero. For $\alpha^2 \leq 1/4$, a series of sudden deaths and revivals of entanglement appear in the evolution. It is worth stressing that, as our results illustrate, when revivals of entanglement after ESD occur, a finite time t after which the entanglement is completely lost does not exist. This is true for both the Markovian case studied in Ref. [98] and our non-Markovian case. The reason for such a behavior is that correlations between the two atoms are continuously created by the common reservoir.

To have a better understanding of the entanglement evolution, we compare our results with i) the Markovian common reservoir case [Fig. 2 (b)] and ii) the non-Markovian independent reservoirs case [Fig. 2 (c)]. On the one hand the comparison with the dynamics in a common Markovian reservoir shows that the non-Markovianity of our model leads to an increase in both the number and the intensity of revivals of entanglement. On the other hand the comparison with the non-Markovian independent reservoirs case shows that, in our model, the reservoir-mediated interaction counters the sudden death of entanglement.

The comparison with those two different models helps in grasping the na-

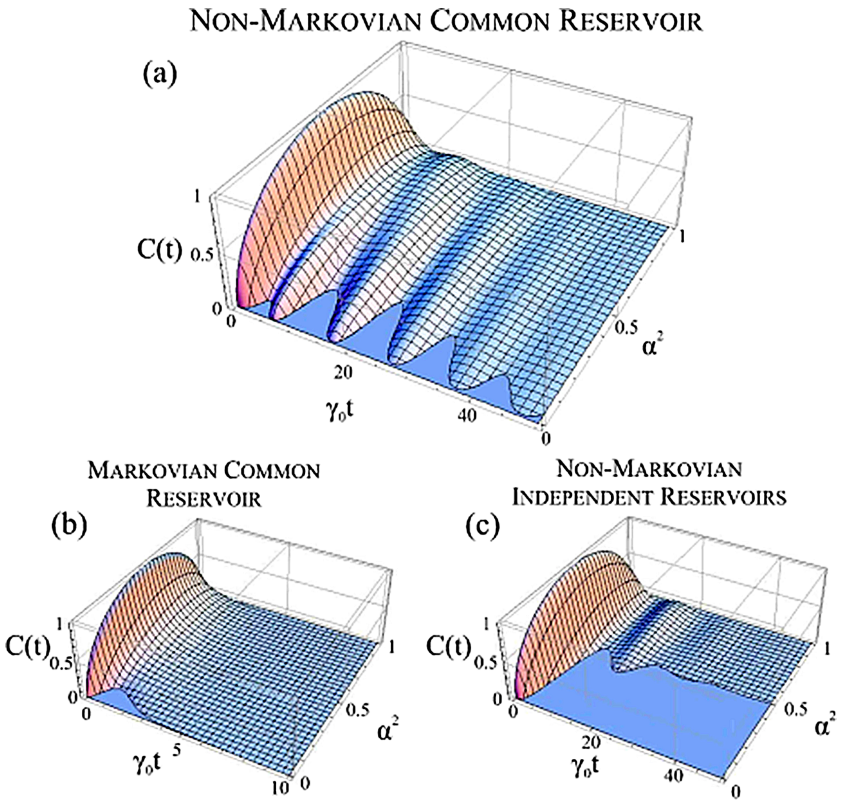

Figure 2: (a) Concurrence as a function of scaled time $\gamma_0 t$ and α^2 for two atoms prepared in the state (29) and interacting with a common Lorentzian structured reservoir. For comparison we show also the equivalent Markovian case (b) and the non-Markovian independent reservoirs situation (c) for the same class of initial states.

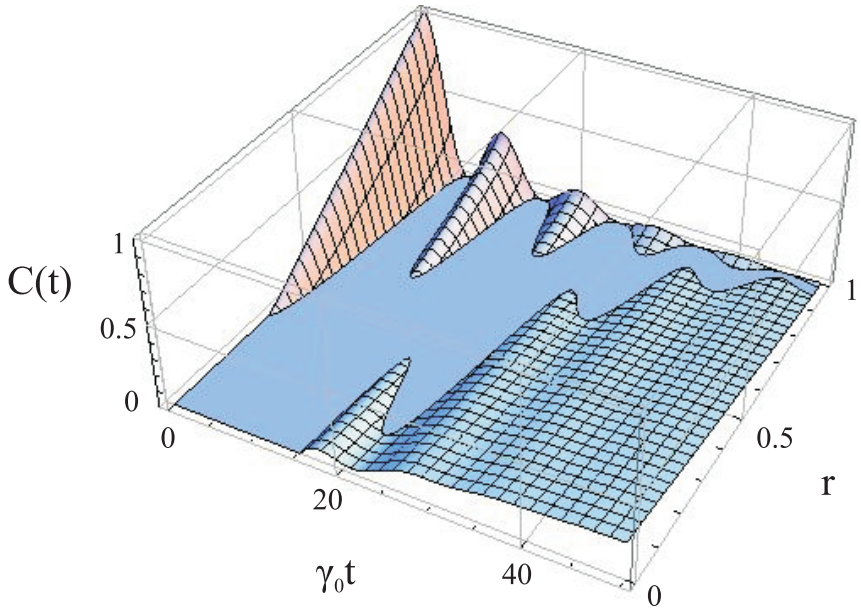

Figure 3: Concurrence as a function of scaled time and r for two atoms prepared in the Werner-like state in Eq. (31) with $|M\rangle$ equal to the superradiant state.

ture of our results. The dynamical scenario emerging from our study of entanglement reveals that there are two main ingredients ruling the dynamics: the backaction of the non-Markovian reservoir and the reservoir mediated interaction.

In paper IV we looked at the interplay between entanglement and mixedness in our model. To this aim we compare the evolution of concurrence and von Neumann entropy finding that the intuitive picture of a simultaneous deterioration of entanglement and purity does not always hold. We focus on the dynamics of a class of mixed states known as Werner-like states. These states have the form

$$\rho_{WL} = r|M\rangle\langle M| + \frac{1-r}{4}\mathbb{I}, \tag{31}$$

with $|M\rangle$ one of the four maximally entangled Bell states. Werner-like states are used in many applications in quantum information processing such as teleportation [99] and entanglement teleportation [100].

In Figure 3 we show the time evolution of entanglement for two atoms prepared in a Werner-like state (with $|M\rangle$ equal to the superradiant state)

as a function of the purity parameter r. We see that the introduction of even a small amount of mixedness modifies qualitatively the evolution of the entanglement. Indeed regions of sudden death and revivals appear. These regions are wider and wider for decreasing value of r, i.e., for more and more mixed initial states.

For $r \leq 1/3$ the initial entanglement becomes zero. In this region the system presents a new interesting dynamical effect: *the sudden birth of entanglement* (ESB). This phenomenon is due to the interaction mediated by the common reservoir, indeed it is present also in the Markovian common reservoir case [101].

In paper VI we have generalized our dynamical model to include the effect of spontaneous emission in free space. From a mathematical point of view this corresponds to adding two Markovian dissipators describing spontaneous emission to Eq. (5). In this more realistic model we show that the dynamics of entanglement has three different dynamical regimes, depending on the relative weight of the Markovian and non-Markovian decay rates. In paper VII this more general model is used to study the dynamics of the maximum of the Bell function, measuring nonlocality.

Concluding, in this section we have seen how a powerful exact approach, namely the peudomode approach, has brought to light a variety of new phenomena characterizing the dynamics of entanglement in a structured reservoir.

4.4 Interplay among entanglement, mixedness and nonlocality

In this section I will describe a compact way of looking at the environment induced dynamics of three important properties of an open quantum system: entanglement (measured by concurrence C), nonlocality (measured by the maximum of the Bell function B) and mixedness (measured by the purity P). More specifically I introduce a new dynamical tool, the CPB parameter space, to study the relationship among these three quantities. As in the previous part of the Thesis I focus on non-Markovian dynamics and investigate the consequences of the reservoir memory comparing the time evolution with the Markovian one.

In paper IV we showed that the dynamical interplay between the entanglement and mixedness strongly depends on the initial state. This is even more strongly emphasized in paper V, where the two quantities above are linked to nonlocality by means of the CPB space. The physical system considered is the same as in the previous section. Not surprisingly, therefore, reservoir-mediated interaction and memory effects induce, with different intensities, revivals of all

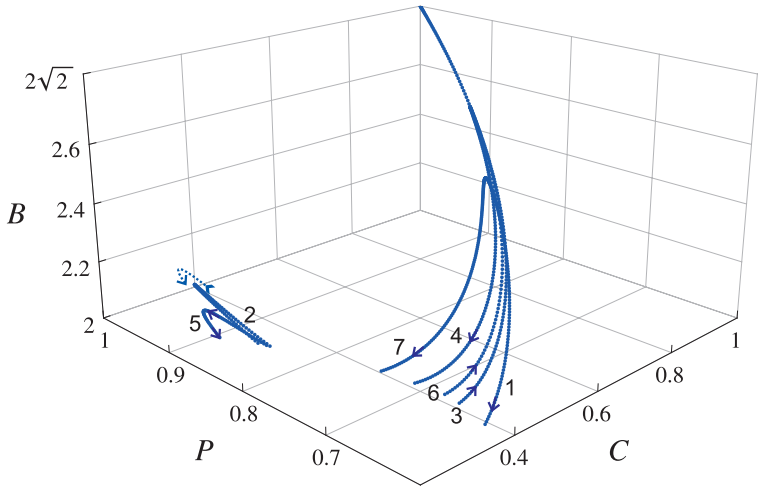

Figure 4: CPB space curve drawn by the system starting from the initial two-excitation Bell state $|\Psi\rangle$ for strong non-Markovian conditions. The arrows indicate the time evolution and the numbering from 1-to-7 indicates the different branches (multi-branch behavior) raising from the dynamics. A one-to-one correspondence among the three quantities is not possible here.

three quantities.

The values of the three quantities C, P, and B, are generally related and connections among pairs of them have been widely investigated. These connections are far from being trivial. As we have mentioned, although for pure states the presence of entanglement implies nonlocality [102], for mixed states a given amount of entanglement does not necessarily guarantee violation of a Bell inequality [59,103,104]. However, we have proven that if the two atoms are initially prepared in the superradiant state, differently from the two-excitation case, a one-to-one correspondence between any two of C, P, and B occurs. More in general when the initial state is a MEMS state, i.e., a state possessing the maximal amount of entanglement for a given degree of mixedness [105,106], a closed analytical relation among C, P, and B can be written. For example, for $2/3 \leq C \leq 1$, one gets

$$P + C^2 - B^2/4 = (1 - C)^2. \tag{32}$$

We now consider an example of a CPB phase curve for two atoms prepared in a Bell state $|\Psi\rangle$ with two excitations. The state of the system is initially pure

($P = 1$), maximally entangled ($C = 1$) and maximally nonlocal ($B = 2\sqrt{2}$). C, P and B deteriorate with time until the representative point has a value of B which satisfies the Bell inequality (branch 1 of Fig. 4). Now, a completely new dynamical feature appears: the curve surfaces from the $B = 2$ plane in a region of small concurrence and high purity (branch 2). This behavior follows from the fact that when the system is almost pure even a small amount of entanglement induces the appearance of nonlocality. After such a revival of purity and nonlocality, the curve sinks again and reappears on the space region with smaller purity (branch 3). However, the system does not pass through the same CPB points of the first branch, but it traces a new branch close to the first one (branch 3). Successively, once again decoherence effects due to the environment lead to deterioration of C, P and B, and a new branch appears (branch 4). The high non-Markovianity of the reservoir again causes Bell violation on the high purity/small concurrence region of space (branch 5). The behavior continues in a similar way and the point draws new branches until a time after which no further violation occurs.

This example demonstrates the interest in the description of non-Markovian dynamics, showing how the structure of the environment can be responsible for the occurrence of new dynamical features. The study of the dynamics of quantum correlations in open quantum systems presents a very rich variety of phenomena, many aspects of which are not yet fully understood. With our work we add another wedge in the mosaic: we solve exactly the dynamics of two qubits in a common non-Markovian reservoir and present the analysis of entanglement and nonlocality, connecting them to purity.

5 Quantum discord in Markovian and non-Markovian systems

In the previous Chapter I have discussed how correlations such as entanglement behave under the action of non-Markovian decoherence. In this Chapter I focus on another type of quantum correlations, the quantum discord. Quantum discord is a young quantity, it has been introduced ten years ago to quantify the total quantum correlations in a quantum system [25, 26].

Very recently the open system dynamics of quantum discord started to be investigated, after the discovery of analytic closed formulas for certain classes of states [107]. In this Chapter I review the state of the art on quantum discord, giving particular relevance to a dynamical effect that we have discovered for depolarizing channels and presented in papers VIII and IX: the sudden transition between classical and quantum decoherence.

5.1 Definition of quantum discord

The definition of quantum discord stems from the difference between the quantum generalizations of two classically equivalent concepts. In particular in classical information theory there are two equivalent expressions for the mutual information of a bipartite system. However, when pursuing the quantum analogue, these two formulations differ, and one can use the mismatch between the two to assess quantum correlations.

The two expressions mentioned above are $I(AB) = H(A)+H(B)-H(AB)$ and $J(AB) = H(A) - H(A|B)$, where A and B are two classical random variables, $H(A)$ and $H(B)$ are Shannon entropies, and $H(A|B)$ is the conditional entropy on A when B is found out.

The first quantum generalization of mutual information is the so called quantum mutual information:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{33}$$

where ρ_{AB} is the density matrix of the total system, $\rho_{A(B)}$ is the reduced density matrix of subsystem $A(B)$, and $S(\rho) = -\mathrm{Tr}\{\rho \log_2 \rho\}$ is the von Neumann entropy. This is generally accepted as the measure for the total amount of correlations (quantum and classical) of a quantum system [108, 109].

The second extension of mutual information requires the generalization of the concept of conditional entropy. Indeed, performing measurements on system B affects our knowledge of system A. In particular, how much system

A is modified depends on the choice of the measurement performed on B. Here the measurement B_k is considered to be of von Neumann type and it is described by a complete set of orthonormal projectors Π_k corresponding to outcome k. So the conditional density operator, which is the quantum state of the total system conditioned on the measurement outcome labeled by k, becomes $\rho_k = (I \otimes B_k)\rho_{AB}(I \otimes B_k)/p_k$ where $p_k = \mathrm{Tr}\{(I \otimes B_k)\rho_{AB}(I \otimes B_k)\}$, and I is the identity operator for subsystem A. Defining the quantum analog of conditional entropy as $S(\rho_{AB}|\{B_k\}) = \sum_k p_k S(\rho_k)$ we can introduce the second quantum extension of mutual information:

$$J(\rho_{AB}|B_k) = S(\rho_A) - S(\rho_{AB}|B_k). \tag{34}$$

In Ref. [26] Henderson and Vedral have shown that the maximum of this quantity over all the possible sets of measurements can be interpreted as a measure of classical correlations of the state,

$$C(\rho_{AB}) \equiv \max_{\{\Pi_j\}}[J(\rho_{AB}|B_k)]. \tag{35}$$

Therefore, the difference between the quantum mutual information $I(\rho_{AB})$, describing total correlations, and the measurement-based definition of quantum mutual information $C(\rho_{AB})$, measuring classical correlations, defines the quantum discord

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}). \tag{36}$$

5.2 State of the art on quantum discord

Quantum discord is an indicator of how strongly "quantum correlated" is a quantum system. The definition of classicality deriving from the entanglement vs separability paradigm, would put separable states in the cauldron of classically correlated states [59]. If it is true that separable states can be prepared via local operations and classical communication, however, it is not guaranteed that they can be represented by classical means, that is via bivariative probability distributions.

Very recently quantum discord has attracted a lot of attention. This quantity has been studied from a mathematical point of view, focusing in particular on its connection to entanglement, its possible operational meaning [110, 111], how it can be witnessed [112, 113], and the state-space geometry of non-zero, zero [114], and constant discord states [115]. Analytical formula for the discord, initially known only for two qubits [107,116], have been recently obtained also for continuous variable systems [117, 118]. The role of discord has been

investigated in many protocols, e.g., in DQC1 (deterministic quantum computation with one quantum bit) [27] or in the Grover search algorithm [119]. The role of discord in the biological world (light-harvesting process) has been also considered [121]. Discord has been found to be important in quantum phase transitions [120] and even relativistic effects have been investigated [122].

One of the most active fields of research on discord is certainly its dynamics under decoherence. The time evolution of the discord has been studied both in the Markovian [123, 124] and in the non-Markovian scenarios for systems of qubits [125], and recently also for bimodal Gaussian states [126]. Already at a first glance the dynamical behaviour of the discord appears to be quite different compared to the one of entanglement. For example, discord never disappears completely after a finite time, in contrast to the ESD phenomenon discussed in the previous Chapter. Moreover, discord can exhibit sudden changes as a function of characteristic parameters of the system, e.g., time. More precisely the derivative of discord with respect to the given parameter may change abruptly.

Perhaps the most striking difference with the dynamics of entanglement is the fact that, under certain circumstances, discord remains constant for long times even in presence of the environment. To be more specific, in this initial time interval, while the initial quantum correlations are preserved, only classical correlations are lost. For Markovian systems there exists a unique time instant $\bar{t}$ after which the system stops losing classical correlations and starts releasing quantum correlations. For this reason we have called this phenomenon sudden transition between classical and quantum decoherence. We have demonstrated this effect for two qubits subjected to depolarizing channels both in the Markovian and in the non-Markovian scenario.

This is the first evidence of a quantum property which is completely unaffected by the presence of the environment. The sudden transition between classical and quantum decoherence has been already detected experimentally with an all-optical setup [127].

As we mentioned before there are indications that quantum discord could be a resource for quantum technologies [27]. If this turns out to be the case one could exploit the constancy of the discord to perform quantum computation or communication tasks without any disturbance from the noisy environment.

5.3 Sudden transition between classical and quantum decoherence

In this section I introduce the physical system studied. Let us consider two qubits under local depolarizing channels. For this type of channels the Marko-

vian dissipator is in the form $\mathcal{L}[\rho_{A(B)}] = \gamma[\sigma_j^{A(B)}\rho_{A(B)}\sigma_j^{A(B)} - \rho_{A(B)}]/2$, with γ the phase damping rate, $\sigma_j^{A(B)}$ the Pauli operator in direction j acting on $A(B)$, and $j = 1, 2, 3$ for the bit, bit-phase, and phase flip cases, respectively. For the sake of simplicity here I focus on the phase flip channel. Let us assume that the qubits are prepared in a state of the form

$$\rho_{AB} = \frac{1}{4}\left(\mathbf{1}_{AB} + \sum_{i=1}^{3} c_i \sigma_i^A \sigma_i^B\right), \tag{37}$$

where c_i is a real number, such that $0 \leq |c_i| \leq 1$ for every i, and $\mathbf{1}_{AB}$ is the identity operator of the total system. For a depolarizing channel the form of the state is maintained during the evolution. The coefficients evolve as $c_1(t) = c_1(0)\exp(-2\gamma t)$, $c_2(t) = c_2(0)\exp(-2\gamma t)$, and $c_3(t) = c_3(0) \equiv c_3$. A state in the form (37) can be conveniently rewritten in the form of a Bell diagonal state

$$\begin{aligned} \rho_{AB}(t) &= \lambda_\Psi^+(t)|\Psi^+\rangle\langle\Psi^+| + \lambda_\Phi^+(t)|\Phi^+\rangle\langle\Phi^+| \\ &+ \lambda_\Phi^-(t)|\Phi^-\rangle\langle\Phi^-| + \lambda_\Psi^-(t)|\Psi^-\rangle\langle\Psi^-|, \end{aligned} \tag{38}$$

where $|\Psi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\Phi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ are the four Bell states, and

$$\lambda_\Psi^\pm(t) = [1 \pm c_1(t) \mp c_2(t) + c_3(t)]/4, \tag{39}$$

$$\lambda_\Phi^\pm(t) = [1 \pm c_1(t) \pm c_2(t) - c_3(t)]/4. \tag{40}$$

For this type of states the evolution of quantum discord and classical correlations is known analytically [107] and equal to

$$C(\rho_{AB}) = \sum_{k=1}^{2} \frac{1 + (-1)^k \chi(t)}{2} \log_2(1 + (-1)^k \chi(t)), \tag{41}$$

$$D(\rho_{AB}) = 2 + \sum_{k,l} \lambda_k^l(t) \log_2 \lambda_k^l(t) - C(\rho_{AB}), \tag{42}$$

where $\chi(t) = \max\{|c_1(t)|, |c_2(t)|, |c_3(t)|\}$, $k = \Psi, \Phi$, and $l = \pm$. Note that $C(\rho_{AB}) + D(\rho_{AB}) = I(\rho_{AB})$, where $I(\rho_{AB})$ is the total mutual information.

The dynamics of classical and quantum correlations for the same classes of states of Eq. (37) was already studied in Ref. [124]. There three different dynamical regimes were found: i) a regime in which discord decays exponentially

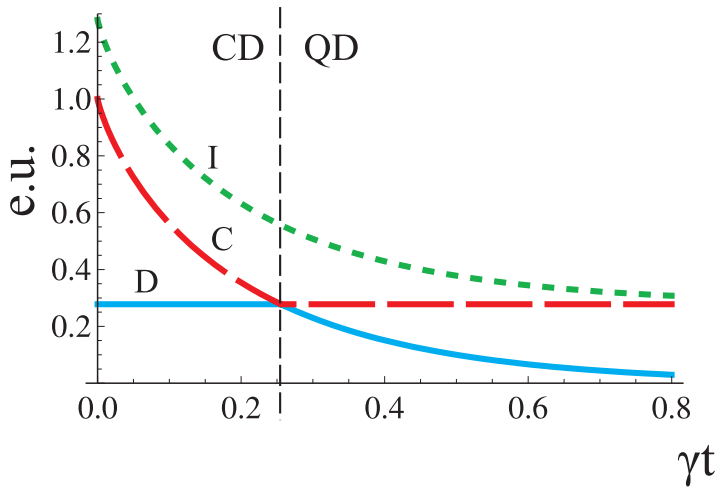

Figure 5: Dynamics of mutual information (green dotted line), classical correlations (red dashed line) and quantum discord (blue solid line) as a function of γt for $c_1(0) = 1$, $c_2(0) = -c_3$ and $c_3 = 0.6$.

and classical correlations are constant; ii) a regime in which both quantum correlations and classical correlations decay exponentially; and iii) another regime in which the quantum discord exhibits a sudden change in the dynamics, i.e., there exists an instant of time $\bar{t}$ in which the time derivative changes abruptly. In the latter regime the discord decays exponentially before *and* after $\bar{t}$, but the decay rate changes. At the same time instant $\bar{t}$ the classical correlations dynamics, passes from an exponential decaying behaviour (for $t < \bar{t}$) to a constant value (for $t > \bar{t}$). In paper VIII we have found that in the last regime and for certain classes of states, before $\bar{t}$ the discord does not decay at all. This can be seen mathematically in a simple way. Let us consider the class of states with $c_{1(2)}(0) = k$, $c_{2(1)}(0) = -c_3 k$ and $c_3(0) = c_3$, with k real and $|k| > |c_3|$. The total quantum mutual information can be written as

$$\begin{aligned} \mathcal{I}[\rho_{AB}(t)] &= \sum_{j=1}^{2} \frac{1+(-1)^j c_3}{2} \log_2[1+(-1)^j c_3] \\ &+ \sum_{j=1}^{2} \frac{1+(-1)^j c_1(t)}{2} \log_2[1+k(-1)^j \exp(-\gamma t)]. \end{aligned} \tag{43}$$

Comparing Eqs. (35) and (42) with Eq. (43) it is easy to see that, for $t < \bar{t} =$

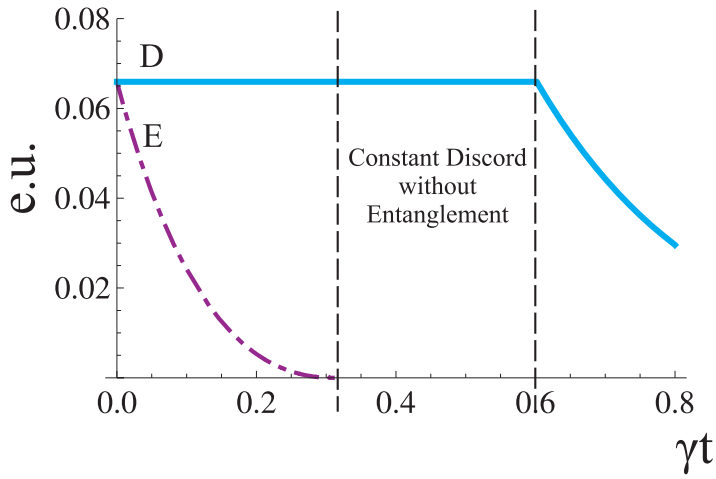

Figure 6: Dynamics of entanglement (violet dashed-dotted line) and quantum discord (blue solid line) as a function of γt for $c_1(0) = 1$, $c_2(0) = -c_3$ and $c_3 = 0.3$.

$-\log_2(c_3/k)/\gamma$, the first constant term and the second decaying term coincide with the discord and the classical correlations, respectively. On the contrary, for $t > \bar{t}$, the two types of correlations swap their role, discord starts to decay (second term) and classical correlations become constant (first term). This result is clearly showed in Figure 5. To elaborate, the system experiences two different dynamical regimes: at the beginning the state of the qubits evolves in a way to lose only classical correlations while quantum correlations are not touched in any way by decoherence. This is what we call the classical decoherence regime. In the second phase, after a sudden change in quantum and classical correlations, quantum correlations are lost while classical correlations are preserved. This is the quantum decoherence phase.

One legitimate question is: what about entanglement? Is the sudden change point related to the occurrence of ESD? We have proven that the entanglement does undergo sudden death at some point but the two phenomena are not connected. Indeed, depending on the state, the sudden death of entanglement can take place either before or after the sudden change time. An example of dynamics of both discord and entanglement, as measured by the relative entropy of entanglement, is given in Figure 6.

Inspired by the discovery of this dynamical effect, Lang and Caves studied

the surfaces of constant discord in the space of parameters of Bell-diagonal states [115]. This study shows that the set of Bell-diagonal states for two qubits can be depicted as a tetrahedron in three dimensions. Level surfaces of entanglement and nonclassical measures can be plotted directly on this three-dimensional geometry. This provides a complete picture of the structure of entanglement and discord for the simple case of Bell-diagonal states. It is interesting to understand how the dynamical effect we have just described is represented on this three-dimensional space.

As we can see from Figure 7, the surface of constant discord are "tube-shaped" therefore discord is neither concave nor convex. The red straight line is a decohering-state trajectory, and it depicts in such constant surface space a constant discord evolution. The line runs along a tube of constant discord, until it encounters an intersecting tube, after which the discord decreases to zero when the state becomes fully classical.

5.4 Explaining the constant discord

In paper IX we continued our investigation on the physical reasons behind the sudden transition and check if this phenomenon is only typical of the Markovian case. An approach which turns out to be useful in the interpretation of our result is proposed in Ref. [128]. There, Modi and collaborators presented a new definition of discord. In analogy to the relative entropy of entanglement, quantum discord is also defined in terms of the relative entropy, where this quantity measures the distance of our state to its closest classical state. In general the new definition of the discord does not equal to the original one. However, the two definitions do coincide in the case of Bell-diagonal states. We can therefore interpret the constant discord evolution in the light of the definition of discord of Ref. [128].

In this framework constant discord means that, in the state space of the qubits, the state of the system has always the same "distance" from its closest classical state. When the discord decays instead the distance to the closest classical state decreases. Interestingly the geometry of the closest classical states shows that, at the transition time, the state under consideration has two closest classical states. This is represented in a pictorial way in Figure 8. There the solid black line represents the trajectory of the state $\rho(t)$ in a schematized state space, and the dotted red line represents the trajectory drawn by the closest classical state $\chi_\rho^{CD}(t)$ (where CD stands for constant discord). This trajectory is parallel to the one traced by $\rho(t)$ meaning that the discord between the two lines is constant. The green square on the $\rho(t)$

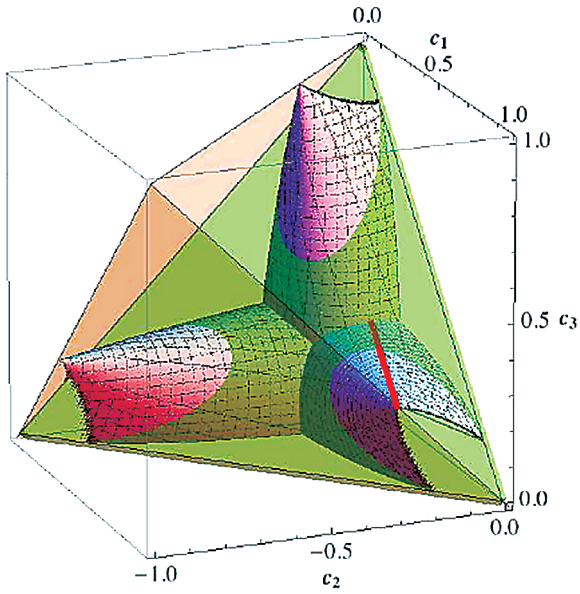

Figure 7: Constant discord surfaces and trajectory under local dephasing channels for a state as in Eq. (37) with initial conditions $c_1(0) = 1$ and $c_2(0) = c_3(0) = 0.3$. This figure is taken from Ref. [115] with the kind permission of the authors.

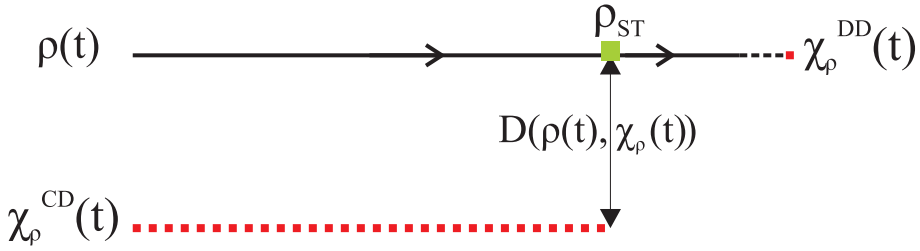

Figure 8: Schematization of the trajectories of both the state of the system and its closest classical state in the state space.

trajectory marks the state ρ_{ST} at which the sudden transition takes place. As mentioned above, the state ρ_{ST} has two different closest classical states having equal distance. One is the state at the end of the red dotted line, i.e., $\chi_\rho^{CD}(t)$, the other one $\chi_\rho^{DD}(t)$ is indicated by the red sphere at the right end of the black line. After the transition $\rho(t)$ keeps traveling the black line from left to right reaching asymptotically the closest classical state.

In paper IX we generalized our study to the non-Markovian scenario. Specifically, we extended the exact model of one qubit under local colored noise depolarizing channel by Daffer *et al.* [129] to the two-qubit case. The phenomenon we have discovered in the Markovian case characterizes also the exact dynamics of the correlations. Specifically, for initial Bell-diagonal states, one can distinguish three different dynamical regimes: (i) constant classical correlations; (ii) damped oscillations of quantum and classical correlations (iii) sudden change behavior with possibility of frozen discord, where the memory effects of the non-Markovian environment bring into play non-trivial dynamical features and allow multiple sudden transitions between the classical and quantum decoherence phases.

To conclude, we demonstrated the existence of a new dynamical effect: the sudden transition between classical and quantum decoherence. This phenomenon is not the result of the Markov approximation but exists also in the exact non-Markovian case. To our knowledge, the constant quantum discord is the first dynamical example of a quantum property which does not decay under decoherence.

6 Conclusions

In this Thesis I have presented my results on different aspects of the dynamics of non-Markovian systems. I have discussed some fundamental problems of the theory of non-Markovian open quantum systems, such as the connection between distinct non-Markovian techniques and the disagreement of different definitions of non-Markovianity. With our study we have gained insight on the role of memory and of the flow-back of information from the environment to the system in different approaches to non-Markovian dynamics.

The dynamics of quantum correlations in non-Markovian systems has been also at the centre of my research. We have solved exactly a fundamental non-Markovian system describing, i.e., two two-level atoms in the same cavity, and investigated the effects that the structured environment has on the dynamics of entanglement. We have also studied the relations between entanglement, purity and nonlocality and generalized the previous model to take into account the effects of spontaneous emission by the atoms.

Studying the dynamics of correlations in a well-known model such as two-qubits in phase flip channels, we have discovered a new dynamical effect entailing the complete preservation of quantum correlations for long intervals of time. This phenomenon is not the result of the Markov approximation but appears also in the non-Markovian scenario.

References

[1] W. H. Zurek, Physics Today **44** (10) 36.

[2] H. - P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, 2002).

[3] U. Weiss, Quantum Dissipative Systems, 3rd ed. (World Scientific, Singapore, 2008).

[4] C.W. Gardiner and P. Zoller, Quantum Noise (Springer- Verlag, Berlin, 2010).

[5] C.W. Lai, P. Maletinsky, A. Badolato, and A. Imamoglu, Phys. Rev. Lett. **96**, 167403, (2006).

[6] A. Pomyalov and D. J. Tannor, J. Chem. Phys. **123**, 204111 (2005).

[7] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[8] V. Gorini, A. Kossakowski, and E. Sudarshan, J. Math. Phys. **17**, 821 (1976).

[9] B. M. Garraway, Phys. Rev. A **55**, 2290 (1997).

[10] J. Piilo, S. Maniscalco, K. Härkönen, and K.-A. Suominen, Phys. Rev. Lett. **100**, 180402 (2008).

[11] S. Nakajima, Progr. Theor. Phys. **20**, 948-959 (1958).

[12] R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960).

[13] H.P. Breuer, B. Kappler, and F. Petruccione, Ann. Phys. **291**, 36-70 (2001).

[14] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).

[15] Á. Rivas, S. Huelga and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).

[16] X.-M. Lu, X. Wang, and C. P. Sun, arXiv:0912.0587.

[17] E.-M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A **81**, 062115 (2010).

[18] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[19] M.A. Nielsen and I.L. Chuang *Quantum Computation and Quantum Information*, (Cambridge, Cambridge University Press, 2000).

[20] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).

[21] B. Bellomo, R. Lo Franco, and G. Compagno, Phys. Rev. Lett. **99**, 160502 (2007).

[22] S. Maniscalco, F. Francica, R. L. Zaffino, N. Lo Gullo, and F. Plastina, Phys. Rev. Lett. **100**, 090503 (2008).

[23] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002).

[24] F. Benatti, R. Floreanini and M. Piani, Phys. Rev. Lett. **91**, 070402 (2003).

[25] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).

[26] L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001).

[27] A. Datta, A. Shaji, and C. Caves, Phys. Rev. Lett. **100**, 050502 (2008).

[28] W.H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).

[29] R. Blatt, Nature **412**, 773 (2001).

[30] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, edited by H. Araki et al., Lecture Notes in Physics Vol. 286 (Springer, Berlin, 1987).

[31] S. Haroche and J.-M. Raimond, *Exploring the Quantum: Atoms, Cavities, and Photons*, (OUP, Oxford, 2006).

[32] P. Lambropoulos and D. Petrosyan, *Fundamentals of Quantum Optics and Quantum Information*, (Springer, Berlin, 2007).

[33] E. Yablonovitch, T. J. Gmitter, and K. M. Leung, Phys. Rev. Lett. **67**, 2295Ð2298 (1991).

[34] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[35] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1998).

[36] W. Nagourney, J. Sandberg, and H. Dehmelt, Phys. Rev. Lett. **56**, 2797 (1986).

[37] J. C. Bergquist, Randall G. Hulet, Wayne M. Itano, and D. J. Wineland, Phys. Rev. Lett. **57**, 1699 (1986).

[38] Th. Sauter, W. Neuhauser, R. Blatt, and P. E. Toschek, Phys. Rev. Lett. **57**, 1696 (1986).

[39] M. Lax, Phys. Rev. **145**, 110 (1966).

[40] B. Misra and E.C.G. Sudarshan, J. Math. Phys. **18**, 756 (1977).

[41] P. Facchi, H. Nakazato, and S. Pascazio, Phys. Rev. Lett. **86**, 2699 (2001).

[42] B. Bellomo, R. Lo Franco, S. Maniscalco, and G. Compagno, Phys. Rev. A **78**, 060302 (2008).

[43] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, Nature **458**, 996 (2009).

[44] P. Rebentrost, R. Chakraborty, A. Aspuru-Guzik, J. Chem. Phys. **131**, 184102 (2009).

[45] R. Alicki, D. A. Lidar, and P. Zanardi, Phys. Rev. A **73**, 052311 (2006).

[46] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **65**, 062101 (2002).

[47] 9 B. M. Terhal and G. Burkard, Phys. Rev. A **71**, 012336 (2005).

[48] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. **101**, 150402 (2008).

[49] B. J. Dalton, S. M. Barnett, and B. M. Garraway, Phys. Rev. A **64**, 053813 (2001).

[50] B. J. Dalton and B. M. Garraway, Phys. Rev. A **68**, 033809 (2003).

[51] J. Piilo, K. Härkönen, S. Maniscalco, and K.A. Suominen, Phys. Rev. A **79**, 062112 (2009).

[52] H.-P. Breuer and J. Piilo, Europhys. Lett. **85**, 50004 (2009).

[53] S. Maniscalco, Phys. Rev. A **72**, 024103 (2005).

[54] S. Barnett and S. Stenholm, Phys. Rev. A **64**, 033808 (2001).

[55] S. Maniscalco and F. Petruccione, Phys. Rev. A **73**, 012111 (2006).

[56] S. Maniscalco, Phys. Rev. A **75**, 062103 (2007).

[57] B. Vacchini and H.-P. Breuer, Phys. Rev. A. **81**, 042103 (2010).

[58] A. Smirne and B. Vacchini, Phys. Rev. A **82**, 022110 (2010).

[59] R.F. Werner Phys. Rev. A **40** 4277 (1989).

[60] Gilles, Brassard, *Quantum cryptography: Public-key distribution and coin tossing* Proceedings of IEEE International Conference on Computers, Systems and Signal Processing 1984. IEEE Computer Society. pp. 175-179, (1984).

[61] P. W. Shor, *Algorithms for quantum computation: Discrete logarithms and factoring*, in Proceedings of the 35th Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, pp. 124Ð134, (1994).

[62] T. Yu and J.H. Eberly, Science **323**, 598 (2009).

[63] A. G. Kofman, A. N. Korotkov, Phys. Rev. A **77**, 052329 (2008).

[64] L. Aolita, R. Chaves, D. Cavalcanti, A. Acín, L. Davidovich, Phys. Rev. Lett. **100**, 080501 (2008).

[65] A. R. Carvalho, A. J. Reid, J. J. Hope, Phys. Rev. A **78**, 012334 (2008).

[66] N. Yamamoto, H. I. Nurdin, M. R. James, I. R. Petersen, Phys. Rev. A **78**, 042339 (2008).

[67] I. Sainz, G. Bjňork, Phys. Rev. A **76**, 042313 (2007).

[68] H. Ian, Z. R. Gong, Y. X. Liu, C. P. Sun, F. Nori, Phys. Rev. A **78**, 013824 (2008).

[69] P. Marek, J. Lee, M. S. Kim, Phys. Rev. A **77**, 032302 (2008).

[70] Y.-X. Gong et al, Phys. Rev. A **78**, 042103 (2008) .

[71] D. Tolkunov, V. Privman, P. K. Aravind, Phys. Rev. A **71**, 060308(R) (2005).

[72] J. P. Paz, A. J. Roncaglia, Phys. Rev. Lett. **100**, 220401 (2008).

[73] C.-S. Chou, T. Yu, B. L. Hu, Phys. Rev. E **77**, 011112 (2008).

[74] R. Vasile, S. Olivares, G. M. Paris, S. Maniscalco, Phys. Rev. A **80**, 062324 (2009).

[75] C. E. Lopez, G. Romero, F. Lastra, E. Solano, J. C. Retamal, Phys. Rev. Lett. **101**, 080503 (2008).

[76] C. Cormick, J. P. Paz, Phys. Rev. A **78**, 012357 (2008).

[77] C.-Y. Lai, J.-T. Hung, C.-Y. Mou, P. Chen, Phys. Rev. B **77**, 205419 (2008).

[78] A. Abliz, H. J. Gao, X. C. Xie, Y. S. Wu, W. M. Liu, Phys. Rev. A **74**, 052105 (2006).

[79] T. Gorin, C. Pineda, T. H. Seligman, Phys. Rev. Lett. **99**, 240405 (2007).

[80] C. Pineda, T. Gorin, T. H. Seligman, New J. Phys. **9**, 106 (2007).

[81] M. Ikram, F.-L. Li, M. S. Zubairy, Phys. Rev. A **75**, 062336 (2007).

[82] K.-L. Liu, H.-S. Goan, Phys. Rev. A **76**, 022312 (2007).

[83] A. Al-Qasimi, D. F. V. James, Phys. Rev. A **78**, 012117 (2008).

[84] A.G. Dijkstra and Y. Tanimura, Phys. Rev. Lett. **104**, 250401 (2010).

[85] G. R. Guthöhrlein, M. Keller, K. Hayasaka, W. Lange and H. Walther, Nature **414**, 49-51 (2001).

[86] M. Keller, B. Lange, K. Hayasaka, W. Lange and H. Walther, Nature **431**, 1075-1078 (2004)

[87] D. L. Moehring, P. Maunz, S. Olmschenk, K. C. Younge, D. N. Matsukevich, L.-M. Duan, C. Monroe , Nature **449**, 68 (2007).

[88] H. Häffner *et al.*, Nature **438**, 643 (2005).

[89] M. A. Sillanpää, J. I. Park, R. W. Simmonds, Nature **449**, 438 (2007).

[90] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.- S. Huang, J. Majer, S. Kumar, S. M. Girvin, R. J. Schoelkopf , Nature **431**, 162 (2004).

[91] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[92] J. Bell, Physics, **1**, 195-200 (1964).

[93] J. F. Clauser, M.A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880-884 (1969).

[94] A. Aspect, P. Grangier and G. Roger, Phys. Rev. Lett. **47**, 460 (1981).

[95] N. Gisin, H. Zbinden, Phys. Lett. A **264**, 103Ð107 (1999).

[96] D. L. Moehring, M. J. Madsen, B. B. Blinov, and C. Monroe, Phys. Rev. Lett. **93**, 090410 (2004).

[97] B. Bellomo, R. Lo Franco, and G. Compagno, Advanced Science Letters **2**, 459 (2009).

[98] Z. Ficek and R. Tanaś, Phys. Rev. A **74**, 024304 (2006).

[99] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. A **60**, 1888 (1999).

[100] J. Lee and M.S. Kim, Phys. Rev. Lett. **84**, 4236 (2000).

[101] Z. Ficek and R. Tanaś, Phys. Rev. A **77**, 054301 (2008).

[102] N. Gisin, Phys. Lett. A **154**, 201 (1991).

[103] J. Barrett, Phys. Rev. A **65**, 042302 (2002).

[104] A. Acin, N. Gisin, and B. Toner, Phys. Rev. A **73**, 062105 (2006).

[105] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, Phys. Rev. A **64**, 030302(R) (2001).

[106] T.-C.Wei, K. Nemoto, P. M. Goldbart, P. G. Kwiat,W. J.Munro, and F. Verstraete, Phys. Rev. A **67**, 022110 (2003).

[107] S. Luo, Phys. Rev. A **77**, 042303 (2008).

[108] B. Groisman, S. Popescu, and A. Winter, Phys. Rev. A **72**, 032317 (2005).

[109] B. Schumacher and M. D. Westmoreland, Phys. Rev. A **74**, 042305 (2006).

[110] D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, A. Winter, arXiv:1008.3205.

[111] V. Madhok, A. Datta, arXiv:1008.4135v2.

[112] B. Dakić, V. Vedral, Č. Brukner, arXiv:1004.0190.

[113] B. Bylicka, D. Chruścínski, Phys. Rev. A **81**, 062102 (2010).

[114] A. Ferraro, L. Aolita, D. Cavalcanti, F.M. Cucchietti, and A. Acín, Phys. Rev. A **81**, 052318 (2010).

[115] M.D. Lang and C.M. Caves, Phys. Rev. Lett. **105**, 150501 (2010).

[116] M. Ali, A.R.P. Rau, and G. Alber, Phys. Rev. A **81**, 042105 (2010).

[117] P. Giorda and M. Paris, Phys. Rev. Lett. **105**, 020503 (2010).

[118] G. Adesso and A. Datta, Phys. Rev. Lett. **105**, 030501 (2010).

[119] J. Cui and H. Fan, J. Phys. A: Math. Theor. **43** 045305 (2010).

[120] T. Werlang, C.Trippe, G.A.P. Ribeiro, G. Rigolin, Phys. Rev. Lett. **105**, 095702 (2010).

[121] K. Bradler, M. M. Wilde, S. Vinjanampathy, D. B. Uskov, arXiv:0912.5112.

[122] A. Datta, Phys. Rev. A **80**, 052304 (2009).

[123] T. Werlang, S. Souza, F. F. Fanchini, C. J. Villas-Boas, Phys. Rev. A **80**, 024103 (2009).

[124] J. Maziero, L. C. Celeri, R. M. Serra, V. Vedral, Phys. Rev. A **80**, 044102 (2009).

[125] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda, A. O. Caldeira, Phys. Rev. A **81**, 052107 (2010).

[126] R. Vasile, P. Giorda, S. Olivares, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **82**, 012313 (2010).

[127] Jin-Shi Xu, Xiao-Ye Xu, Chuan-Feng Li, Cheng-Jie Zhang, Xu-Bo Zou, Guang-Can Guo , Nat. Commun. **1**, 7 (2010).

[128] K. Modi, T. Patarek, W. Son, V. Vedral, and M. Williamson, Phys. Rev. Lett. **104**, 080501 (2010).

[129] S. Daffer, K. Wódkiewicz, J. D. Cresser, and J. K. McIver, Phys. Rev. A **70**, 010304(R) (2004).